Math 269 Homework

There is a firm 'no late homework' policy. I will often assign harder optional problems. I recommend doing (but not turning in) the optional problems. They often involve useful concepts that will come in handy as the semester progresses.

Assignment 1 (assigned 2017-01-18, due 2017-01-25).

1. For every $a \in \mathbb{R}^d$ show $\lim_{x\to a}|x|^{1/2} = |a|^{1/2}$.
2. (a) Let $f\colon \mathbb{R}^d \to \mathbb{R}$, and suppose $\lim_{x\to a} f(x) = \ell$. Given $v \in \mathbb{R}^d$, consider the one variable function g_v defined by $g_v(t) = f(a+tv)$. True or false: For any $v \in \mathbb{R}^d - \{0\}$, $\lim_{t\to 0} g_v(t) = \ell$? Prove it.

   (b) Is the converse true? Namely, if for every $v \in \mathbb{R}^d$ we know $\lim_{t\to 0} g_v(t) = \ell$, then must $\lim_{x\to a} f(x)$ exist and equal ℓ? Prove it.
3. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$. True or false: $\lim_{x\to a} f(x) = \ell$ if and only if for every $i \in \{1,\dots,n\}$ we have $\lim_{x\to a} f_i(x) = \ell_i$?
4. In this question $x = (x_1, x_2) \in \mathbb{R}^2$. Determine whether the following limits exist, and find them explicitly if they do exist. [Whether the limit exists or not, you should prove your answer. You may of course use the ε-δ definition directly, or rely on results that have already been proved.]

   (a) $\displaystyle\lim_{x\to 0} \frac{x_1x_2}{|x|^2}$.   (b) $\displaystyle\lim_{x\to 0} \frac{x_1x_2}{|x|}$.
5. If $f, g\colon \mathbb{R}^d \to \mathbb{R}$ are such that $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$ and $m \neq 0$, then show that $\lim_{x\to a} f(x)/g(x)$ exists and equals l/m.
6. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that for every $a \in \mathbb{R}$, $\lim_{x\to a} f(x)$ does not exist? Find f, or prove it doesn't exist.

Assignment 2 (assigned 2017-01-25, due 2017-02-01).

1. For $n \in \mathbb{N}$ show directly that the function $f\colon (0,\infty) \to \mathbb{R}$ defined by $f(x) = x^{1/n}$ is continuous.
2. For each of the following functions defined on $\mathbb{R}^2 - \{(0,0)\}$ determine if the function has a limit as $(x,y) \to (0,0)$. Prove your answer.

   (a) $\displaystyle f(x,y) = \frac{3x^2y^2}{x^2+y^2}$ (c) $\displaystyle f(x,y) = \frac{x^3y^4}{|x|^5+y^6}$

   (b) $\displaystyle f(x,y) = \frac{xy(x^2-y^2)}{x^4+y^4}$ (d) $\displaystyle f(x,y) = \frac{x^2y^3}{(x^4+y^6)^{1/3}}$
3. Let $U \subseteq \mathbb{R}^m$, and suppose $f\colon U \to \mathbb{R}^n$ is such that for every $V \subseteq \mathbb{R}^n$ open, $f^{-1}(V) \subseteq U$ is also open. Show that f is continuous.
4. Let $U \subseteq \mathbb{R}^m$ be a domain, $f\colon U \to \mathbb{R}^n$ be a function, and $a \in U$. Show that f is continuous at a if and only if for *every* sequence $(a_n) \to a$ we have $(f(a_n)) \to f(a)$.
5. Let C be the Cantor set.

   (a) Let (a_k) be any sequence such that $a_k \in \{0,2\}$ for all k. Show that the point $\sum_1^\infty a_k 3^{-k} \in C$. [In fact, every point in C can be expressed in this form. If you know a bit about cardinality, you can use this to quickly show that C is uncountable.]

   (b) Recall we constructed C by starting from $[0,1]$ and deleting $(1/3, 2/3)$. Then of the remaining two intervals, we again deleted the middle third, and so on. Let E be the set of all endpoints of the intervals we deleted (i.e. $E = \{1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \dots\}$). Does C simply consist of 0, 1 and the interval endpoints E? Prove or disprove it. If your answer is no, explicitly find a point $c \in C - E$, instead of an abstract existence argument.

Optional problems, and details I omitted in class.

- ∗ Show that f is continuous if and only if the inverse image of closed sets is closed.
- ∗ Show that a arbitrary union of open sets is open.
- ∗ Show that a arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed. Show however, that an infinite union of closed sets need not be closed.
- ∗ Decide if the following are true or false. If true, prove them. If not, provide counter examples.

  (a) The continuous image of a closed set is necessarily closed. (I.e. if f is a continuous function and C is a closed set, then $f(C) \stackrel{\text{def}}{=} \{f(c) \mid c \in C\}$ is necessarily closed).

  (b) The continuous image of open sets is necessarily open.

Assignment 3 (assigned 2017-02-01, due 2017-02-08).

1. A sequence (a_n) is called *Cauchy* if for any $\varepsilon > 0$ there exists $N > 0$ such that whenever $m, n > N$, we have $|a_m - a_n| < \varepsilon$.
   (a) Is any convergent sequence Cauchy? Prove it, or find a counter example.
   (b) Is any Cauchy sequence bounded? Prove it, or find a counter example.
   (c) If (a_n) is a Cauchy sequence that has a convergent subsequence, then must (a_n) be convergent? Prove it, or find a counter example.
   (d) Is any Cauchy sequence (in $\mathbb{R}^d$) convergent (in $\mathbb{R}^d$)? Prove it, or find a counter example.
2. Let (a_n) be a sequence. The *partial sums* of (a_n) are defined to be the sequence (s_n) defined by $s_n = \sum_1^n a_k$. Recall that the *series* $\sum_1^\infty a_k$ is convergent (or the sequence (a_n) is sumable) if the sequence of partial sums (s_n) is convergent. Here are a few facts that you would have encountered (possibly without proof) in your previous Calculus courses.
   (a) *(Comparison test)* Let (a_n) be a sequence in $\mathbb{R}^d$ and (b_n) a sequence in $\mathbb{R}$. If $|a_n| \leqslant b_n$ for all n and the series $\sum b_n$ is convergent, then show that the series $\sum a_n$ is convergent. [Hint: Cauchy sequences are your friend . . .]

   Using the comparison test, the other standard tests (ratio, root, alternating, etc.) can be deduced quickly. This is usually done in the standard one variable calculus courses that you should have already taken. If you haven't seen them (or have forgotten them), look these up as we will use them extensively. To keep the length manageable, we will only cover absolute convergence on here.

   (b) Recall a series $\sum_1^\infty a_k$ is *absolutely convergent* if $\sum_1^\infty |a_k|$ is convergent. Show that an absolutely convergent series is convergent, and $|\sum_1^\infty a_k| \leqslant \sum_1^\infty |a_k|$.
   (c) Is any convergent series absolutely convergent? Prove it, or find a counter example.
   (d) If $(|a_{n+1} - a_n|) \to 0$, must $\sum_1^\infty a_n$ be convergent? Prove it, or find a counter example.
3. A set K is called *sequentially compact* if any sequence in K has a convergent subsequence in K. (That is if (a_n) is any sequence with $a_n \in K$ for all n, then there exists a subsequence (a_{n_k}) such that $\lim a_{n_k}$ exists *and belongs to* K.) Show that a set is sequentially compact if and only if it is closed and bounded.
4. (a) *(Intermediate value theorem)* If $f : [a, b] \to \mathbb{R}$ is continuous, $f(a) < 0$ and $f(b) > 0$ show that there exists $x \in (a, b)$ such that $f(x) = 0$.
   (b) True or false: Any polynomial $f : \mathbb{R} \to \mathbb{R}$ of odd degree has a root in $\mathbb{R}$. Prove it, or find a counter example.
5. True or false: For any polynomial $f : \mathbb{R} \to \mathbb{R}$, $|f|$ attains a minimum in $\mathbb{R}$. Prove it, or find a counter example.

Assignment 4 (assigned 2017-02-08, due 2017-02-15).

1. (a) If $f : \mathbb{Q} \to \mathbb{R}$ is uniformly continuous, then show that it extends to a continuous function $F : \mathbb{R} \to \mathbb{R}$. (That is, show that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(q) = f(q)$ for all $q \in \mathbb{Q}$.) [Hint: Cauchy sequences help. One way of defining exponentials for all real numbers is using this result.]
   (b) Does the previous part fails if we only assume f to be continuous? Prove it, or find a counter example.
2. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and bijective. Show that f is strictly monotone. [A function is strictly monotone if it is either always strictly increasing, or always strictly decreasing.]
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bijective, show that f^{-1} is also continuous.
   (c) ~~*(Optional)* Find an example of $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is continuous and bijective such that f^{-1} is *NOT* continuous.~~
3. (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a, show that f is continuous at a.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' *need not be continuous*. Here is an example: Let $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, and $f(x) = 0$ if $x = 0$. Show that f is differentiable, however f' is *not* continuous (let alone differentiable).
   (c) *(Optional)* Let $n \geqslant 2$. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is n times differentiable, however the n^{th} derivative of f is not continuous?
4. (a) If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at a, and $g(a) \neq 0$, then show that f/g is differentiable at a, and $(f/g)'(a) = [g(a)f'(a) - f(a)g'(a)]/g(a)^2$.
   (b) If $f, g : \mathbb{R} \to \mathbb{R}$ are such that f is differentiable at $g(a)$ and g is differentiable at a, then prove $f \circ g$ is differentiable at a and $(f \circ g)'(a) = f'(g(a))g'(a)$.
5. (a) Suppose f and g are n times differentiable at a. Show that fg is also n times differentiable at a, and find a formula for $(fg)^{(n)}(a)$ in terms of derivatives of f and g.
   (b) Let $n \in \mathbb{N}$. Suppose f is n times differentiable at $g(a)$ and g is n times differentiable at a. Show that $f \circ g$ is n times differentiable at a. [I don't recommend trying to find a formula for $(f \circ g)^{(n)}$.]
6. (a) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable, and f' is increasing. Show that for all $x, y \in [a, b]$, $\theta \in [0, 1]$ we have $f(\theta x + (1 - \theta)y) \leqslant \theta f(x) + (1 - \theta)f(y)$.
   (b) Conversely, suppose for all $x, y \in [a, b]$, $\theta \in [0, 1]$ we have $f(\theta x + (1 - \theta)y) \leqslant \theta f(x) + (1 - \theta)f(y)$. If f is differentiable, show that f' is increasing.

   A function that satisfies either of the above properties is called convex. (The second form is preferable since it doesn't assume differentiability.)

   (c) If $p, q > 1$ with $1/p + 1/q = 1$ and $x, y \in \mathbb{R}$ show that $xy \leqslant |x|^p/p + |y|^q/q$. [Hint: This is part (c) of a question. Google "Young's inequality" if you need more help.]
   (d) Does $\lim_{x \to 0} \frac{x_1 x_2}{(|x_1|^{4/3} + x_2^4)^{.99}}$ exist? How about $\lim_{x \to 0} \frac{x_1 x_2}{(|x_1|^{4/3} + x_2^4)^{1.01}}$ exist? Prove it.

Assignment 5 (assigned 2017-02-15, due Never).

In light of your midterm on 2017-02-22, this homework is optional. It will not be graded, and scanned solutions will not be posted. These are good practice problems, however, and a few of them will make their way to your next homework.

1. If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at a, then we know that $f(x) \approx f(a) + (x-a)f'(a)$. The function $f(a) + (x-a)f'(a)$ is a "first order" approximation of f. The point of this question is to find higher order approximations of f, provided the higher order derivatives of f exist.

   Let $a \in \mathbb{R}$ be fixed, and define $P_{n,f}$, the n^{th} Taylor approximation of f to be the polynomial
   $$P_{n,f}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\,.$$
   (a) For $n \geqslant 1$, show that $P'_{n,f}(x) = P_{n-1,f'}(x)$.

   (b) *(Taylor's theorem.)* Suppose f is n times differentiable at a. Show that $\lim\limits_{x\to a} \frac{f(x)-P_{n,f}(x)}{(x-a)^n} = 0$. [HINT: L'Hospital's rule]

   The above says that $f(x) - P_{n,f}(x)$ is "of smaller order" than $(x-a)^n$. We'd expect it to be "of order" $(x-a)^{n+1}$. This is indeed the case, under a stronger assumption.

   (c) *(A.k.a Taylor's theorem.)* Suppose f is n times differentiable at a, and $f^{(n)}$ is continuous at a. Further, suppose there exists $\varepsilon > 0$ such that $f^{(n)}$ is differentiable on $B(a,\varepsilon) - \{a\}$. For all $x \in B(a,\varepsilon) - \{a\}$, show that there exists ξ between x and a such that
   $$f(x) = P_{n,f}(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}\,.$$
   [HINT: Apply the Cauchy mean value theorem repeatedly to $\frac{f(x)-P_{n,f}(x)}{(x-a)^{n+1}}$.]

2. (a) Suppose f is twice differentiable at a, and f attains a local maximum at a, show that $f'(a) = 0$ and $f''(a) \leqslant 0$.

   (b) Give an example of a function that is twice differentiable at a, has $f'(a) = f''(a) = 0$, however does not attain a local maximum at a.

   (c) Suppose f is twice differentiable at a, $f'(a) = 0$ and $f''(a) < 0$. Show that f attains a local maximum at a.

3. (a) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable and bijective, however f^{-1} is *not* differentiable.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, bijective and f' is never 0, then show that f^{-1} is also differentiable.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable, and $a < b$. If α is such that $f'(a) < \alpha < f'(b)$ then show that there exists $\xi \in (a,b)$ such that $f'(\xi) = \alpha$. [WARNING: f' need not be continuous, so you can't apply the intermediate value theorem.]

5. If $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is linear, show that $DT_a = T$.

6. (a) If $f, g\colon \mathbb{R}^m \to \mathbb{R}^n$ are differentiable, show that $D(f\cdot g) = (Df)^T g + (Dg)^T f$.

   (b) If $a \in \mathbb{R}^m$, $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^m \to \mathbb{R}$ is such that $g(a) \neq 0$ then show that
   $$D\Big(\frac{f}{g}\Big)_a = \frac{g(a)\,Df_a + f(a)\,Dg_a}{g(a)^2}$$

7. Let $f(x,y) = \frac{x^2 y}{x^2+y^2}$ when $(x,y) \neq 0$ and $f(0) = 0$.

   (a) Show that that all directional derivatives of f at 0 exist.

   (b) Explicitly verify that $\partial_x f$ and $\partial_y f$ are not continuous at 0.

   (c) Is f differentiable at 0? Prove it.

8. Here is an example of a function whose partial derivatives are *not continuous*, but the function is differentiable anyway. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x) = |x|^2 \sin(1/|x|)$ when $x \neq 0$, and $f(0) = 0$.

   (a) Show that $\partial_1 f$ and $\partial_2 f$ are continuous in $\mathbb{R}^2$ except at 0.

   (b) Show that f is differentiable at all points in $\mathbb{R}^2$, including at $a = 0$.

9. If $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, then show that f is continuous.

10. (a) Let $F(x,y) = xy$. Given two differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$, let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be defined by $\gamma(t) = (f(t), g(t))$. Observe $\frac{d}{dt}(fg) = D(F \circ \gamma)$. Compute this using the chain rule, and derive the product rule.

    (b) Derive the quotient rule from the chain rule using a method similar to the previous question.

11. Let $U = \mathbb{R}^2 - \{(x,0) \mid x \leqslant 0\}$, and $V = \{(r,\theta) \mid r > 0, \theta \in (-\pi,\pi)\}$. Given a differentiable function f defined on U, we treat it as a function of the coordinates x and y. Using the relation $x = r\cos\theta$ and $y = r\sin\theta$ for $(r,\theta) \in V$, we can now treat f as a function of r and θ.

    (a) Express $\partial_r f$ and $\partial_\theta f$ in terms of $\partial_x f$, $\partial_y f$, r and θ.

    (b) Let $u = x^2 + y^2$ and $v = y/x$. Explicitly express u, v in terms of r and θ and compute $\partial_r u$, $\partial_\theta u$, $\partial_r v$ and $\partial_\theta v$ directly. Verify that this agrees with the formulae in the previous part.

    (c) If further r, θ are functions functions of variables s and t, compute $\partial_s f$ in terms of $\partial_x f$, $\partial_y f$, r, θ, $\partial_s r$ and $\partial_s \theta$.

    (d) Express r, θ in terms of x and y.

    (e) Suppose now g is a differentiable function defined on V, which we treat as a function as a function of r and θ. Using the previous part, we can treat g as a function of x and y. Compute $\partial_x g$ and $\partial_y g$ in terms of $\partial_r g$, $\partial_\theta g$, x and y. Verify your formula is correct for the function $g(r,\theta) = r\theta$.

Assignment 6 (assigned 2017-02-22, due 2017-03-01).

1. Do questions 1, 4, 7, 8 and 11 from Assignment 5.

Assignment 7 (assigned 2017-03-01, due 2017-03-08).

1. Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 \neq 0$. Find all critical points of $ax^2 + 2bxy + cy^2$. Find conditions on a, b, c that would classify this as a local minimum, maximum or saddle.
2. Find the critical points of each of these functions. For each critical point, determine whether it is a local maximum, local minimum, saddle or neither.
   (a) $\frac{x}{x^2+y^2}$
   (b) $[x^2 + (y+1)^2][x^2 + (y-1)^2]$
   (c) $\sin x \cosh y$
   (d) $x^2 - 2xy + y^2$
3. Given $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ define the *divergence* and *curl* of u by
$$\nabla \cdot u \stackrel{\text{def}}{=} \sum_{i=1}^{3} \partial_i u_i \qquad \text{and} \qquad \nabla \times u \stackrel{\text{def}}{=} \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix},$$
respectively. (Note the gradient ∇u is $(Du)^T$, and is notationally different from the divergence $\nabla \cdot u$ because of the missing dot.)
   (a) Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^2. We can combine the divergence, gradient and curl to form a few second order operators. For instance, $\nabla \cdot (\nabla f) = \operatorname{tr} Hf$ is known as the *Laplacian* of f, denoted by Δf. Which of the 9 second order combinations of divergence, gradient and curl make sense? Of the combinations that make sense, exactly one must *always* be 0. Which one?
   (b) Let $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ be C^2. Which of the 9 second order combinations of divergence, gradient and curl make sense? Of the combinations that make sense, exactly one must *always* be 0. Which one?
   (c) Let $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 function. Show that $\nabla \times \nabla \times u = -\Delta u + \nabla \nabla \cdot u$. Here Δu is called the Laplacian of u, and defined to be the column vector $(\nabla \cdot \nabla u_1, \nabla \cdot \nabla u_2, \nabla \cdot \nabla u_3)^T$. [In fact, for a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$, the Laplacian of f (denoted by the same symbol Δf) is defined to be $\nabla \cdot \nabla f$.]
4. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $f\colon \mathbb{R}^d \to \mathbb{R}$ and define $D^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} f$. (We define $\partial_i^0 f = f$ in case any of the coordinates α_i are 0.) In this context, α is called a *multi-index*, and we define $|\alpha| = \sum_i \alpha_i$ to be the *order* of α. Also define $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_d!$, and for $x \in \mathbb{R}^d$ define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$.
   Let $a \in \mathbb{R}^d$ and $f\colon \mathbb{R}^d \to \mathbb{R}$ be C^k. Show that there exists a function R_k such that
$$f(a+h) = \sum_{|\alpha| \leqslant k} \frac{D^\alpha f(a)}{\alpha!} h^\alpha + R_k(h) \quad \text{and} \quad \lim_{h \to 0} \frac{|R_k(h)|}{|h|^k} = 0\,.$$
5. Suppose $f\colon \mathbb{R}^d \to \mathbb{R}$ is C^2. True or false: f is convex if and only if the Hessian Hf is always positive semi-definite? Prove it, or find a counter example.
6. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Recall a domain is a open connected set, $\bar{\Omega}$ is the closure of Ω, and $\partial\Omega = \bar{\Omega} - \Omega$ is the boundary of Ω.
   (a) If $\Delta u > 0$ in Ω, show that $\sup\{u(x) \mid x \in \Omega\} = \sup\{u(x) \mid x \in \partial\Omega\}$.
   (b) If $\Delta u \geqslant 0$ in Ω, show that $\sup\{u(x) \mid x \in \Omega\} = \sup\{u(x) \mid x \in \partial\Omega\}$. [Hint: Let $\varepsilon > 0$ and set $v(x) = u(x) + \varepsilon x_1^2$. Google "maximum principle" if you're stuck.]

Assignment 8 (assigned 2017-03-08, due 2017-03-22).

1. Complete the following details of the proof of the inverse function theorem.
   (a) If $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is any C^1 function such that Df_a is invertible, show that there exists $\varepsilon > 0$ such that whenever $|x - a| < \varepsilon$, Df_x is also invertible.
   (b) Recall we assume $U \subseteq \mathbb{R}^d$, $f\colon U \to \mathbb{R}^d$ is C^1, $a \in U$ and $Df_a = I$. We showed there exist U', V open such that $a \in U'$ and $f\colon U' \to V$ is bijective. Let $g\colon V \to U'$ be the inverse of f. Show that g is C^1.
   Hint: Write $f(x) - f(x_0) - Df_{x_0}(x - x_0) = e(x - x_0)$ for some function e. Set $y = f(x)$, $y_0 = x_0$. Then $x = g(y)$ and $x_0 = g(y_0)$, and write the above in terms of y. You will need an ε-δ argument and the lower bound on $|f(x) - f(x')|$ we had in class.
2. *(Spherical coordinates)* Let $V = \{(r, \theta, \phi) \mid r > 0, \theta \in (-\pi, \pi), \phi \in (0, \pi)\}$ and define $\varphi(r, \theta, \phi) = (x, y, z)$ where $x = r \sin\phi \cos\theta$, $y = r \sin\phi \sin\theta$ and $z = r \cos\phi$. (Geometrically, ϕ is the angle between (x, y, z) and the positive z-axis, and θ is the angle between the projection (x, y) and the positive x-axis.)
   (a) Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0 \text{ or } y \neq 0\}$. Verify that the inverse function theorem applies at all points in U, and r, θ, φ can be (locally) expressed as C^1 functions of x, y, z.
   (b) Explicitly express r, θ, ϕ as functions of x, y, z and show that $\varphi : V \to U$ is bijective. Let $(r, \theta, \phi) = \psi(x, y, z)$ denote the inverse function. Compute $\det D\psi$ explicitly.
   (c) If f is a differentiable function compute $\partial_x f$, $\partial_y f$ and $\partial_z f$ in terms of $\partial_r f$, $\partial_\theta f$, $\partial_\phi f$, r, θ and ϕ.
   (d) If g is a differentiable function compute $\partial_r g$, $\partial_\theta g$ and $\partial_\phi g$ in terms of $\partial_x g$, $\partial_y g$ and $\partial_z g$, x, y and z.
3. We proved the 2D implicit function theorem in class. Here is the higher dimensional version.
   Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open, and $f : U \times V \to \mathbb{R}^n$ be C^1. Let $x_0 \in U$, $y_0 \in V$ and $a = f(x_0, y_0)$. Let $\mathcal{D}_y f = (\partial_{y_j} f_i)$ be the sub-matrix obtained by taking all the rows and the last n columns of Df. If $\mathcal{D}_y f_{(x_0, y_0)}$ is invertible, show that there exists $\varepsilon > 0$, open sets $W \ni a$, $U' \ni x_0$ and a C^1 function $g\colon U' \to V$ such that
$$\{(x, y) \in U \times V \mid f(x, y) = a\} \cap W = \{(x, g(x)) \mid x \in U'\}.$$
   Moreover, show that $Dg_x = -(\mathcal{D}_y f_{(x, g(x))})^{-1} (\mathcal{D}_x f_{(x, g(x))})$, where $\mathcal{D}_x f$ is the sub-matrix obtained by taking all the rows and the first m columns of Df.
4. For each of the equations below near the given point, which variables can be solved for and expressed as differentiable functions of the remaining variables (according to the implicit function theorem). For each of these variables, compute all the partials at the given point. (That is, if you say w, x can be expressed as differentiable functions of y, z, compute $\partial_y w$, $\partial_z w$, $\partial_y x$ and $\partial_z x$ at the given point.)
   (a) $x^2 + y^2 - \cos(xy) = 0$ near $(1, 0)$.
   (b) $e^{xz} + y \sin(yz) + z = 0$ near $(0, 0, -1)$.
   (c) $\sin(xy) + \sin(yz) + \sin(xz) = 0$ and $e^{xyz} + x + y + z = 2$, near $(0, 0, 1)$.

Assignment 9 (assigned 2017-03-22, due 2017-03-29).

1. Parametrize the following curves.
   (a) $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$, where $a, b > 0$ are constants.
   (b) $\cos x \cos y = 1/2$ for $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
2. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a differentiable function so that $\gamma(t)$ represents the position of a particle at time t. Let $v(t) = \gamma'(t)$ denotes the velocity, and $a(t) = v'(t)$ denote the acceleration. We'll see shortly that if γ parametrises a curve Γ, then $v(t)$ is tangent to Γ at the point $\gamma(t)$.
   (a) If $|v(t)| = 1$ for all t, show that $a(t)$ and $v(t)$ are orthogonal vectors.

   If a particle moves in a manner that keeps the *magnitude* of the velocity constant but not the direction, then it will experience acceleration. The magnitude of the acceleration experienced is directly related to the curvature of the path taken by the particle (think about the force you feel when driving around a sharp curve).

   Definition: If a curve Γ is parametrized by the function γ, then we define the *curvature* at the point $\gamma(t)$ by $\kappa = \frac{1}{|\gamma'(t)|} |(\frac{\gamma'(t)}{|\gamma'(t)|})'| = \frac{1}{|v|} |(\frac{v}{|v|})'|$.

   Note that if $|\gamma'(t)| = 1$ for all t, then the curvature is exactly the magnitude of the acceleration. Also, as γ' appears in the denominator, we'll assume in this question that all parametrizations always have non-zero derivative.

   (b) Show that the curvature κ is independent of the parametrization.
   HINT: Suppose β is another parametrization Γ, and set $\varphi = \gamma^{-1} \circ \beta$. Show that the (one variable) function φ is C^1 and φ' has constant sign. Now use $\beta = \gamma \circ \varphi$ and compute the curvature using the parametrization β instead, and verify it is the same as the curvature with the parametrization γ.
   (c) Compute the curvature at any point on a circle of radius R.
   (d) Compute the curvature of the curve $y^2 = x^2 - 1$ at the point $(x, 1)$.
   (e) If $d = 3$, show $\kappa^2 = |v|^{-6}(|a|^2|v|^2 - (a \cdot v)^2)$, and hence conclude $\kappa = \frac{|a \times v|}{|v|^3}$.
3. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be C^2, and $\Gamma = \{(x, f(x)) \mid x \in \mathbb{R}\}$ be the graph of f. Show that the curvature of Γ is $|f''|/(1 + (f')^2)^{3/2}$.
   (b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be C^2, $c \in \mathbb{R}$ and Γ be the curve $\{(x, y) \mid g(x, y) = c\}$. Suppose further $\nabla g \neq 0$ on the curve Γ. Show that the curvature of Γ is

$$\frac{|\partial_x^2 g(\partial_y g)^2 + \partial_y^2 g(\partial_x g)^2 - 2\partial_x \partial_y g \partial_x g \partial_y g|}{|\nabla g|^3}.$$

4. (a) If $n > 1$, and $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ is C^1, show that φ is not injective.
   (b) If $n > 1$, and $\varphi \colon \mathbb{R} \to \mathbb{R}^n$ is C^1, show that φ is not surjective.
   (c) *(Hard optional challenge)* Do the previous two parts with $n > m$, and $\varphi \colon \mathbb{R}^n \to \mathbb{R}^m$ (or $\varphi \colon \mathbb{R}^m \to \mathbb{R}^n$, respectively) instead.
5. Let $M = \{x \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$. Show that M is not a 1-dimensional manifold.

Assignment 10 (assigned 2017-03-29, due Never).

1. Let $V \subseteq \mathbb{R}^m$ be open, and $\varphi\colon \to \mathbb{R}^d$ be a C^1 embedding (i.e. φ is injective, and for all $x \in V$, $\operatorname{rank}(D\varphi) = m$). For every $x_0 \in V$ show that there exists $V' \subseteq V$ open, $U \subseteq \mathbb{R}^d$ open and a C^1 function $\Psi\colon U \to \mathbb{R}^m$ such that $x_0 \in V'$, and for all $x \in V'$, we have $\psi(\varphi(x)) = x$.

   NOTE: The above simply says that ψ is the inverse of φ, when restricted to $\varphi(V')$. Defining ψ on $\varphi(V')$ isn't a problem, of course. The point of this question is that ψ is a C^1 function defined on an open neighborhood of $\varphi(V')$, which is the inverse of φ when restricted to $\varphi(V')$. We stated this in class as a consequence of the local flattening lemma. While you can prove this quickly using the local flattening lemma, I recommend doing it directly instead since it helps with understanding the proof of the local flattening lemma.

2. Let $M \subseteq \mathbb{R}^d$ be a m dimensional manifold and $f\colon M \to \mathbb{R}$ be some function. Given $a \in M$, we say F is a local C^1 extension of f if there exists $U \ni a$ open and a C^1 function $F\colon U \to \mathbb{R}$ such that for every $x \in U \cap M$ we have $F(x) = f(x)$.

   Suppose F, G are two local C^1 extensions of f. Show that DF_a need not equal DG_a in general, however, for every $v \in TM_a$ we must have $DF_a v = DG_a v$.

Let $M \subseteq \mathbb{R}^3$ be a surface, and $a \in M$. We know TM_a is a two dimensional subspace of $\mathbb{R}^3$. We say $\hat{n}(a) \in \mathbb{R}^3$ is a *unit normal* at the point a if $\hat{n}(a) \cdot v = 0$ for all $v \in TM_a$, and $|\hat{n}(a)| = 1$. By counting dimensions we know that at any point there exists exactly two unit normal vectors.

3. (a) Let M be the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Find the "outward pointing" unit normal to M at the point $(x_0, y_0, z_0) \in M$.

   (b) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^1, and $M = \{(x, y, f(x, y))\} \subseteq \mathbb{R}^3$ is the graph of f. Given $a \in M$, find a formula for the unit normal at a that points upwards.

   (c) More generally, suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ is C^1, $M = \{f = 0\}$ and $\nabla f \neq 0$ at all points in M. Given $a \in M$ find a formula for a unit normal at the point a.

   (d) Let M be the Möbius strip parametrized as follows. Let $U = \{(r, \theta) \mid r \in (-1/2, 1/2),\ \theta \in \mathbb{R}\}$. Define $u(\theta) = \cos\theta e_1 + \sin\theta e_2$, and $v(\theta) = \cos(\theta/2) u(\theta) + \sin(\theta/2) e_3$, where $e_i \in \mathbb{R}^3$ are the standard basis vectors. Let $\varphi(r, \theta) = u(\theta) + r v(\theta)$ and $M = \varphi(U)$. At a point $a \in M$, find a unit normal.

   (e) Let M be the Möbius strip above. Prove that there *does not* exist a continuous function $\nu\colon M \to \mathbb{R}^3$ such that for every $a \in M$, $\nu(a)$ is a unit normal at the point a.

Let $M \subseteq \mathbb{R}^3$ be a surface, $a \in M$ and $\hat{n}$ be a unit normal at a. Let P be any plane in $\mathbb{R}^3$ passing through a and $a + \hat{n}$. Then $P \cap M$ is some curve, and let κ_P denote the *signed* curvature of this curve at a. As P is rotated (so that it still passes through a and $a + \hat{n}$), the curve $P \cap M$, and consequently κ_P changes. The *mean curvature*, denoted by H, is defined to be average of the maximum and minimum curvature.

Suppose there exists $U \supseteq M$ open and a C^1 function $\hat{n}\colon U \to \mathbb{R}^3$ such that for all $a \in M$, $\hat{n}(a)$ is a unit normal at the point a. In this case, a standard theorem in differential geometry guarantees $2H = -\nabla \cdot \hat{n}$. The choice of the normal vector affects the sign of the curvature: positively curved surfaces "curve towards" the unit normal.

4. (a) Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ is C^2, $M = \{f = 0\}$ and $\nabla f \neq 0$ at all points in M. Given $a \in M$ show that the mean curvature is given by

   $$2H = \frac{\sum_{i,j=1}^3 \partial_i \partial_j f\, \partial_i f\, \partial_j f - |\nabla f|^2 \Delta f}{|\nabla f|^3}.$$

   (b) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^2, and $M = \{(x, y, f(x, y))\} \subseteq \mathbb{R}^3$ is the graph of f. Show that the mean curvature is given by

   $$2H = \frac{(1 + (\partial_y f)^2) \partial_x^2 f + (1 + (\partial_x f)^2) \partial_y^2 f - 2 \partial_x f\, \partial_y f\, \partial_x \partial_y f}{(1 + |\nabla f|^2)^{3/2}}$$

5. Suppose $f\colon \mathbb{R}^d \to \mathbb{R}^n$ is C^1, $a \in \mathbb{R}^d$, and $\operatorname{rank} Df_a = n$. True or false: There must exist $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subseteq f(\mathbb{R}^d)$. Prove it, or find a counter example.

6. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be two differentiable functions, $c, d \in \mathbb{R}$. Consider the two surfaces $\Gamma = \{x \in \mathbb{R}^3 \mid f(x) = c\}$ and $\Delta = \{x \in \mathbb{R}^3 \mid g(x) = d\}$. Suppose $C = \Gamma \cap \Delta$ is a curve, $a \in C$ and that the vectors $\nabla f(a)$ and $\nabla g(a)$ are linearly independent. Find the tangent space of C at a in terms of ∇f and ∇g. Verify your formula by explicitly computing it when $f(x, y, z) = z^2 - x^2 - y^2 + 1$ and $g(x, y, z) = 2(x - 1) + y - z$ at the point $(1, 0, 0)$.

7. For each of the following implicitly defined sets S, decide if there is an open neighborhood U of the given point such that $S \cap U$ is a manifold. If yes, find a basis of the tangent space, and as many many linearly independent normal vectors as possible at the given point. Also find the tangent line (or tangent plane).

   (a) $x \sin(x) = y + x e^y$ at $(0, 0)$.

   (b) $\ln(xy) = y - x$ at $(1, 1)$

   (c) $x \sin y + y \sin z + z \sin x = 0$ at $(0, \pi, 2\pi)$.

   (d) $z^2 = x^2 + y^2 - 1$ and $2(x - 1) + y - z = 0$ at $(1, 0, 0)$.

Let $M, N \subseteq \mathbb{R}^d$ be two manifolds and $a \in M \cap N$. We say M, N *intersect transversely* at a if for all $u \in TM_a$ and $v \in TN_a$ we have $u \cdot v = 0$.

8. (a) Suppose $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ are C^1. Given $c, d \in \mathbb{R}$, let $M = \{u = c\}$, $N = \{v = d\}$ and assume ∇u and ∇v are nonzero on M and N respectively. Let $a \in M \cap N$. Find a necessary and sufficient condition on ∇u and ∇v that guarantees M and N intersect transversely at a.

   (b) Suppose instead $u\colon \mathbb{R}^3 \to \mathbb{R}^2$, and $v\colon \mathbb{R}^3 \to \mathbb{R}$. As before $c \in \mathbb{R}^2$, and $d \in \mathbb{R}$, let $M = \{u = c\}$, $N = \{v = d\}$ and assume Du and Dv have rank 2 and rank 1 on M and N respectively. Show that M, N intersect transversely at $a \in M \cap N$ if and only if $\nabla u_1(a) \times \nabla u_2(a) \in \operatorname{span}\{\nabla v(a)\}$.

9. *(Challenge)* If Γ is any C^2 curve contained on the sphere $x^2 + y^2 + z^2 = 1$, then show that the curvature at any point on Γ is at least 1.

Assignment 11 (assigned 2017-04-05, due 2017-04-12).

1. Do problems 3(a)–(d), 4, 7(c)–(d), and 8 from the previous homework.

   NOTE: Q1 is also helpful, so I recommend doing it if you have time. Q3(e) is a little harder, but also a fun (possibly surprising) thing to think about. Q9 is worth a reward.

2. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Sometimes when maximising a function in the region $\{g \leqslant c\}$, the maximum (or minimum) could be attained on the boundary $\{g = c\}$. In this case you can of course find all *interior* local maxima and minima by solving $\nabla f = 0$ and looking at Hf. For maxima and minima on the boundary, it is often convenient to use Lagrange multipliers.

3. Find the absolute maxima and minima of e^{-xy} on the set $x^2 + 4y^2 \leqslant 1$.

Optional problems

* For each of the curves below, find the local maxima and minima of the curvature.
  (a) A circle of radius R.
  (b) A straight line.
  (c) The parabola $y = x^2$.
  (d) The ellipse $x^2/a^2 + y^2/b^2 = 1$, for $a, b > 0$.
  (e) The curve $\cos x \cos y = 1/2$ in the region $x, y \in (-\pi/2, \pi/2)$.
* Maximise the volume of an open box given the surface area is $3a^2$. (That is, maximise xyz under the constraint $xy + 2(yz + zx) = 3a^2$.)
* Maximise the volume of a cylinder given that the total surface area is $6\pi a^2$.

  For fun, check if the proportions of your optimal cylinder agrees with your standard coke can; if not, write to Coco-cola with a proposal to save money and the environment. . .
* *(Young's inequality)* Let $p, q > 1$ be such that $1/p + 1/q = 1$. In the region $x, y > 0$ maximise xy subject to the constraint $x^p/p + y^q/q = C$. Use this to give a different proof of Young's inequality (from Homework 4, 6(c)).
* *(Cauchy-Schwartz inequality)* Maximise $x \cdot y$ for $x, y \in \mathbb{R}^n$, subject to the constraint $|x| = a$ and $|y| = b$. Use this to give a different proof of the Cauchy-Schwartz inequality: $|x \cdot y| \leqslant |x|\,|y|$.

Assignment 12 (assigned 2017-04-12, due 2017-04-19).

1. Compute the integrals integrals:

   (a) $\displaystyle\int_{x=-1}^{1} \int_{y=1}^{2} \sin(xy^2)\,dy\,dx.$ (b) $\displaystyle\int_{y=0}^{1} \int_{x=-\cos^{-1} y}^{\cos^{-1} y} e^{\sin x}\,dx\,dy.$

   [Just for fun, try plugging these into a computer, and see if it can spit out the correct answer.]

2. Compute both iterated integrals of the function
$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$
   on the region $[0, 1] \times [0, 1]$. Are they equal? [HINT: First compute $\partial_x \partial_y \tan^{-1}(y/x)$.]

3. Compute $\int_0^\infty \frac{\sin x}{x}\,dx$.

   HINT: Substitute $1/x = \int_0^\infty e^{-xy}\,dy$ above and switch the order of integration. This is one situation where the hypothesis of Fubini's theorem *won't* be satisfied directly; so you will have to work a little to justify your steps.

4. Define $f(x, y) = \dfrac{\exp(-\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}$, and let $I = \displaystyle\int_{\mathbb{R}^2} f(x, y)\,dx\,dy$.
   (a) Compute I by transforming to polar coordinates.
   (b) Compute I by making the substitution $x = u^2 - v^2$ and $y = 2uv$.

5. Let $U \subseteq \mathbb{R}^2$ be a (bounded) oddly shaped region containing the origin. Define $C \subseteq \mathbb{R}^3$ to be the cone with vertex $(0, 0, h)$ and base U given by
$$C = \Big\{(x, y, z) \Bigm| 0 < z < h,\ \Big(\frac{hx}{h - z}, \frac{hy}{h - z}\Big) \in U\Big\}$$
   Find $\operatorname{vol}(C)$ in terms of h and $\operatorname{area}(U)$.

   HINT: Let $u = hx/(h - z)$, $v = hy/(h - z)$ and $w = z$, and transform $\int_C 1\,dx\,dy\,dz$ into u, v, w coordinates.

6. Let $D \subseteq \mathbb{R}^2$ represent an irregular plate whose density is given by $\rho(x, y)$. Let $\ell \subseteq \mathbb{R}^2$ be a straight line, representing a knife edge upon which D is balanced. The magnitude of the *torque* experienced when D is balanced on ℓ is given by $T_\ell = \int_D \rho d\,dx\,dy$. Here $d = p \cdot \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to ℓ, and $p = p(x, y)$ is the vector from (x, y) to the closest point on ℓ. The plate D will balance on a knife edge along ℓ if (and only if) $T_\ell = 0$.
   (a) Define $a = (\int_D x\rho\,dA)/\int_D \rho dA$ and $b = (\int_D y\rho\,dA)/\int_D \rho dA$. If ℓ is any line passing through (a, b) show that $T_\ell = 0$.

   The point (a, b) above for which $T_\ell = 0$ is called the *center of mass* of D.

   (b) Find the center of mass of the triangle with vertices $(0, 0)$, (a, b), $(0, c)$ that has a uniform density. [Assume $0 < a < c$ and $b > 0$.]

7. Let $R > 0$, $\alpha \in (0, \pi)$ and $U \subseteq \mathbb{R}^3$ be the set of all points x such that $|x| < R$ and the angle between x and e_3 is at most α. Find the volume of U.

Optional problems

* If $U \subseteq \mathbb{R}^3$ is a (closed) cuboid and $f \colon U \to \mathbb{R}$ is continuous, then show that f is Riemann integrable.

Assignment 13 (assigned 2017-04-19, due 2017-04-26).

1. (a) *(Leibniz's rule)* Suppose $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is C^1. Show that $\partial_y \int_a^b f(x,y)\,dx = \int_a^b \partial_y f(x,y)\,dx$. [HINT: Use Fubini's theorem and the fundamental theorem of calculus.]

   (b) For $t>0$ compute $\displaystyle \frac{d}{dt}\int_1^t \frac{e^{-s^2 t}}{s}\,ds$.

2. *(Co-area formula)* Let $U\subseteq\mathbb{R}^2$ be a bounded region, and $h\colon U\to[0,1]$ be C^1. For $t\in[0,1]$, let $C_t=\{h=t\}$. Assume that for all $t>0$, C_t is a C^1 curve, and $C_1=\partial U$. If $f\colon U\to\mathbb{R}$ is continuous, show that

$$\int_U f\,dA = \int_{t=0}^1 \int_{C_t} f\,\frac{|d\ell|}{|\nabla h|}\,dt\,.$$

[For this problem assume that there exists a C^1 curve $\Gamma\subseteq U$, and a function $\theta\colon U-\Gamma\to(0,2\pi)$ such that $\nabla h\cdot\nabla\theta=0$, and $\varphi=(h,\theta)\colon U-\Gamma\to(0,1)\times(0,2\pi)$ is a C^1 diffeomorphism.]

3. Let $P\subseteq\mathbb{R}^2$ be a (not necessarily convex) polygon whose vertices, ordered counter clockwise, are (x_1,y_1), …, (x_N,y_N). Show that

$$\operatorname{area}(P)=\frac{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+\cdots+(x_Ny_1-x_1y_N)}{2}.$$

[This is called the *surveyors formula*. While the statement involves only elementary coordinate geometry, it isn't as easy to prove directly this way. Hint: Use Greens theorem.]

4. Let $\Gamma\subseteq\mathbb{R}^2-0$ be a piecewise C^1 parametric curve, and $\gamma\colon[0,1]\to\Gamma$ be a parametrization. Define the *winding number* of Γ about the origin to be

$$W(\Gamma)=\frac{1}{2\pi}\int_\Gamma \frac{1}{x^2+y^2}\begin{pmatrix}-y\\ x\end{pmatrix}\cdot d\ell=\frac{1}{2\pi}\int_0^1\frac{-\gamma_2\gamma_1'+\gamma_1'\gamma_2}{|\gamma|^2}\,dt$$

   (a) If Γ is a closed, piecewise C^1 curve that encloses an open set U and $0\notin U$, show that $W(\Gamma)=0$.

   (b) If $0\in U$, show that the previous part is false. [HINT: Choose $U=B(0,1)$.]

   (c) If Γ is a closed (parametric) curve (i.e. $\gamma(0)=\gamma(1)$), show that $W(\Gamma)\in\mathbb{Z}$.
   HINT: The intuition here is as follows: Let $\varphi(x,y)=\tan^{-1}(y/x)$, and observe $2\pi W(\Gamma)$ should be $\oint_\Gamma \nabla\varphi\cdot d\ell$. If φ was a C^1 function (which it most certainly is not), then we could use the fundamental theorem for line integrals and say that $2\pi W(\Gamma)$ is the angle swept out by the curve, and if Γ is closed, this should be an integer multiple of 2π. The way you make this rigorous is to divide Γ into finitely many pieces where each piece is contained in a half plane, and connects a point on the x axis to point on the y axis (or vice versa). Now show that W of each piece is $\pm 1/4$, and show that when combined you get an integer.

Optional problems.

∗ Let $U,V\subset\mathbb{R}^2$ be domains, $\varphi:U\to V$ be C^2, and $F:V\to\mathbb{R}^2$ be C^1. Define $G:U\to\mathbb{R}^2$ by $G(x)=(D\varphi_x)^T(F\circ\varphi(x))$. Show that

$$\partial_1G_2-\partial_2G_1=[(\partial_1F_2-\partial_2F_1)\circ\varphi]\det(D\varphi)$$

[This was a detail used in the proof of Greens theorem.]

Assignment 14 (assigned 2017-04-26, due 2017-05-03).

1. Let $\Sigma=\{(x,y,z)\mid y^2+z^2=1,\ -1<x<1\ \&\ z>0\}$, and $F=e_3$. Compute $\int_\Sigma F\cdot\hat n\,dS$, where at any point on Σ, $\hat n$ is the upward pointing unit normal.

2. Let $a>b>0$, and Σ be the torus obtained by rotating a circle with center $(a,0,0)$ and radius b about the z axis. Parametrize the surface, and evaluate the surface integral that computes $\operatorname{area}(\Sigma)$.

3. (a) Let $f:[a,b]\to(0,\infty)$ be C^1, and $\Sigma\subset\mathbb{R}^3$ be the surface formed by rotating the graph of f about the x-axis. Explicitly,

$$\Sigma=\{(x,y,z)\mid x\in[a,b]\text{ and } y^2+z^2=f(x)^2\}.$$

   Show that $\operatorname{area}(\Sigma)=2\pi\displaystyle\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$.

   (b) Find the surface area of a cone with base radius r and height h.

4. Let $(\Sigma,\hat n)$ be a bounded oriented surface in $\mathbb{R}^3$. Assume $\partial\Sigma$ is the union of finitely many piecewise C^1 curves. Orient the outermost curve counter clockwise (with respect to the normal vector $\hat n$), and all the inner curves clockwise (with respect to the normal vector $\hat n$). Let $U\supseteq\bar\Sigma$ be open and $F\colon U\to\mathbb{R}^3$ be C^1.

   (a) Suppose $V\subseteq\mathbb{R}^2$ is a domain and $\varphi\colon V\to\Sigma$ is C^2. Define $G\colon\bar V\to\mathbb{R}^2$ by $G=(D\varphi)^TF\circ\varphi$. Show that

$$\partial_1G_2-\partial_2G_1=[(\nabla\times F)\circ\varphi]\cdot(\partial_1\varphi\times\partial_2\varphi)$$

   (b) *(Stokes Theorem)* Show that $\displaystyle\oint_{\partial\Sigma}F\cdot d\ell=\int_\Sigma\nabla\times F\cdot\hat n\,dS$.
   HINT: Assume there exists $V\subseteq\mathbb{R}^2$ open, and a C^1 bijective function $\varphi\colon\bar V\to\bar\Sigma$ such that $\hat n\cdot(\partial_1\varphi\times\partial_2\varphi)>0$. Apply Greens theorem and use the previous part.

5. (a) Let A be a 3×3 matrix, and $u,v\in\mathbb{R}^3$. Show $Au\times Av=\operatorname{adj}(A)^T(u\times v)$, where $\operatorname{adj}(A)$ is the adjoint of the matrix A.

   (b) Let $(\Sigma,\hat n)$ be an oriented surface in $\mathbb{R}^3$, and $U\supseteq\bar\Sigma$ be a domain. Let $\psi\colon U\to\mathbb{R}^3$ be an injective C^1 function such that $\det(D\psi)>0$ on all of U, and $F:\psi(\Sigma)\to\mathbb{R}^3$ be a continuous function. Show that

$$\int_{\psi(\Sigma)}F\cdot\hat n\,dS=\int_\Sigma(\operatorname{adj}(D\psi)(F\circ\psi))\cdot\hat n\,dS$$

6. Suppose $u\colon\mathbb{R}^3\to\mathbb{R}^3$ is a C^1 function such that $\nabla\cdot u=0$. Show that there exists $v\colon\mathbb{R}^3\to\mathbb{R}^3$ such that $u=\nabla\times v$.
   NOTE: We've seen before that if v is C^2 and $u=\nabla\times v$, then $\nabla\cdot u=0$. This problem addresses the converse.