Stochastic Calculus: TA: Yan Xu (Ryan).

30% (1) Midterm: Feb 8^{th} (in class).

50% (2) Final Mar 4^{th}.

(3) Makeup Final Mar 13^{th}. → Max Grade B−
only improve by 1.

HW: 20% (capped by exam avg).

Class website: Google "GAUTAM IYER" → 944.
(Subscribe mailing list).

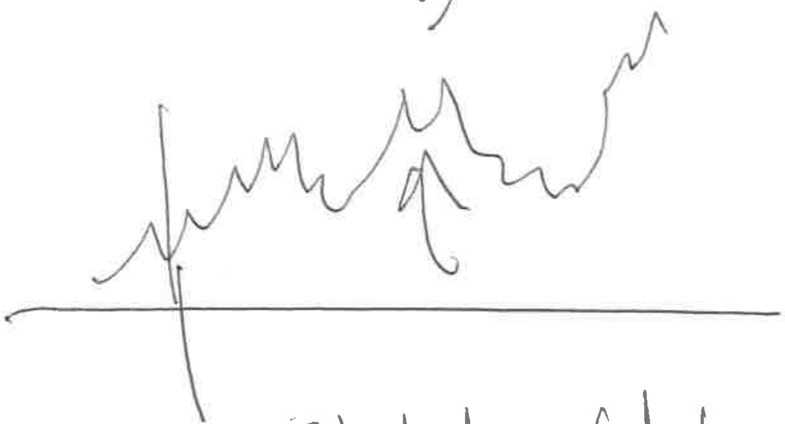
Friday: 1:30 → Make up class.

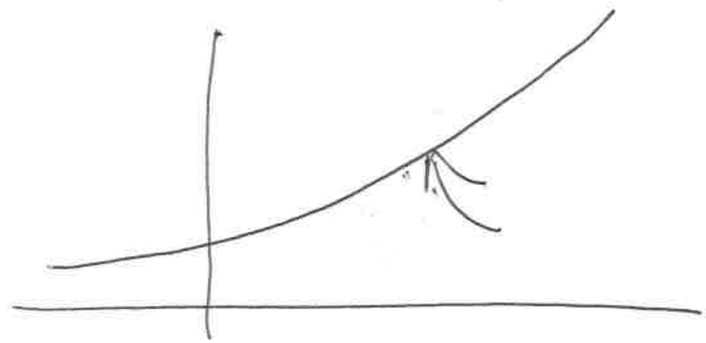
3:00 (4:20). → Extra recitation (this week only)
Usually $\boxed{\text{Fri } 2{:}30}$

HW Due Monday: (Blackdn., in class).

Continuous time.

$$(S_{n+1} - S_n) = \alpha S_n \longrightarrow \boxed{\frac{dS}{dt} = \alpha S.} \rightarrow S(t) = e^{\alpha t} S(0)$$





Stochastic Calculus.

Model stock price. $dS(t) = \alpha S(t)dt + \underbrace{\sigma S(t)\, dW(t)}_{\text{noisy fluctuations}}$.

① Understand "noisy fluctuations" : $\boxed{\text{Brownian Motion}}$.

② Use : Price Securities (not Prediction).

Eg: Call option: strike price K & maturity T.

What is a fair price for this option;

Black Scholes Merton Eq:

$c(t, x) \longrightarrow$ fair price of the option at time $t \leq T$.

given the the stock price $= x$.

$$c(T, x) = (S - K)^{+} = \max\{S-K, 0\}.$$

BSM: $c(t,x) = x \cdot N\left(d_{+}(T-t, x)\right) - K e^{-r(T-t)} N\left(d_{-}(T-t, x)\right).$

$N \longrightarrow$ Normal density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

$\tau = T - t$

$$d_{\pm} = \frac{1}{\sigma \sqrt{\tau}} \left[\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau \right].$$

interest rate.

Tools : B.M. Itô integrals, $\boxed{\text{Itô formula}}$ SDE.

$\boxed{\text{Goal } 2:}$ → "Risk Neutral Measures."

Fundamental theorems of Asset Pricing:

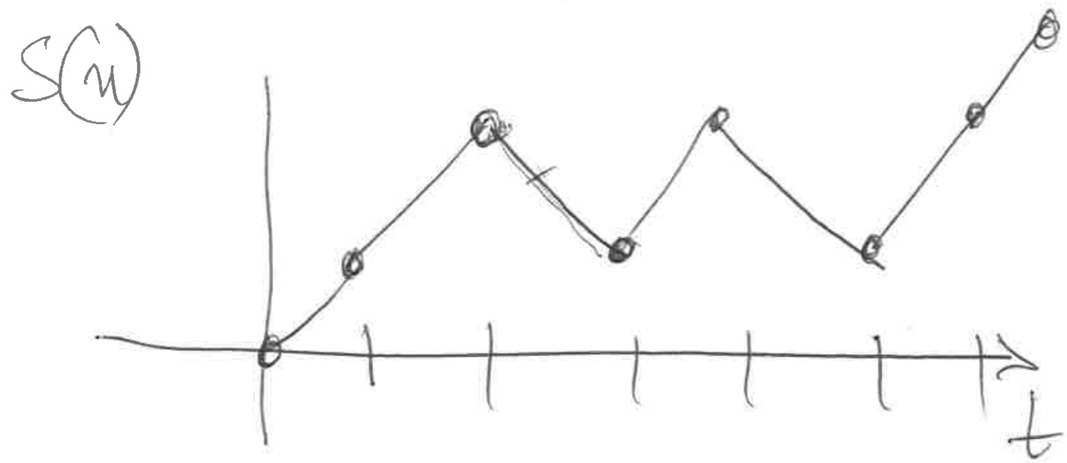
1. Existence of RNM ⟹ no arbitrage.
2. Uniqueness of RNM ⟹ all derivative securities can be hedged.

BROWNIAN MOTION:

(Weiner Process).

Random Walks: $\{X_n\} \longrightarrow \boxed{\text{i.i.d}}$ Random variables.

$$S(n) = \sum_{i=1}^{n} X_i \qquad \left(S(n+1) = \boxed{S(n)} + X_{n+1} \right.$$



~~$S(t) = S(n). \quad t \in [n, n+1)$~~

$$S(t) = S(n) + (t-n) X_{n+1}. \qquad t \in [n, n+1),$$

Rescale: $S_\varepsilon(t) = \alpha_\varepsilon S\left(\frac{t}{\varepsilon}\right)$.

Rescaled step size.

$\varepsilon \to$ time step.

Will chose α_ε ~~so~~ that S_ε "conges" as $\varepsilon \to 0$.

Compute $\mathrm{Var}\left(S_\varepsilon(t)\right)$.

① ~~$\mathrm{Var}\, S(t) =$~~

$\boxed{\text{Assume } X_n \text{ mean } 0 \text{ \& variance } 1}$

② Variance $S(n) = n$. $\mathrm{Var}(S(t)) = n + (t-n)^2$.

③ Variance $S_\varepsilon(t) = \alpha_\varepsilon^2 \left(\left\lfloor \frac{t}{\varepsilon} \right\rfloor + \left(\frac{t}{\varepsilon} - \left\lfloor \frac{t}{\varepsilon} \right\rfloor \right)^2 \right)$.

Choose x_ε so that $\lim_{\varepsilon \to 0} \mathrm{Var}(S_\varepsilon(t))$ exists.

$x_\varepsilon = \sqrt{\varepsilon}$ works. You check: $x_\varepsilon \ll \sqrt{\varepsilon}$.

$\Rightarrow \mathrm{Var}(S_\varepsilon) \longrightarrow 0$

If $x_\varepsilon \gg \sqrt{\varepsilon}$.. $\mathrm{Var}(S_\varepsilon) \longrightarrow \infty$.

Theorem: If $x_\varepsilon = \sqrt{\varepsilon}$ then $S_\varepsilon(t)$ "converges" as $\varepsilon \longrightarrow 0$.

Let $W(t) = \lim_{\varepsilon \to 0} S_\varepsilon(t)$. W is called a B.M.

$W(t)$ is a R.V for every $t \in \mathbb{R}$.

Measure Theoretic Probability:

Sample Space: Set Ω. (Fixed, actual choice unimportant).

<u>Def:</u> (σ-algebra). We say $\mathcal{G} \subseteq \mathcal{P}(\Omega) \leftarrow$ power set.

$\mathcal{G}$ is a σ-alg if (0) $\mathcal{G} \neq \{\phi\}$.

(1) If $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$.

(2) If $A_1, A_2, \ldots \in \mathcal{G} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$.

Termlogy: Elents of $\mathcal{G}$ are called events. [σ alg $\equiv$ infomtion]

Note $\mathcal{G}$ is a σ-alg: then (1) $\phi \in \mathcal{G}$ & $\Omega \in \mathcal{G}$.

Def: $(\Omega, \mathcal{G})$ Wee say $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{G})$ if (1) $\mathbb{P}: \mathcal{G} \to [0,1]$ $(\forall G \in \mathcal{G}, \mathbb{P}(G) \in [0,1])$

(2) $\mathbb{P}$ is countably additive: If $G_1, G_2, \ldots \in \mathcal{G}$, pair one pair-wise disjoint (i.e. $i \neq j \Rightarrow G_i \cap G_j = \phi$) then $\mathbb{P}\left(\bigcup_{i=1}^{\infty} G_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(G_i)$ AND (3)

Rmk: $\mathbb{P}$ is a prob meas. $\Rightarrow$ $\mathbb{P}(\phi) = 0$, $\boxed{\mathbb{P}(\Omega) = 1}$

(2) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. $A \subseteq B, \Rightarrow \mathbb{P}(B - A) = \mathbb{P}(B) - \mathbb{P}(A)$.

(4) $A_1 \subseteq A_2 \subseteq A_3 \cdots$ $\mathbb{P}(\bigcup A_i) = \lim_{i \to \infty} \mathbb{P}(A_i)$.

5 $A_1 \supseteq A_2 \supseteq A_3 \cdots \quad P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i=\infty} P(A_i)$.

Random Variable: Given $(\Omega, \mathcal{G}, P)$

A random variable is a FUNCTION $X : \Omega \longrightarrow \mathbb{R}$

such that $\forall x \in \mathbb{R}$, $\{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{G}$.

[X called $\mathcal{G}$-meas R.V, or measurable]

Notation: Always suppress ω. $\{\omega \in \Omega \mid X(\omega) \leq x\} = \{X \leq x\}$.

$\{X \leq x\} \in \mathcal{G}$. $\quad \{X > x\}, \quad \{X \in (a,b)\} \in \mathcal{G}$.