Stochastic Calculus for Financial

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$$\frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^{\lfloor t-s/\varepsilon \rfloor} X_i \to N(0, t-s)$$

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$$\frac{\sum_{i=1}^{n} X_i - \mu}{\sigma/\sqrt{n}} \to N(0, \sigma)^2.$$

$$\varepsilon = \frac{1}{n}, \quad n \to \infty$$

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$$\left\lfloor \frac{t-s}{\varepsilon} \right\rfloor$$

Fubini

$$\int_Y \int_X f(x,y)\,dx\,dy = \int_X \int_Y f(x,y)\,dy\,dx$$

- Stochastic Calculus
  - Brownian Motion
    - martingale
    - Markov Process.
  - Itô Calculus. – Black-Scholes PDE
- Financial application
  - Risk-Neutral pricing theory
    - Girsanov Thm.
    - Martingale Representation Thm.
  - fundamental thm of asset pricing. : relation between arbitrage opportunity & existence neutral of risk neutral measure

1° Probability measure.

Given $(\overset{\text{Sample space}}{\Omega}, \underset{\sigma\text{-algebra}}{\mathcal{G}})$, a prob measure $\mathbb{P}: \mathcal{G} \to [0,1]$.

① $\mathbb{P}(\Omega) = 1$

② Countable-additivity: $\{A_n\} \subseteq \mathcal{G}$, $A_n \cap A_m \neq \phi$,

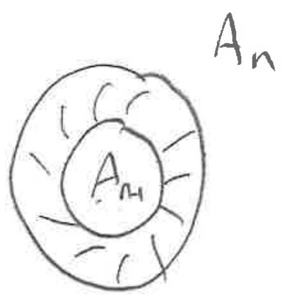
$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

Ex 1: $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n \subseteq \cdots$, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \underline{\lim_{n\to\infty} \mathbb{P}(A_n)}$$

$B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_n = A_n \setminus A_{n-1}$, ...

$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, $\{B_n\}$ are pairwise disjoint.



$$\begin{aligned}\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mathbb{P}(A_n \setminus A_{n-1}) \\ &= \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} (\mathbb{P}(A_n) - \mathbb{P}(A_{n-1}))} \\ &= \underline{\lim_{N\to\infty} \mathbb{P}(A_N)}\end{aligned}$$

Subadditive of $\mathbb{P}$: $\{A_n\} \in \mathcal{F}$,

① $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \mathbb{P}(A_i)$ ✓

② $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

① $\mathbb{E}\left(\mathbb{1}_{\bigcup_{i=1}^{n} A_i}\right) \le \mathbb{E}\left(\sum_{i=1}^{n} \mathbb{1}_{A_i}\right)$ (You check!)

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \qquad\qquad \sum_{i=1}^{n} \mathbb{P}(A_i)$$

② $B_1 = A_1$, $B_2 = A_1 \cup A_2$, ..., $B_n = \bigcup_{i=1}^{n} A_i$, ...

$$\underline{\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right)} = \mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n) = \underline{\lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)}$$

$$\le \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_n) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

2° Change of measure

Given $(\Omega, \mathcal{G}, \mathbb{P})$.

Given $Z \geq 0$, $\mathbb{E}(Z) = 1$

(*) $\mathbb{Q}(A) = \int_A Z(\omega) \, d\mathbb{P}(\omega) = \mathbb{E}(Z \mathbb{1}_A)$, $A \in \mathcal{G}$

You can check that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{G})$.

(*) is called change of measure formula.

$$\mathbb{P} \xrightarrow{(*)} \mathbb{Q}$$

$\underline{Y \sim N(\mu, 1)}$, $\underline{X = Y - \mu \sim N(0,1)}$ under $\mathbb{P}$.

$$\mathbb{P}(Y \leq t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\mu)^2} dy \quad , \quad \mathbb{P}(X \leq t) = N(t)$$

$$Z = e^{-\mu X - \frac{1}{2}\mu^2} > 0 , \quad \mathbb{E}(Z) = 1$$

$$Q(A) = \underline{\mathbb{E}}(Z \mathbb{1}_A) \qquad \mathbb{E}^{Q}(\cdot)$$

$\boxed{Q}$: What's the distribution of Y under Q ??

$$Q(Y \leq t) = \mathbb{E}(Z \mathbb{1}_{\{Y \leq t\}})$$

$$= \mathbb{E}\left(e^{-\mu X - \frac{1}{2}\mu^2} \mathbb{1}_{\{X \leq t - \mu\}}\right)$$

$$= \int_{\mathbb{R}} e^{-\mu x - \frac{1}{2}\mu^2} \mathbb{1}_{\{x \le t-\mu\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx =$$

$$= \int_{-\infty}^{t-\mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+\mu)^2} dx \overset{z = x+\mu}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}z^2} dz = N(t)$$

$$\mathbb{P} \xrightarrow{(*)} \mathbb{Q}$$

$$Y \sim N(\mu, 1) \Longrightarrow N(0,1)$$

$$\boxed{dS_t} = S_t \underbrace{\mu(t, S_t)}_{\text{drift}} dt + \overbrace{S_t \underbrace{\sigma(t, S_t)}_{\text{Volatility}} dW_t}$$

$$Y = \underbrace{\cancel{\mu}}_{\text{drift}} + \underbrace{X}_{\text{vol}}$$

3° R.V. & Expectation:

Ex 1: $\Omega = [0,1]$, $\mathcal{G} = \mathcal{B}([0,1])$, $\mathbb{P}$ - Lebesgue measure

$\mathbb{P}([a,b]) = b-a$

$\{\omega\} = [\omega,\omega]$.

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in \mathbb{Q}^c \text{ irrational} \\ 0 & \text{if } \omega \in \mathbb{Q} \text{ - rational} \end{cases}$$

Show X is a r.v. and compute $\mathbb{E}(X)$.

$\{\omega : X(\omega) \le a\} \in \mathcal{G}$

$\{\omega : X(\omega) = 0\} \in \mathcal{G}$ ✓

$\{\omega : X(\omega) = 0\} = \mathbb{Q} \cap [0,1]$.

In HW, $\{0\} \in \mathcal{B}$

$$\mathbb{Q} \cap [0,1] = \boxed{\bigcup_{\underbrace{\omega \in \mathbb{Q} \cap [0,1]}_{\text{Countable.}}} \{\omega\}} \in \mathcal{B}([0,1])$$

$$\mathbb{E}(X) = \int_{[0,1] \cap \mathbb{Q}^c} \mathbb{1} \, d\mathbb{P}(\omega) = \mathbb{P}([0,1] \cap \mathbb{Q}^c) \overset{?}{=}$$

$$1 - \mathbb{P}(\underline{[0,1] \cap \mathbb{Q}})$$

$$= 1 - \sum_{\omega} \underline{\mathbb{P}(\{\omega\})} = 1 - 0 = \underline{1}$$

Ex 2

$$\mathbb{E}(X) = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$$

Show $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geq t)\, dt$

$$= \int_0^\infty \int_\Omega \mathbb{1}_{\{X \geq t\}}\, d\mathbb{P}(\omega)\, dt$$

$$= \int_\Omega \underbrace{\int_0^\infty \mathbb{1}_{\{X \geq t\}}\, dt}\, \underline{d\mathbb{P}}$$

$$\int_0^{X(\omega)} \mathbb{1}\, dt = t \Big|_0^{X(\omega)} = X(\omega)$$

$$= \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$$

4° Brownian Motion :

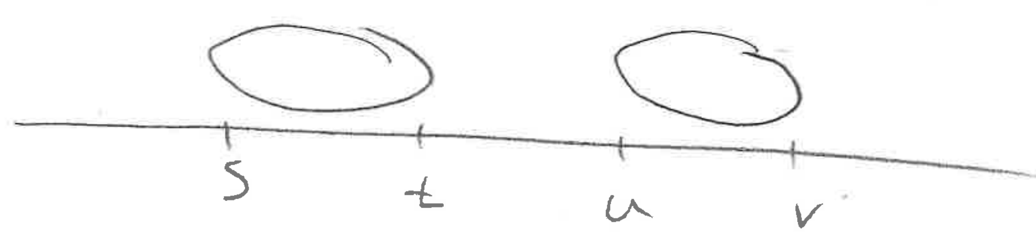
Recall: A stochastic process $(B_t)_{t\geq 0}$ is a B.M.

1. $B_0 = 0$ (a.s.)
2. $t \mapsto B_t(\omega)$ is cont (a.s.)
3. (Stationary increments) : $0 \leq t < t+h$,

$$B_{t+h} - B_t \sim N(0, h)$$

4. (independent increments), $0 \leq s < t \leq u < v$

$$B_t - B_s \perp\!\!\!\perp B_v - B_u$$



Ex1: If B is a B.M.

(1) then $X_t = \frac{1}{\sqrt{\lambda}} B_{\lambda t}, \ t \geq 0$ is B.M. (You check)

(2) $$Y_t = \begin{cases} t B_{\frac{1}{t}}, & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$ is a B.M. (You check)

Def: A stochastic process X is called Guassian if for $t_1 < t_2 < \cdots < t_k$, $(X_{t_1}, X_{t_2}, \cdots, X_{t_k})$ is jointly normal distributed.

Fact: finite dimensional distribution of a Guassian process X is characterized by.

$$m(t) = \mathbb{E}(X_t).$$

$$\rho(s,t) = \mathbb{E}\big((X_t - m(t))(X_s - m(s))\big).$$

$X = B$,

$$m(t) = \mathbb{E}(B_t) = 0 \qquad B_t \sim N(0,t).$$

$$\rho(s,t) = \mathbb{E}(B_s B_t) = \min(s,t)$$

<u>Conclusion</u>: A process B is a B.M. iff

B is a Guassian process, with

$$m(t) = 0$$

$$\rho(s,t) = s \wedge t.$$