21-269 Vector Analysis: Midterm 2.

2017-04-05

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 50 minutes. The exam has a total of 4 questions and 40 points.*
- *You may use any result proved in class or any regular homework problem* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *The questions are roughly in increasing order of difficulty. Good luck* ◡̈.

[10] 1. Show that there exists an open set $U \subseteq \mathbb{R}$ and a C^1 function $g\colon U \to \mathbb{R}$ such that

$$0 \in U\,, \quad g(0) = 0\,, \quad \text{and} \quad e^{2g(x)} + xg(x) - 1 = x \text{ for all } x \in U?$$

Moreover compute $g'(0)$.

[10] 2. Let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function such that

$$u(0,0) = 0\,, \quad Du_{(0,0)} = \begin{pmatrix} 0 & \pi \end{pmatrix}\,, \quad Hu_{(0,0)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\,.$$

(Recall $Hu_{(0,0)}$ is the Hessian of u at $(0,0)$.) Let $v(t) = u(2t, 3t - t^2)$. Compute $\partial_t^2 v(0)$. Your answer should be a number and not involve u, v or t.

[10] 3. Let $a \in \mathbb{R}^d$ and $f\colon \mathbb{R}^d \to \mathbb{R}$ be a C^2 function such that $Df_a = 0$ and $Hf_a = I$. (Here Hf_a is the Hessian of f at a and I is the $d \times d$ identity matrix.) Show that f attains a local minimum at a.

NOTE: This is a special case of a theorem we proved in class. Please provide a complete proof here, and not just deduce it from the general theorem we proved. You may, however, use any result from class/homework that is independent of the above result.

[10] 4. Let $M \subseteq \mathbb{R}^d$ be a m-dimensional manifold and $a \in M$. Does there exist an open set $V \subseteq \mathbb{R}^m$ and continuous functions $\varphi, u\colon V \to \mathbb{R}^d$ such that $0 \in V$, $\varphi(0) = a$, and for every $x \in V$ we have $\varphi(x) \in M$, $u(x) \in TM_{\varphi(x)}$ and $u(x) \neq 0$? If yes, prove it. If no, find a counter example.