$$S(t) = S(n) + (t-n)X_{n+1} \qquad t \in [n, n+1)$$

$$S_\varepsilon(t) = \sqrt{\varepsilon}\, S\left(\frac{t}{\varepsilon}\right)$$

$S_\varepsilon \xrightarrow[\varepsilon \to 0]{} W$ ← called Brownian Motion.

---

$\Omega$ ← Sample Space. $\mathcal{G} = \sigma$-alg → ① Closed under comp ② Countable unions.

$\mathbb{P} \to$ Pr measure: $\forall G \in \mathcal{G},\ P(G) \in [0,1]$.

① $P$ is ctably add. $A_i \in G$, $A_i \cap A_j = \emptyset$ for $i \neq j$

$$\Rightarrow P\left(\bigcup_1^\infty A_i\right) = \sum P(A_i).$$

② $P(\Omega) = 1$.

Random Variables: $X$ is a Random Variable if

① $X$ is a FN from $\Omega \longrightarrow \mathbb{R}$.

② $\forall \alpha \in \mathbb{R}$, $\{X \leq \alpha\} \in \mathcal{G}$.

$\{X \leq \alpha\} = \{\omega \in \Omega \mid X(\omega) \leq \alpha\}$.

Notation: $X$ is a $\mathcal{G}$-meas R.V. (meas wrt $\mathcal{G}$)

Remk: $\{X = \alpha\}$, $\{X > \alpha\}$, $\{X \geq \alpha\}$, $\{X \in (\alpha, \beta)\} \in \mathcal{G}$.

Remk: If $X$ & $Y$ are R.V.'s. $X+Y$, $XY$, $\frac{X}{Y}$.

$X \vee Y$ (max), $X \wedge Y$, $e^X$, ---, are all RV's.

Ex: $A \in \mathcal{G}$. Let $X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$

Notation $\mathbb{1}_A(\omega) =$

Check: If $A \in \mathcal{G}$, then $\mathbb{1}_A$ is a RV.

Note $\{\mathbb{1}_A \leq \alpha\} = \begin{cases} \phi & \alpha < 0 \\ A^c & \alpha \in [0,1) \\ \Omega. & \alpha \geq 1 \end{cases}$ $\in \mathcal{G}$. ✓

Eg: $A_1, \dots A_N \in \mathcal{G}$, <u>disjoint</u>. $a_1, \dots a_N \in \mathbb{R}$.

Define $X = \sum a_i \mathbb{1}_{A_i}$ $X$ is a RV.

(called a Simple RV).

Expectations $\longrightarrow$ Lebesgue Integral.

↳ $X = \sum a_i \mathbb{1}_{A_i}$ , $\{A_i\}$ disj.

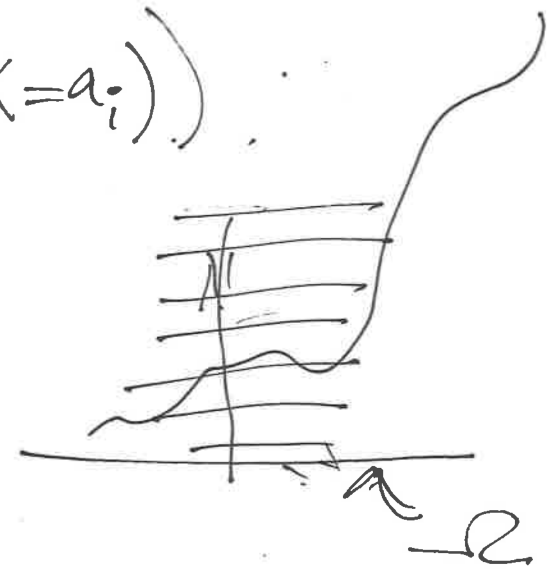
$a_i$'s are unique.

$(i \neq j,\ a_i \neq a_j)$.

$\{X = a_i\} = A_i$

Define: $EX = \sum a_i P(A_i) \quad \left(= \sum a_i P(X = a_i)\right)$.

Def: Say $Y$ is a RV. $Y \geq 0$




Define $EY = \lim_{n \to \infty} EX_n$, where

$$X_n = \sum_{k=0}^{n^2-1} \frac{k}{n} \mathbb{1}_{\left\{\frac{k}{n} \leq Y < \frac{k+1}{n}\right\}}.$$

Def: $Y$ any RV: $Y^+ = \max\{Y, 0\}$
$Y^- = -\min\{Y, 0\}$ } ~~Define~~

Define $EY = EY^+ - EY^-$

Notation: $\int_\Omega Y \, dP \stackrel{def}{=} EY$ $\left( \int_A Y \, dP = E(1_A Y) \right)$

Properties: (1) linearity: $E(X+Y) = EX + EY$.

(2) (Positivity): $X \geq 0$ (almost surely) $\Rightarrow EX \geq 0$

(i.e. $P(X \geq 0) = 1$).

$\Rightarrow$ If $X \leq Y$ a.s. $\Rightarrow EX \leq EY$

# A first Characterization of B.M.

**Def:** A Brownian Motion is a

1. Continuous Process.
2. Stationary Increments
3. Independent Increments.

$\Updownarrow$

**Def 2:** A B.M. is

1. continuous Process.
2. Independent Increments
3. If $s<t$ $\quad W(t)-W(s) \sim N(0, \sigma^2(t-s))$.

**Rmk:** A Standard BM is a B.M. with $W(t)-W(s) \sim N(0, t-s)$

① Continuous Process:

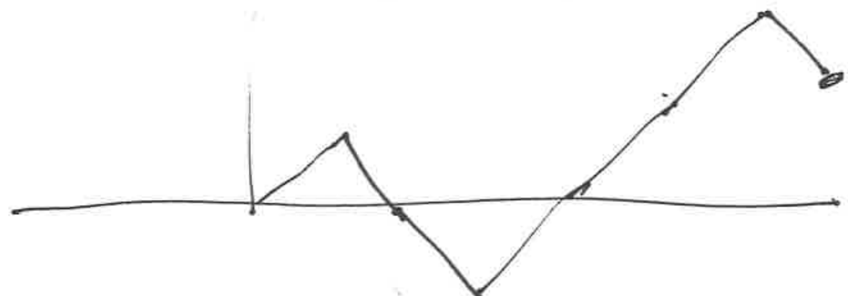
ⓐ Process (aka Stochastic Process) is a fn

$$X : \underset{\uparrow}{\Omega} \times \underbrace{[0,\infty)}_{\text{time.}} \longrightarrow \mathbb{R} \quad \text{such that}$$

$\forall t \in [0,\infty)$, $X(t) : \Omega \longrightarrow \mathbb{R}$ is a Random Var.

fix $\omega \in \Omega$.

ⓑ Continuous Process:



Process ~~for which~~ $\cancel{t \longmapsto X(}$

$\forall \omega \in \Omega$, the fn $t \longmapsto X(\omega, t)$ is cts

(No jumps).

Stationary Increments: $X$ a Stochastic Process.

Pick $t, h$ ~~$\in \mathbb{R}$~~ $\geq 0$ $\qquad \underbrace{X(t+h) - X(t)}_{\text{increment}}$.

Stationary increments: Dist of $(X(t+h) - X(t))$.

ONLY depends on $h$ & not on $t$.

$S(n+1) = S(n) \cancel{*} + X(n+1)$. $\qquad S(n+1) - S(n) = \underbrace{X(n)}_{\text{iid}}$.

Independent Increments:

$X$ has Ind Inc if for any $0 \le t_0 < t_1 < t_2 \cdots < t_N$.

The RV's $\{X(t_0), X(t_1) - X(t_0), X(t_2) - X(t_1), \cdots X(t_N) - X(t_{N-1})\}$.

---

Independence (Measure theoretic way).

<u>Def</u>: $X \to$ a R.V. $(\Omega \to \mathbb{R}$ + $\{X \le \alpha\} \in \mathcal{G})$.

Define $\sigma(X) = \{\sigma\text{-alg generated by } \{X \le \alpha\}\ \forall \alpha \in \mathbb{R}\}$.

$= \text{smallest } \sigma\text{-alg that contains } \{X \le \alpha\}\ \forall \alpha \in \mathbb{R}$.

$\sigma(X)$ = Info you can learn by observing $X$.

$\Omega = \{1, 2, \dots 52\}$. Black $B = \{1, \dots 26\}$ Red $R = \{27, \dots 52\}$.

Red $\rightarrow$ \$1

Black $\rightarrow$ $-$\$1.

Dealer only tells you your winnings.

Know P(picking any individual card) $= \frac{1}{52}$.

By observing only my winnings: can only deduce $P(\text{Red}) = P(\text{Black}) = \frac{1}{2}$.

$$X = \mathbb{1}_R - \mathbb{1}_B \qquad \sigma(X) = \{\emptyset, R, B, \Omega\}.$$

Independence: X, Y.

① X & Y are Independent if. $\forall A \in \sigma(X)$ & $\forall B \in \sigma(Y)$.

$$P(A \cap B) = P(A)\,P(B).$$

② $X_1, X_2, \cdots X_N$. are independent if

$\forall A_i \in \sigma(X_i)$,   ~~$P(A_i \cap A_j) = P(A_i)P(A_j)$ $i \neq j$.~~

$$P(A_1 \cap A_2 \cap \cdots A_N) = P(A_1)\,P(A_2) \cdots P(A_N).$$

Theorem: $\{X_1 \dots X_N\}$ are RV's.

① $\{X_1, \dots X_N\}$ indep't.

$\Leftrightarrow$ ② $\forall \alpha_1, \dots \alpha_N \in \mathbb{R}$, $P\left(\bigcap_i \{X_i \le \alpha_i\}\right) = \overset{\text{Product}}{\prod_{i=1}^{N}} P(\{X_i \le \alpha_i\})$.

$\Leftrightarrow$ ③ $E\left(\prod_{i=1}^{N} f_i(X_i)\right) = \prod_{i=1}^{N} E f_i(X_i)$.

FOR EVERY $f_1, \dots f_N$ cts (bdd).

$\Leftrightarrow$ $E \exp\left(\sqrt{-1} \sum_{i=1}^{N} \xi_i X_i\right) = \prod_{i=1}^{N} E \exp\left(\sqrt{-1}\, \xi_i X_i\right)$.

$(i = \sqrt{-1})$

Eg: $W \longrightarrow$ Stand BM. $\boxed{\begin{matrix} W(t)-W(s) \sim N(0, t-s) \\ W(0) = 0. \end{matrix}}$

Compute $\underline{E}$ ~~W(t)~~ $W(s)W(t)$. $(s<t)$.

$$E\,W(s)W(t) = E\left(W(s)\big[\underbrace{W(t)-W(s)}\,+W(s)\big]\right).$$

$$= \underbrace{E\,W(s)\big(W(t)-W(s)\big)}_{\underbrace{\underline{E}\,W(s)\,\underline{E}\big(W(t)-W(s)\big)}_{0}} + \underbrace{E\big(W(s)^2\big)}_{s}$$

$$0 \quad + $$

$$E W_s W_t = s \qquad (s < t).$$

In general. $E W_s W_t = s \wedge t$.

$$P(A \cap A) = P(A)^2,$$