Math 269 exam like questions

Here are a few questions that appeared on previous exams on the material related to midterm 2 (chain rule, inverse / implicit function theorems, manifolds and tangent spaces). This is not a comprehensive review: there are many topics covered (e.g. parametrizations, Taylor's theorem, coordinate changes, etc.) that aren't in this list, so I recommend reviewing your homework and notes thoroughly. This is just to give you an idea of the level of questions you are expected to solve on the exam.

As an example, something similar in difficulty and length to your midterm would have questions 2, 4, 5(c) and 7 on your midterm. In this case I would give a C to anyone who gets two questions, B to anyone who gets 3 questions and an A to anyone who gets all four questions. (Also, I don't recommend preparing for your midterm expecting it to be a subset of these questions with the fonts and margins changed...).

1. Let $f(x,y) = 2x^2 + 2xy + 2x + 5y^2 - 8y$. Find all points at which $Df = 0$. Classify these points as local maxima, local minima or saddles.

2. Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $2\partial_x u - \partial_y u = 1$. Let $g(t) = u(4t, -2t)$. Compute $\frac{dg}{dt}$. Your answer should simplify to a number.

3. Let $u(x,y) = \ln(x^2 + y^2)$. Compute Δu when $(x,y) \neq (0,0)$. [Recall $\Delta u = \partial_x^2 u + \partial_y^2 u$.]

4. Find the tangent space and the tangent plane of the surface defined by $x \sin y + y \sin z + z \sin x = 0$ at the point $(0, \pi, 2\pi)$.

5. Suppose $f \colon \mathbb{R}^2 \to \mathbb{R}$ is C^1 and $a = (a_1, a_2) \in \mathbb{R}^2$.
   (a) State (without proof) a condition involving only $\partial_1 f(a)$ and $\partial_2 f(a)$ that guarantees the existence of an open set $U \ni a_2$ and a C^1 function $g \colon U \to \mathbb{R}$ such that $g(a_2) = a_1$ and $f(g(t), t) = f(a)$ for all $t \in U$.
   (b) Assuming f, g are both C^2, compute $g''(a_2)$ in terms of $f(a)$, $\partial_i f(a)$ and $\partial_i \partial_j f(a)$ where $i, j \in \{1, 2\}$.
   (c) If is C^2, prove that g is also C^2.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = (x^2 - y^2, 2xy)$. For what points $(x_0, y_0) \in \mathbb{R}^2$ can you find (small) open sets U, V such that $(x_0, y_0) \in U$, $f(x_0, y_0) \in V$ and $f : U \to V$ is bijective?

7. Suppose $f \colon \mathbb{R}^d \to \mathbb{R}^n$ is C^1, $a \in \mathbb{R}^d$, and $\operatorname{rank} Df_a = n$. True or false: There must exist $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subseteq f(\mathbb{R}^d)$. Prove it, or find a counter example.