Assignment 15 (assigned 2017-05-03, due Never).

1. Let $u, v : \mathbb{R}^3 \to \mathbb{R}^3$ be C^1 vector functions, $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^1 scalar functions. Prove the following identities.
   (a) $\nabla(fg) = f\nabla g + g\nabla f$.
   (b) $\nabla \cdot (fu) = (\nabla f) \cdot u + f\nabla \cdot u$.
   (c) $\nabla \times (fu) = f\nabla \times u + (\nabla f) \times u$
   (d) $\nabla \times (u \times v) = u(\nabla \cdot v) - v(\nabla \cdot u) + (v \cdot \nabla)u - (u \cdot \nabla)v$
2. Let $F = (2x, y^2, z^2)$.
   (a) Compute $\int_\Sigma F \cdot \hat{n}\, dS$, where $\Sigma \subseteq \mathbb{R}^3$ is the sphere of radius 1.
   (b) Compute $\oint_\Gamma F \cdot d\ell$, where Γ is the intersection of Σ above and the plane $x + 2y + 3z = 0$.
3. In each of the following cases show that
$$\int_U (\nabla \cdot v)\, dV \neq \int_{\partial U} v \cdot \hat{n}\, dS.$$
   Also explain why this does not contradict the divergence theorem.
   (a) $U = B(0,1) \subseteq \mathbb{R}^3$, and $v = x/|x|^4$.
   (b) $U = \{x \in \mathbb{R}^3 \mid z > x^2 + y^2\}$, and $v(x,y,z) = (0, 0, e^{-z})$.

The remaining problems are questions of difficulty comparable to that of the 'longish' questions that will be on your exam. It is ***not*** *a comprehensive review. Your actual exam may (or may not 😱😈) contain a few easier questions, a few questions you've already seen in class / homework, and or a few computational questions.*

4. Let (a_n) be a Cauchy sequence. Show that $\limsup a_n = \liminf a_n$, and use this to conclude (a_n) is convergent.
   NOTE: This gives an alternate proof of convergence of Cauchy sequences. You should do this problem without using the fact that Cauchy sequences are convergent.
5. Let $f\colon [0,1] \to \mathbb{R}$ be a function, and define $G \subseteq \mathbb{R}^2 = \{(x, f(x) \mid x \in [0,1]\}$.
   (a) If f is continuous, must G be closed? Prove it, or find a counter example.
   (b) If G is closed, must f be continuous? Prove it, or find a counter example.
   (c) If G is closed, and f is bounded, must f be continuous? Prove it, or find a counter example.
   [On an actual exam, I would usually only put one of the above parts in the interest of time.]
6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function.
   (a) If $\lim_{x\to 0}\Bigl(\lim_{y\to 0} f(x,y)\Bigr)$ and $\lim_{y\to 0}\Bigl(\lim_{x\to 0} f(x,y)\Bigr)$ both exist, must they be equal? If yes, prove it. If no find a counter example.
   (b) If $\lim_{x\to 0}\Bigl(\lim_{y\to 0} f(x,y)\Bigr)$, $\lim_{y\to 0}\Bigl(\lim_{x\to 0} f(x,y)\Bigr)$ and $\lim_{(x,y)\to 0} f(x,y)$ all exist, must they be equal? If yes, prove it. If no find a counter example.
7. Let $f(x) = \exp(-1/x^2)$ for $x > 0$ and $f(x) = 0$ for $x \leqslant 0$. True or false: for every $n \in \mathbb{N}$, f is C^n. Prove your answer.
8. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be C^2, $a \in \mathbb{R}^d$ be a critical point of f and assume $\det(Hf_a) \neq 0$. True or false: There exists $\varepsilon > 0$ such that f has exactly one critical point in $B(a, \varepsilon)$. Prove it, or find a counter example.
9. Let $V = \{(r,\theta) \mid r > 0, \theta \in (-\pi, \pi)\}$, and set $x = r\cos\theta$, $y = r\sin\theta$. Given a C^2 function f of x and y, express Δf in terms of r, θ and derivatives of f with respect to r and θ. Recall $\Delta f = \partial_x^2 f + \partial_y^2 f$. [ANSWER: $\Delta f = \partial_r^2 f + \partial_r f/r + \partial_\theta^2 f/r^2$.]
10. Let $M \subseteq \mathbb{R}^d$ be a m-dimensional manifold, and $a \in \mathbb{R}^d - M$. Assume further M is a closed (but not necessarily bounded) subset of $\mathbb{R}^d$.
    (a) True or false: there exists $a_0 \in M$ such that $|a - a_0| = \inf\{|a - x| \mid x \in M\}$. Prove it, or find a counter example.
    (b) Suppose $a_0 \in M$ as in the previous part exists. True or false: for all $v \in TM_{a_0}$ we have $v \cdot (a - a_0) = 0$. Prove it, or find a counter example.
11. Let $B = B(0,R) \subseteq \mathbb{R}^3$. Evaluate the surface integral $\int_{\partial B} 1\, dS$, and derive a formula for the surface area of a sphere.
12. *(3D Co-area formula)* Let $U \subseteq \mathbb{R}^3$ be a bounded region, and $h\colon U \to [0,1]$ be C^1. For $t \in [0,1]$, let $\Sigma_t = \{h = t\}$. Assume that for all $t > 0$, Σ_t is a C^1 surface, and $\Sigma_1 = \partial U$. If $f\colon U \to \mathbb{R}$ is continuous, show that
$$\int_U f\, dV = \int_{t=0}^1 \int_{\Sigma_t} f\, \frac{dS}{|\nabla h|}\, dt\,.$$
    [For this problem assume that there exists a C^1 surface $\Sigma' \subseteq U$, and a function $\theta\colon U - \Sigma' \to (0,1)^2$ such that $D\theta\, \nabla h = 0$, and $\varphi = (h, \theta)\colon U - \Sigma' \to (0,1)^3$ is a C^1 diffeomorphism.]
13. *(Greens identity)* Let $U \subseteq \mathbb{R}^3$ be a bounded domain whose boundary is a C^1 surface. If $f, g : U \to \mathbb{R}$ be C^2, show
$$\int_U (f\Delta g)\, dV = \int_{\partial U} f\nabla g \cdot \hat{n}\, dS - \int_U (\nabla f \cdot \nabla g)\, dV.$$
    Recall $\Delta g = \sum_i \partial_i^2 g = \nabla \cdot (\nabla g)$.
14. Define the *Newton potential* $N : \mathbb{R}^3 - 0 \to \mathbb{R}$ by $N(x) = -1/(4\pi|x|)$.
    (a) Let Σ be a closed surface, $\hat{n}$ be the outward pointing unit normal, and U be the region enclosed by U. Show that $\int_\Sigma \nabla N \cdot \hat{n}\, dS = 1$ if $0 \in U$ and 0 otherwise. [HINT: Compute ΔN.]
    (b) *(Gauss's law)* Let $\rho\colon \mathbb{R}^3 \to \mathbb{R}$ be C^1, and define
$$F(x) = \int_{y \in \mathbb{R}^3} \rho(x - y) \nabla N(y)\, dy\,.$$
    (This is a volume integral. I wrote dy instead of dV for clarity, since y is the integration variable and x is fixed.) Show that $\int_\Sigma F \cdot \hat{n}\, dS = \int_U \rho\, dV$.
    NOTE: The first part is fair game as an exam question, but this part is probably a bit harder than what I would put on an exam if you haven't seen it before.
15. Does there exist a C^1 function $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $x \cdot \nabla \times F(x) > 0$ for all $x \neq 0$? If yes, find one such function. If no, prove it doesn't exist.