Unless otherwise stated, $W$ denotes a standard (one dimensional) Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geq 0\}$ (if not otherwise specified) is the Brownian filtration.

5. 1. If $0 \leq s < t$ compute
   
   $$E\left(W(s)^3 \int_0^t (r + W(r))^2 \, dW(r) \mid \mathcal{F}_s\right).$$
   
   Your answer may involve $W$ and Itō integrals, but not any expectations or conditional expectations.

5. 2. Let $X(t) = e^{3t}W(t)^2$. Explicitly find adapted processes $b, \sigma$ such that
   
   $$X(t) = X(0) + \int_0^t b(r) \, dr + \int_0^t \sigma(r) \, dW(r).$$

5. 3. Let $W_1$ and $W_2$ be two independent standard one-dimensional Brownian motions. Find an adapted process $\sigma$ such that the process $B$ defined by
   
   $$B(t) = \int_0^t \frac{1}{1 + W_1(s)^2} \, dW_1(s) + \int_0^t \sigma(s,W_1(s),W_2(s)) \, dW_2(s)$$
   
   is also a standard one-dimensional Brownian motion.

5. 4. Compute
   
   $$E\left(W(t)^3 \int_0^t e^{3W(s)} \, dW(s)\right).$$
   
   You may leave your answer as a Riemann integral of a function that does not involve $W$ or expectations.

5. 5. Consider the statement “The replicating portfolio of an European put is always long on cash.” We interpret this statement mathematically as follows: Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate $r$, and the stock price follows a geometric Brownian motion with mean return rate $\alpha$ and volatility $\sigma$. Here $\alpha$, $\sigma$ and $r > 0$ are constants. Consider a European put with strike $K$ and maturity $T$. Let $p(t,x)$ be the price of this option at time $t$ given that the stock price is $x$. In order to price this option, we construct a replicating portfolio that at time $t$ holds $\Delta(t)$ shares of the stock and has $\Gamma(t)$ invested in the money market account. Write down a formula for $\Gamma(t)$ and use your formula to determine whether $\Gamma(t) \geq 0$ for all $t < T$ or not.
   [You do NOT have to re-derive the Black-Scholes formula, and or any of the Greeks, and may use whatever you know here. In order to get full credit you only need to produce a (correct) formula for $\Gamma(t)$, and use this formula to determine whether $\Gamma(t) \geq 0$ or not. If you have either an incorrect formula with no explanation, or an incorrect explanation (even with a correct formula) you will get no partial credit whatsoever. If you have a correct explanation with a slightly incorrect formula, you might get some partial credit.]

5. 6. Compute
   
   $$E\left[\left|\int_0^t W(s) \, ds\right|^{1/2}\right].$$
   
   Your answer may involve $t$ and Riemann integrals, but may not involve $W$ or expectations.

5. 7. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate $r$, and the stock price follows a geometric Brownian motion with mean return rate $\alpha$ and volatility $\sigma$. Here $\alpha$, $\sigma$ and $r > 0$ are constants. Let $T > 0$ and consider a derivative security that pays
   
   $$V(T) = \left(\ln\left(\frac{S(T)}{S(0)}\right)\right)^+ = \max\left\{0, \ln\left(\frac{S(T)}{S(0)}\right)\right\},$$
   
   where $S(t)$ denotes the stock price at time $t$. Compute $V(T).$
at maturity $T$. Compute the arbitrage free price of this security at any time $t \in [0, T)$. Your answer may involve $\alpha, \sigma, r, t, S(t), T$ and or Riemann integrals. However, your answer should not involve $W, \tilde{W}$, expectations or conditional expectations.