21-269 Vector Analysis: Midterm 2.

Mar 23rd, 2012

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 5 questions and 50 points.*
- *You may use without proof any result that has been proved in class or on the homework, provided you **CLEARLY** state the result you are using.*
- *Difficulty wise: #1 $\leqslant$ #2 $\leqslant$ #3 $\leqslant$ #4 $<$ #5. The last inequality is strict.*

[10] 1. Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $2\partial_x u - \partial_y u = 1$. Let $g(t) = u(4t, -2t)$. Compute $\frac{dg}{dt}$. [Hint: Your final answer will simplify to a number.]

[10] 2. State whether true or false. No justification is required. A correct answer is worth 2 points, blank answer worth 1 point and incorrect answer worth 0 points.

(a) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a. Then $\partial_1 f$ and $\partial_2 f$ necessarily exist and are continuous at a.

(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is such that both $\partial_1 f$ and $\partial_2 f$ exist at a. Then f is necessarily differentiable at a.

(c) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is such that both $\partial_1 f$ and $\partial_2 f$ exist at a. Then f is necessarily continuous at a.

(d) Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}$ are such that the product fg is differentiable at a. Then f and g are both differentiable at a, and further $D(fg)_a(v) = Df_a(v)g(a) + f(a)Dg_a(v)$.

(e) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ be two differentiable functions such that for all $x \in \mathbb{R}$, $f'(x) \leqslant g'(x)$. Then necessarily $f(x) \leqslant g(x)$ for all $x \in \mathbb{R}$.

[10] 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$, and $a = (x, y) \in \mathbb{R}^2$ be some fixed point. Suppose for all $s, t \in \mathbb{R}$ we have

$$f(x + s, y + t) - f(x, y) = 2x^2 t + 6yst + 9xt^2.$$

Is f differentiable at a? If yes, what is Df_a?

[10] 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f(1/n) = 0$ for all natural numbers n. Suppose further $\lim_{x \to 0} f'(x)$ exists. Must $\lim_{x \to 0} f'(x) = 0$? Provide a proof, or a counter example.

[10] 5. Let $C \subseteq \mathbb{R}$ be compact. Let $D = \{x + y \mid x, y \in C\}$. Is D compact? Prove or provide a counter example.