Handbook of Set Theory

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Contents

I. A Core Model Toolbox and Guide

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1. Introduction

The subject of this chapter is core model theory at a level where it involves iteration trees. Our toolbox includes a list of fundamental theorems that set theorists can use off the shelf in applications; see Section 3. It also contains a catalog of applications of this sort of core model theory; see Section 5. The odd sections have no proofs and are basically independent of the even sections. For those interested in the nuts and bolts of core model theory, we offer a guide to the monograph *The Core Model Iterability Problem* [42] by John Steel in Section 2. We also provide an outline of the paper *The covering lemma up to a Woodin cardinal* by William Mitchell, John Steel and the author [20] in Section 4.

What developed into the theory of core models began in earnest with theorems of Ronald Jensen on L under the hypothesis that $0^{\#}$ does not exist. Jensen showed that if $0^{\#}$ does not exist, then L is the canonical core model, which is written $K = L$. He also showed that if $0^{\#}$ exists but $0^{\#\#}$ does not exist, then $K = L[0^{\#}]$ is the canonical core model. In general, K^V is the canonical core model (if there is one) whereas W is a core model if $W = K^M$ where M is a transitive class model of ZFC. Unfortunately, we must ask the reader to pay close attention to articles in the sense of grammar.

Whether or not it is possible to give a definition of K that allows us to make sense of K^M for all M is unknown. Up until recently, for those M for which K^M has been defined, K^M has turned out to be an extender model. Backing up slightly, recall that the existence of $0^{\#}$ is equivalent to the existence of an ordinal κ and an ultrafilter F over $\wp(\kappa) \cap L$ that gives rise to a non-trivial elementary embedding from L to itself. *Large cardinal axioms* such as the existence of $0^{\#}$ can all be phrased in terms of the existence of filters or systems of filters. Some of these systems are

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known as *extenders*. A *model* is a transitive set or proper class transitive model of ZFC. *Extender models* are models of the form J_Ω^E where $\Omega \leq \mathrm{On}$, E is a sequence of length Ω and E_α is an extender for each $\alpha < \Omega$.[1]

Statements asserting that certain models with large cardinals do not exist are called *anti-large cardinal hypotheses*. Instead of making this precise, we list the four examples most relevant to our introduction.

- $0^\#$ does not exist.
- There is no proper class model with a measurable cardinal.
- There is no proper class model with a measurable cardinal κ with Mitchell order $o(\kappa) = \kappa^{++}$.
- There is no proper class model with a Woodin cardinal.

For the last three examples, it would be equivalent to replace "model" by "extender model" although this is not obvious.

Much of core model theory deals with generalizations of Jensen's theorems about L. The core model theorist adopts an anti-large cardinal hypothesis, possibly for the sake of obtaining a contradiction. Then he defines K and shows that K has many of the same useful properties that L has if $0^\#$ does not exist. With some exceptions, these properties fall into the following categories.

- Fine structure with the consequence, for example, that GCH and combinatorial principles such as $\diamondsuit$ and $\square$ hold in K.
- Universality with the consequence, for example, that the existence of certain extender models is absolute to K.
- Maximality with the consequence, for example, that certain large cardinal properties of κ are downward absolute to K.
- Definability in a way that makes K absolute to set forcing extensions.
- Covering with the consequence, for example, that K computes successors of singular cardinals correctly.

Often, such properties of K are used in elaborate proofs by contradiction. In order to prove that a principle P implies the existence of a model with large cardinal C, one may assume that there is no model of C and use P to show that one of the basic properties of K fails. When this accomplished, it follows that the large cardinal consistency strength of P is at least C.

[1] Is every core model an extender model? Since we do not know how to define K in the abstract, it is impossible to answer this question. There are models that are not extender models that most likely will be accepted as core models but these are beyond the scope of this introduction.

Dodd and Jensen developed the theory of K under the anti-large cardinal hypothesis that there is no proper class model with a measurable cardinal. Mitchell did this under the hypothesis that there is no proper class model with a measurable cardinal κ of order κ^{++}. Steel did this under the hypothesis that there is no proper class model with a Woodin cardinal except that he added a technical hypothesis, which we discuss momentarily.

It is important to emphasize that we do not know how to define K without an anti-large cardinal hypothesis. We do not refer to the Dodd-Jensen, Mitchell or Steel core model without the corresponding anti-large cardinal hypothesis. It is also important to know that the various definitions of K are consistent with each other. For example, if there is no transitive class model with a measurable cardinal, then the Dodd-Jensen, Mitchell and Steel definitions of K coincide. Quite reasonably, if $0^{\#}$ does not exist, then $K = L$ under all three definitions.

For all but the last section of this paper we assume:

Anti-large cardinal hypothesis. *There is no proper class model with a Woodin cardinal.*

From what we said about the core model theories predating Steel's, the reader might expect that we could go straight into a discussion of K. But it is not known if the theory of K can be developed under this anti-large cardinal hypothesis alone. Following Steel, we add:

Technical hypothesis. Ω *is a measurable cardinal and* U *is a normal measure over* Ω.

This means that U is a non-principal Ω-complete normal ultrafilter on $\wp(\Omega)$. Except in the last section, we also assume this technical hypothesis throughout this paper. Of course, by adding the technical hypothesis to ZFC we obtain a stronger theory. But, in this setting, it is not much stronger as measurable cardinals are much weaker than Woodin cardinals.

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2. Basic Theory of K

2.1. Second-Order Definition of K

All of the results and proofs in Sections 2.1 and 2.2 are due to Steel and come from [42]. But we only assume that the reader is familiar with [41] through the theory of countably certified construction.[2] Recall from §6 of [41] that

[2] Countably certified constructions are called K^c*-constructions* in [41]. There, K^c-constructions are studied in generality before a particular maximal K^c-construction

$$\langle \mathcal{N}_\alpha \mid \alpha \leq \Omega \rangle$$

a countably certified construction is a sequence of premice $\langle \mathcal{N}_\alpha \mid \alpha \leq \Omega \rangle$ where either

$$\mathcal{N}_{\alpha+1} = \mathrm{rud}(\mathfrak{C}(\mathcal{N}_\alpha))$$

or

$$\mathcal{N}_{\alpha+1} = \mathrm{rud}(\mathfrak{C}(\mathcal{N}_\alpha)^\frown\langle F \rangle),$$

where the second option (adding an extender) is permitted if

$$\mathfrak{C}(\mathcal{N}_\alpha)^\frown\langle F \rangle$$

is a countably certified mouse. When β is a limit ordinal, we define

$$\mathcal{N}_\beta = \liminf \langle \mathcal{N}_\alpha \mid \alpha < \beta \rangle.$$

The gist of §6 of [41] as it applies to us is that the following statements hold for all $\gamma \leq \Omega$.

1. $\mathcal{N}_\gamma$ is a 1-small premouse. In other words, no initial segment of $\mathcal{N}_\gamma$ has the first-order properties of a sharp for an inner model with one Woodin cardinal.

2. If $\mathcal{P}$ is a countable premouse that embeds into $\mathcal{N}_\gamma$, then $\mathcal{P}$ is $\omega_1 + 1$ iterable.

3. Let $\alpha < \gamma$. Suppose that $\kappa \leq \rho_\omega^{\mathcal{N}_\beta}$ for all β such that $\alpha < \beta < \gamma$. Then $\mathcal{N}_\alpha$ and $\mathcal{N}_\gamma$ agree below $(\kappa^+)^{\mathcal{N}_\alpha}$.

The proof of the clause (1) uses our anti-large cardinal hypothesis. Countable certificates are used in the proof of the clause (2). Clause (3) implies that $\mathcal{N}_\Omega$ has height Ω. Another important fact that we revisit in this paper is Theorem 6.19 of [41], which implies that if $\langle \mathcal{N}_\alpha \mid \alpha \leq \Omega \rangle$ is a maximal countably certified construction, then $\mathcal{N}_\Omega$ computes κ^+ correctly for U almost all $\kappa < \Omega$. In this context, *maximal* means that at all successor stages of the construction, if it is possible to add an extender, then we do.

To define K^c, we consider a kind of countably certified construction that is not maximal but still computes the successors of U almost all cardinals correctly. The new condition is that we add an extender to form

$$\mathcal{N}_{\alpha+1} = \mathrm{rud}(\mathfrak{C}(\mathcal{N}_\alpha)^\frown\langle F \rangle)$$

whenever it is permitted so long as

$$\mathrm{crit}(F) \text{ is an inaccessible cardinal}$$

and, if

$$(\mathrm{crit}(F)^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = \mathrm{crit}(F)^+$$

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is fixed, at which point K^c is defined to be $\mathcal{N}_\Omega$. The terminology here is slightly different in this respect, and so is the definition of K^c.

then

$$\{\kappa < \mathrm{crit}(F) \mid \kappa \text{ is an inaccessible cardinal and } (\kappa^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = \kappa^+\}$$

is stationary in $\mathrm{crit}(F)$. For the rest of this section, fix such a countably certified construction and let $K^c = \mathcal{N}_\Omega$.

2.1 Definition. A *weasel* is an $(\omega, \Omega + 1)$ iterable premouse of height Ω.

This is slightly different from the notation in [42] where weasels are not required to be iterable at all.[3] The meaning of $(\omega, \Omega + 1)$ *iterable* is given by Definition 4.4 of [41]. It says that there is a strategy for picking cofinal branches at limit stages that avoids illfounded models at all stages when building almost normal iteration trees. These are iteration trees obtained as follows.

- Build a normal iteration tree $\mathcal{T}_0$ of length $\leq \Omega + 1$.
- If $\mathcal{T}_n$ has successor length $\theta_n + 1 < \Omega + 1$, then build a normal iteration tree $\mathcal{T}_n$ on an initial segment of $\mathcal{M}^{\mathcal{T}_n}_{\theta_n}$.
- If $\mathcal{T}_n$ is defined for all $n < \omega$, then form the concatenation

$$\mathcal{T}_0{}^\frown \mathcal{T}_1{}^\frown \cdots {}^\frown \mathcal{T}_n{}^\frown \cdots$$

In particular, the unique cofinal branch of the infinite concatenation may have only finitely many drops and its corresponding direct limit must be wellfounded. The original *raison d'être* for almost normal iteration trees is the Dodd-Jensen lemma, Theorem 4.8 of [41]. The reader must forgive us for not saying whether we mean *normal* or *almost normal* when we write *iteration tree* in this basic account except at key places when the difference is most pronounced.

By the next theorem, the only way to iterate a weasel is to pick the unique cofinal wellfounded branch through an iteration tree of limit length $< \Omega$.

2.2 Theorem. *Let $\mathcal{P}$ be a premouse with no Woodin cardinals. Suppose that $\mathcal{T}$ is an iteration tree of limit length on $\mathcal{P}$. Assume that*

$$\delta(\mathcal{T}) < \mathrm{On} \cap \mathcal{P}.$$

Then $\mathcal{T}$ has at most one cofinal wellfounded branch.

[3] Following the convention on premice versus mice, a structure with the first-order properties of a weasel should have been called a *preweasel*.

Sketch. Let $\theta = \mathrm{lh}(\mathcal{T})$. Recall from Definition 6.9 of [41] that

$$\delta(\mathcal{T}) = \sup(\{\mathrm{lh}(E^{\mathcal{T}}_\eta) \mid \eta < \theta\})$$

and $\mathcal{M}(\mathcal{T})$ is the unique passive mouse of height $\delta(\mathcal{T})$ that agrees with $\mathcal{M}^{\mathcal{T}}_\eta$ below $\mathrm{lh}(E^{\mathcal{T}}_\eta)$ for all $\eta < \theta$. Our anti-large cardinal hypothesis implies that $\delta(\mathcal{T})$ is not a Woodin cardinal in $L[\mathcal{M}(\mathcal{T})]$. Let $\mathcal{Q}(\mathcal{M}(\mathcal{T}))$ be the premouse $\mathcal{R}$ of minimum height such that

$$\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{R} \triangleleft L[\mathcal{M}(\mathcal{T})]$$

and $\delta(\mathcal{T})$ is not a Woodin cardinal in $\mathrm{rud}(\mathcal{R})$. By Theorem 6.10 of [41], there is at most one cofinal branch b of $\mathcal{T}$ with the property that

$$\mathcal{Q}(\mathcal{M}(\mathcal{T})) \trianglelefteq \mathrm{wfp}(\mathcal{M}^{\mathcal{T}}_b).^4$$

Our assumptions about $\mathcal{P}$ and $\mathcal{T}$ imply that if b is a cofinal wellfounded branch of $\mathcal{T}$, then

$$\mathcal{Q}(\mathcal{M}(\mathcal{T})) \trianglelefteq \mathcal{M}^{\mathcal{T}}_b.$$

⊣

2.3 Theorem. *K^c is a weasel.*

Sketch. We already know that K^c is a premouse of height Ω. It remains to see that K^c is $(\omega, \Omega + 1)$ iterable. Here we show that it is Ω iterable. Our strategy is to pick the unique cofinal wellfounded branch through iteration trees of length $< \Omega$ and to use the fact that Ω is measurable to find a branch through iteration trees of length Ω.

First suppose that $\mathcal{T}$ is an iteration tree on K^c of length $\theta < \Omega$. Recalling that $H(\lambda)$ denotes the collection of sets hereditarily of cardinality $< \lambda$, let

$$\pi : N \to H(\Omega^+)$$

be an elementary embedding with N countable and transitive. Say $\pi(\mathcal{P}) = K^c$ and $\pi(\mathcal{S}) = \mathcal{T}$. By Theorem 6.16 of [41], $\mathcal{P}$ has an $\omega_1 + 1$ iteration strategy. Then $\mathcal{S}$ is consistent with this strategy because there is only one strategy: by Theorem 2.2, $[0, \eta)_S$ is the unique cofinal wellfounded branch of $\mathcal{S}{\restriction}\eta$ whenever η is a limit ordinal $< \mathrm{lh}(\mathcal{S})$.

Assume that $\theta = \eta + 1$ and F is an extender from the $\mathcal{M}^{\mathcal{T}}_\eta$ sequence such that $\mathrm{lh}(F) > \mathrm{lh}(E^{\mathcal{T}}_\zeta)$ for all $\zeta \leq \eta$. We claim that

$$\mathrm{Ult}(\mathcal{M}^*_{\zeta+1}, F)$$

[4] We define $\mathcal{M}^{\mathcal{T}}_b$ to be the Mostowski collapse of the direct limit of $\mathcal{M}^{\mathcal{T}}_\eta$ for $\eta \in b$ even if this direct limit is illfounded. By $\mathrm{wfp}(\mathcal{M}^{\mathcal{T}}_b)$ we mean the wellfounded part of $\mathcal{M}^{\mathcal{T}}_b$. In this case, the wellfounded part and the transitive part are the same because $\mathcal{M}^{\mathcal{T}}_b$ is its own Mostowski collapse.

is wellfounded where $\zeta \leq \eta$ is least so that $\text{crit}(F) < \nu(E^{\mathcal{T}}_{\zeta})$, $\mathcal{M}^*_{\zeta+1}$ is the maximal level of $\mathcal{M}^{\mathcal{T}}_{\zeta}$ that is measured by F, and the degree of the ultrapower is as large as possible. Otherwise, there exists such an F and a witness to illfoundedness in the range of π, so the corresponding extension of $\mathcal{S}$ using $\pi^{-1}(F)$ is also illfounded. This contradicts that $\mathcal{P}$ is $\text{lh}(\mathcal{S}) + 1$ iterable.

Now assume that θ is a limit ordinal $< \Omega$. Let b be the unique cofinal wellfounded branch of $\mathcal{S}$. We know that b is the unique cofinal branch of $\mathcal{S}$ with the property that

$$\mathcal{Q}(\mathcal{M}(\mathcal{S})) \trianglelefteq \text{wfp}(\mathcal{M}^{\mathcal{S}}_b).$$

By our technical hypothesis,

$$\mathcal{Q}(\mathcal{M}(\mathcal{T})) \triangleleft L_{\Omega}[\mathcal{M}(\mathcal{T})].$$

Therefore,

$$\mathcal{Q}(\mathcal{M}(\mathcal{S})) = \pi^{-1}(\mathcal{Q}(\mathcal{M}(\mathcal{T}))) \in N.$$

Let $\kappa < \pi^{-1}(\Omega)$ be a regular cardinal of N greater than the cardinality of $\mathcal{Q}(\mathcal{M}(\mathcal{S}))$ in N. For example, we may simply take

$$\kappa = (|\delta(\mathcal{S})|^+)^N.$$

Let $\mathcal{S}^*$ be $\mathcal{S}$ construed as an iteration tree on $\mathcal{J}^{\mathcal{P}}_{\kappa}$ and G be an N-generic filter over $\text{Col}(\omega, \kappa)$.[5] Then $\mathcal{S}^*$ and $\mathcal{Q}(\mathcal{M}(\mathcal{S})) = \mathcal{Q}(\mathcal{M}(\mathcal{S}^*))$ are hereditarily countable in $N[G]$. Moreover, in $N[G]$, there is a set Z and a subtree $\mathcal{U}$ of ${}^{<\omega}Z$ whose infinite branches correspond to picking an ordinal $\eta < \pi^{-1}(\theta)$ and a level $\mathcal{Q} \triangleleft \mathcal{M}^{\mathcal{S}^*}_{\eta}$, then, in infinitely many steps, picking a cofinal branch c of $\mathcal{S}^*$ and simultaneously defining an isomorphism

$$f : \mathcal{Q}(\mathcal{M}(\mathcal{S})) \simeq i^{\mathcal{S}^*}_{\eta,c}(\mathcal{Q}).$$

By being slightly more precise about the definition of $\mathcal{U}$, we guarantee that $\mathcal{U}$ has a unique branch, namely the one determined by b and the least ordinal $\eta \in b$ such that

$$\mathcal{Q}(\mathcal{M}(\mathcal{S})) \in \text{ran}(i_{\eta,b}).$$

By the absoluteness of wellfoundedness, $b \in N[G]$. Then $b \in N$ by the uniqueness of b and the homogeneity of the poset $\text{Col}(\omega, \kappa)$. The fact that b is a cofinal wellfounded branch of $\mathcal{S}$ is absolute to N. Therefore, $\pi(b)$ is a cofinal wellfounded branch of $\mathcal{T}$.

Finally, suppose that $\mathcal{T}$ is an iteration tree on K^c of length Ω. Let

$$b = [0, \Omega)_{j(T)}$$

where j is the ultrapower map corresponding to U. There is an elementary embedding from $\mathcal{M}^{\mathcal{T}}_b$ to $\mathcal{M}^{j(\mathcal{T})}_{\Omega}$. Since $\mathcal{M}^{j(\mathcal{T})}_{\Omega}$ is wellfounded, so is $\mathcal{M}^{\mathcal{T}}_b$. $\dashv$

[5] We recall that $\text{Col}\,(\omega, \kappa)$ is the collapsing poset consisting of the finite partial functions from ω to κ.

2.4 Theorem. $\{\kappa < \Omega \mid (\kappa^+)^{K^c} = \kappa^+\} \in U$.

Sketch. Let $V' = \mathrm{Ult}(V, U)$ and $j : V \to V'$ be the ultrapower embedding. Then for all $\mathcal{A} \subseteq \wp(\Omega)$, if $|\mathcal{A}| \leq \Omega$, then $j\restriction\mathcal{A} \in V'$. Assume for contradiction that

$$(\Omega^+)^{j(K^c)} < \Omega^+.$$

Let F be the extender of length $j(\Omega)$ derived from $j\restriction j(K^c)$. Then $F \in V'$ and F is countably certified in V'. Now an elaborate induction similar to the proof of Theorem 6.18 of [41] shows that for all $\nu < j(\Omega)$, either the trivial completion of $F\restriction\nu$ is on the $j(K^c)$ sequence, or something close enough that still implies

$$F\restriction\nu \in j(K^c).$$

We could add F itself to $j(K^c)$ to get a model with a superstrong cardinal but it is enough to note that the initial segments of F witness that Ω is a Shelah cardinal in $j(K^c)$ for a contradiction. ⊣

2.5 Definition. Let

$$A_1 = \{\kappa < \Omega \mid \kappa \text{ is an inaccessible cardinal and } (\kappa^+)^{K^c} = \kappa^+\}$$

and

$$A_0 = \{\lambda \in A_1 \mid A_1 \cap \lambda \text{ is not stationary in } \lambda\}.$$

2.6 Theorem. *The following hold.*

(1) $A_0 \notin U$.

(2) A_0 *is stationary.*

(3) *If $\lambda \in A_0$, then there are no total-on-K^c extenders on the K^c sequence with critical point λ.*

Sketch. From Theorem 2.4 it follows that $A_1 \in U$. Suppose for contradiction that $A_0 \in U$. Then $\Omega \in j(A_0)$. So $j(A_0) \cap \Omega$ is not stationary in Ω. But $A_0 = j(A_0) \cap \Omega$. Thus A_0 is not stationary in Ω. Since U is normal, $A_0 \notin U$, which is a contradiction.

Suppose for contradiction that A_0 is not stationary. Then there exists a C club in Ω such that for all $\lambda < \Omega$, if $\lambda \in A_1 \cap C$, then $A_1 \cap \lambda$ is stationary in λ. Let λ be the least element of $A_1 \cap \lim(C)$. Then $C \cap \lambda$ is club in λ and λ is an inaccessible cardinal, so $\lim(C) \cap \lambda$ is club in λ. Since $A_1 \cap \lambda$ is stationary in λ, there exists a $\kappa < \lambda$ such that $\kappa \in A_1 \cap \lim(C)$, which is a contradiction.

Let $\lambda \in A_0$. For all sufficiently large $\alpha < \Omega$,

$$(\lambda^+)^{\mathcal{N}_\alpha} = (\lambda^+)^{K^c}$$

and $\mathcal{N}_\alpha$ and K^c agree below their common λ^+. For such α,

$$(\lambda^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = (\lambda^+)^{\mathcal{N}_\alpha}$$

and $\mathfrak{C}(\mathcal{N}_\alpha)$ and $\mathcal{N}_\alpha$ agree below their common λ^+. Therefore,

$$(\lambda^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = \lambda^+$$

and

$$\{\kappa < \lambda \mid \kappa \text{ is an inaccessible cardinal and } (\kappa^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = \kappa^+\}$$

is not stationary in λ. By the definition of K^c, we cannot add an extender with critical point λ to $\mathfrak{C}(\mathcal{N}_\alpha)$ in forming $\mathcal{N}_{\alpha+1}$. It follows that if $\lambda^+ < \xi < \Omega$, then $\mathrm{crit}(E_\xi^{K^c}) \neq \lambda$. Thus there are no total-on-K^c extenders on the K^c sequence with critical point λ. $\dashv$

Next we discuss some basic facts about coiteration. Suppose that $(\mathcal{P}, \mathcal{Q})$ is a pair of mice. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}, \mathcal{Q})$ determined by their respective iteration strategies. Say $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$. By Theorem 3.11 of [41], there are two possibly overlapping cases.

1. $\mathcal{P} \leq^* \mathcal{Q}$. That is, $[0, \eta]_S$ does not drop in model or degree and
$$\mathcal{M}_\eta^{\mathcal{S}} \trianglelefteq \mathcal{M}_\theta^{\mathcal{T}}.$$
2. $\mathcal{P} \geq^* \mathcal{Q}$. That is, $[0, \theta]_T$ does not drop in model or degree and
$$\mathcal{M}_\eta^{\mathcal{S}} \trianglerighteq \mathcal{M}_\theta^{\mathcal{T}}.$$

Moreover, by the proof of Theorem 3.11 of [41],

$$\eta, \theta < \max(|\mathcal{P}|, |\mathcal{Q}|)^+.$$

We continue the discussion above but assume instead that $\mathcal{P}$ and $\mathcal{Q}$ both have height $\leq \Omega$ and are $\Omega+1$ iterable. Using the fact that Ω is inaccessible, we can modify the proof of Theorem 3.11 of [41] to show that the coiteration of $(\mathcal{P}, \mathcal{Q})$ is successful. Moreover, with the same notation as above, $\eta, \theta \leq \Omega$ and, if

$$\max(\eta, \theta) = \Omega,$$

then at least one of the following holds.

1. $\mathcal{P} \leq^* \mathcal{Q}$, $\mathcal{P}$ is a weasel and $i_{0,\eta}^{\mathcal{S}}``\Omega \subseteq \Omega$.
2. $\mathcal{P} \geq^* \mathcal{Q}$, $\mathcal{Q}$ is a weasel and $i_{0,\theta}^{\mathcal{T}}``\Omega \subseteq \Omega$.

We leave it to the reader to fill in these details.

2.7 Definition. A weasel $\mathcal{Q}$ is *universal* iff $\mathcal{P} \leq^* \mathcal{Q}$ for all $\Omega+1$ iterable premice $\mathcal{P}$ of height $\leq \Omega$.

By the next theorem, K^c is a universal weasel.

2.8 Theorem. *If $\mathcal{Q}$ is a weasel and $\{\kappa < \Omega \mid (\kappa^+)^{\mathcal{Q}} = \kappa^+\}$ is stationary, then Q is universal.*

Sketch. Otherwise, there is an $\Omega+1$ iterable mouse $\mathcal{P}$ of height $\leq \Omega$ such that not $\mathcal{P} \leq^* \mathcal{Q}$. Therefore, $\mathcal{P} \geq^* \mathcal{Q}$ and, with notation as in our discussion on coiteration, $\eta = \Omega$. Moreover, for some $\xi \in [0, \Omega]_S$ and $\kappa < \Omega$,

$$i^{\mathcal{S}}_{\xi,\Omega}(\kappa) = \Omega.$$

Then the set

$$\{\lambda \in (\xi, \Omega)_S \mid i^{\mathcal{S}}_{\xi,\lambda}(\kappa) = \lambda\}$$

is club. Let us assume for simplicity that $\theta = \Omega$. Then also

$$\{\lambda \in (0, \Omega)_T \mid i^{\mathcal{T}}_{0,\lambda}``\lambda \subseteq \lambda\}$$

is club because $i^{\mathcal{T}}_{0,\Omega}``\Omega \subseteq \Omega$. Let λ be a regular cardinal in both these clubs with

$$(\lambda^+)^{\mathcal{Q}} = \lambda^+.$$

Then

$$i^{\mathcal{T}}_{0,\lambda}(\lambda) = \sup(i^{\mathcal{T}}_{0,\lambda}``\lambda) = \lambda$$

so

$$i^{\mathcal{T}}_{0,\lambda}(\lambda^+) = \lambda^+$$

and

$$(\lambda^+)^{\mathcal{M}^{\mathcal{T}}_\lambda} = \lambda^+.$$

On the other hand,

$$(\lambda^+)^{\mathcal{M}^{\mathcal{S}}_\lambda} = i^{\mathcal{S}}_{\xi,\lambda}((\kappa^+)^{\mathcal{M}^{\mathcal{S}}_\xi}) = \sup(i^{\mathcal{S}}_{\xi,\lambda}``(\kappa^+)^{\mathcal{M}^{\mathcal{S}}_\xi}) < \lambda^+.$$

Because $i^{\mathcal{S}}_{\lambda,\Omega}$ has critical point λ and $i^{\mathcal{T}}_{\lambda,\Omega}$ has critical point $\geq \lambda$,

$$(\lambda^+)^{\mathcal{M}^{\mathcal{S}}_\Omega} = (\lambda^+)^{\mathcal{M}^{\mathcal{S}}_\lambda} < (\lambda^+)^{\mathcal{M}^{\mathcal{T}}_\lambda} = (\lambda^+)^{\mathcal{M}^{\mathcal{T}}_\Omega}.$$

This contradicts that $\mathcal{M}^{\mathcal{T}}_\Omega \trianglelefteq \mathcal{M}^{\mathcal{S}}_\Omega$. $\dashv$

We are leading up to the definitions of the definability and hull properties for weasels. Historically, these derive from familiar properties of mice that have gone unnamed. Before dealing with weasels, we digress to discuss the analogous properties of mice as motivation. The fundamental intuition from fine-structure theory of mice is that cores and ultrapowers are inverse operations. Let us give an illustrative example. Suppose that $\mathcal{Q}$ is a 1-sound mouse, $E = \dot{F}^{\mathcal{Q}}$ and

$$\rho_1^{\mathcal{Q}} \leq \mathrm{crit}(E) = \kappa < \mathrm{On} \cap \mathcal{Q}.$$

Let

$$i : \mathcal{Q} \to \mathcal{R} = \mathrm{Ult}(\mathcal{Q}, E)$$

be the ultrapower map. Then i is a Σ_1 elementary embedding and cofinal in the sense that

$$\mathrm{On} \cap \mathcal{R} = \sup(i``(\mathrm{On} \cap \mathcal{Q})).$$

Moreover, $\rho_1^{\mathcal{R}} = \rho_1^{\mathcal{Q}}$ and $p_1^{\mathcal{R}} = i(p_1^{\mathcal{Q}})$. By the definition of 1-soundness,

$$\mathcal{Q} = \mathrm{Hull}_1^{\mathcal{Q}}(\rho_1^{\mathcal{Q}} \cup p_1^{\mathcal{Q}}).$$

By definition, $\mathrm{Hull}_1^{\mathcal{Q}}(X)$ has elements $\tau^{\mathcal{Q}}[c]$ where τ is a Σ_1 Skolem term and $c \in X^{<\omega}$. Therefore,

$$\mathrm{ran}(i) = \mathrm{Hull}_1^{\mathcal{R}}(\rho_1^{\mathcal{R}} \cup p_1^{\mathcal{R}}).$$

The moral is that by deriving an extender from the inverse of the Mostowski collapse of this hull, we recover E. We abstract two key notions from this example. Observe that κ is the least ordinal α such that

$$\alpha \notin \mathrm{Hull}_1^{\mathcal{R}}(\alpha \cup p_1^{\mathcal{R}}).$$

This says that κ is the least ordinal $\alpha \geq \rho_1^{\mathcal{R}}$ such that $\mathcal{R}$ fails to have a certain *definability property* at α. Observe also that

$$\wp(\kappa) \cap \mathcal{R} \subseteq \text{the Mostowski collapse of } \mathrm{Hull}_1^{\mathcal{R}}(\kappa \cup p_1^{\mathcal{R}}).$$

This says that $\mathcal{R}$ has a certain *hull property* at κ. The combination of the two observations above is the minimum required to derive an extender over $\mathcal{R}$ with critical point κ from the inverse of the Mostowski collapse of

$$\mathrm{Hull}_1^{\mathcal{R}}(\kappa \cup p_1^{\mathcal{R}}).$$

Of course, $\mathcal{Q}$ has the definability and hull properties at all $\alpha \geq \rho_1^{\mathcal{Q}}$ since we assumed that $\mathcal{Q}$ is 1-sound. We could go on to show that for all $\alpha \geq \rho_1^{\mathcal{R}}$, $\mathcal{R}$ fails to have the definability property at α iff α is a generator of E. And that $\mathcal{R}$ has the hull property at α iff $\alpha \leq \kappa$ or $\alpha \geq \nu(E)$ where

$$\nu(E) = \sup(\{(\kappa^+)^{\mathcal{Q}}\} \cup \{\xi + 1 \mid \xi \text{ is a generator of } E\}).$$

Taking our discussion to the next level, suppose instead that $\mathcal{Q}$ is a weasel. This is fundamentally different because

$$\rho_1^{\mathcal{Q}} = \mathrm{On} \cap \mathcal{Q} = \Omega.$$

Nevertheless, it is important to find an analogous way of undoing iterations of $\mathcal{Q}$. What we need are versions of the definability and hull properties that are appropriate for weasels. And we need a way to take hulls in K^c that produces weasels with these properties.

2.9 Definition. Let $\mathcal{Q}$ be a weasel and $\Gamma \subseteq \mathcal{Q}$. Then Γ *is thick in* $\mathcal{Q}$ iff there is a club C in Ω such that for all $\lambda \in A_0 \cap C$,

1. $(\lambda^+)^{\mathcal{Q}} = \lambda^+$,
2. λ is not the critical point of a total-on-$\mathcal{Q}$ extender on the $\mathcal{Q}$ sequence, and
3. there is a λ-club in $\Gamma \cap \lambda^+$.

2.10 Definition. $\mathcal{Q}$ is a *thick* weasel iff Ω is thick in $\mathcal{Q}$.

The reader will not find the expression *thick weasel* in the literature but the concept needed a name so we picked one. Clearly K^c is a thick weasel. The next three results are useful closure properties of thick sets.

2.11 Theorem. *Let $\mathcal{Q}$ be a thick weasel. Then*

$$\{\Gamma \subseteq \mathcal{Q} \mid \Gamma \textit{ is thick in } \mathcal{Q}\}$$

is an Ω-complete filter.

2.12 Theorem. *Suppose that $\pi : \mathcal{P} \to \mathcal{Q}$ is an elementary embedding and* $\mathrm{ran}(\pi)$ *is thick in $\mathcal{Q}$. Let*

$$\Phi = \{\alpha < \Omega \mid \pi(\alpha) = \alpha\}.$$

Then Φ is thick in both $\mathcal{P}$ and $\mathcal{Q}$.

2.13 Theorem. *Let $\mathcal{T}$ be an iteration tree on a thick weasel $\mathcal{Q}$ with*

$$\mathrm{lh}(\mathcal{T}) = \theta + 1 \leq \Omega + 1.$$

Assume that there is no dropping along $[0, \theta]_T$ and $i^{\mathcal{T}}_{0,\theta}$"$\Omega \subseteq \Omega$. Let

$$\Phi = \{\alpha < \Omega \mid i^{\mathcal{T}}_{0,\theta}(\alpha) = \alpha\}.$$

Then Φ is thick in both $\mathcal{Q}$ and $\mathcal{M}^{\mathcal{T}}_{\theta}$.

The proofs of the previous three theorems are reasonable exercises for the reader. The $\theta = \Omega$ case of the Theorem 2.13 is why we used A_0 instead of A_1.

2.14 Definition. A thick weasel $\mathcal{Q}$ has the *definability property* at α iff

$$\alpha \in \mathrm{Hull}^{\mathcal{Q}}(\alpha \cup \Gamma)$$

whenever Γ is thick in $\mathcal{Q}$.

By definition, the elements of $\mathrm{Hull}^{\mathcal{Q}}(X)$ are those of the form $\tau^{\mathcal{Q}}[c]$ where τ is a Skolem term and $c \in X^{<\omega}$. Equivalently, $a \in \mathrm{Hull}^{\mathcal{Q}}(X)$ iff $\{a\}$ is first-order definable over $\mathcal{Q}$ with parameters from X.

2.15 Definition. A thick weasel $\mathcal{Q}$ has the *hull property* at α iff

$$\wp(\alpha) \cap \mathcal{Q} \subseteq \text{the Mostowski collapse of } \mathrm{Hull}^{\mathcal{Q}}(\alpha \cup \Gamma)$$

whenever Γ is thick in $\mathcal{Q}$.

2.16 Theorem. *Let $\beta < \Omega$ and $\mathcal{Q}$ be a thick weasel with the definability and hull properties for all $\alpha < \beta$. Suppose that $\mathcal{T}$ is an iteration tree on $\mathcal{Q}$ with*

$$\mathrm{lh}(\mathcal{T}) = \theta + 1 \leq \Omega + 1.$$

Assume that there is no dropping along $[0, \theta]_T$ and $i^{\mathcal{T}}_{0,\theta}``\Omega \subseteq \Omega$. Then the following hold for all $\alpha < \beta$.

(1) *$\mathcal{M}^{\mathcal{T}}_{\theta}$ does not have the definability property at α iff there exists an*

$$\eta + 1 \in [0, \theta]_T$$

such that α is a generator of $E^{\mathcal{T}}_{\eta}$.

(2) *$\mathcal{M}^{\mathcal{T}}_{\theta}$ does not have the hull property at α iff there exists an $\eta + 1 \in [0, \theta]_T$ such that*

$$(\mathrm{crit}(E^{\mathcal{T}}_{\eta})^{+})^{\mathcal{M}^{\mathcal{T}}_{\theta}} \leq \alpha < \nu(E^{\mathcal{T}}_{\eta}).$$

Sketch. For simplicity, we deal only with the case of a single ultrapower. In other words, $\theta = 2$. Let $E = E^{\mathcal{T}}_{0}$ and consider the following diagram.

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "j"] \arrow[dr, "i"'] & \mathrm{Ult}(\mathcal{Q}, E) \\
& \mathrm{Ult}(\mathcal{Q}, E \restriction \alpha) \arrow[u, "k"']
\end{tikzcd}$$

Then $\mathrm{crit}(k) = \alpha$ iff α is a generator of E. Let $\Phi = \{\xi < \Omega \mid j(\xi) = \xi\}$. Then Φ is thick in all three models. Of course, $j(\xi) = \xi$ implies $k(\xi) = \xi$ and $i(\xi) = \xi$.

First we prove the *if* direction of (1). Assume that α is a generator of E. Equivalently, that $\alpha = \mathrm{crit}(k)$. Suppose for contradiction that $\mathrm{Ult}(\mathcal{Q}, E)$ has the definability property at α. Then there is a Skolem term τ and a parameter $c \in (\alpha \cup \Phi)^{<\omega}$ such that

$$\alpha = \tau^{\mathrm{Ult}(\mathcal{Q},E)}[c] = k(\tau^{\mathrm{Ult}(\mathcal{Q},E\restriction\alpha)}[c]\,).$$

This is a contradiction since $\alpha \notin \mathrm{ran}(k)$.

Second we prove the *if* direction of (2). Assume that

$$(\mathrm{crit}(E)^{+})^{\mathcal{Q}} \leq \alpha < \nu(E).$$

The main point is that

$$E\restriction\alpha \in \mathrm{Ult}(\mathcal{Q}, E)$$

whereas

$$E\restriction\alpha \notin \mathrm{Ult}(\mathcal{Q}, E\restriction\alpha).$$

We know this because E is on the $\mathcal{Q}$ sequence, which is a good extender sequence. Since

$$(\mathrm{crit}(E)^{+})^{\mathcal{Q}} \leq \alpha,$$

it is possible to code $E\restriction\alpha$ by $A \subseteq \alpha$ with $A \in \mathrm{Ult}(\mathcal{Q}, E)$. Suppose for contradiction that $\mathrm{Ult}(\mathcal{Q}, E)$ has the hull property at α. Then there is a Skolem term τ and a parameter $c \in (\alpha \cup \Phi)^{<\omega}$ such that

$$A = \tau^{\mathrm{Ult}(\mathcal{Q},E)}[c] \cap \alpha.$$

Since $\mathrm{crit}(k) \geq \alpha$,

$$A = \tau^{\mathrm{Ult}(\mathcal{Q},E\restriction\alpha)}[c] \cap \alpha \in \mathrm{Ult}(\mathcal{Q}, E\restriction\alpha),$$

so

$$E\restriction\alpha \in \mathrm{Ult}(\mathcal{Q}, E\restriction\alpha),$$

which is a contradiction.

Notice that the two *if* directions did not use the hypothesis that $\mathcal{Q}$ has the definability and hull properties at all ordinals $< \beta$. These are used for the two *only if* directions, which we leave to the reader. $\dashv$

The next theorem explains how the definability property and hull property are related, and its proof is a good example of how they are used.

2.17 Theorem. *For all $\beta < \Omega$, if $\mathcal{Q}$ has the definability property at all $\alpha < \beta$, then $\mathcal{Q}$ has the hull property at β.*

Sketch. By induction, we may assume that $\mathcal{Q}$ has the definability and hull properties at all $\alpha < \beta$. Suppose that Γ is thick in $\mathcal{Q}$. Let

$$\pi : \mathcal{P} \simeq \mathrm{Hull}^{\mathcal{Q}}(\beta \cup \Gamma)$$

be the inverse of the Mostowski collapse. We must show that

$$\wp(\beta) \cap \mathcal{P} = \wp(\beta) \cap \mathcal{Q}.$$

If Δ is thick in $\mathcal{P}$, then $\{\xi \in \Delta \mid \pi(\xi) = \xi\}$ is thick in $\mathcal{Q}$. From this it follows that $\mathcal{P}$ has the definability and hull properties at all $\alpha < \beta$. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}, \mathcal{Q})$. Both $\mathcal{P}$ and $\mathcal{Q}$ are universal, so

$$\mathcal{M}^{\mathcal{S}}_{\eta} = \mathcal{M}^{\mathcal{T}}_{\theta}$$

where $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$. Moreover, there is no dropping along $[0, \eta]_S$ and $[0, \theta]_T$. It is enough to see that

$$\mathrm{crit}(i^{\mathcal{S}}_{0,\eta}), \mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) \geq \beta.$$

For contradiction, suppose that

$$\mathrm{crit}(i^{\mathcal{S}}_{0,\eta}) < \beta.$$

Apply Theorem 2.16 to $\mathcal{S}$ to see that $\mathrm{crit}(i^{\mathcal{S}}_{0,\eta})$ is equal to the least α such that $\mathcal{M}^{\mathcal{S}}_{\eta}$ does not have the definability property at α. And apply Theorem 2.16 to $\mathcal{T}$ to see that

$$\mathrm{crit}(i^{\mathcal{S}}_{0,\eta}) = \mathrm{crit}(i^{\mathcal{T}}_{0,\theta}).$$

Call this ordinal α and let

$$\alpha^* = \min(i^{\mathcal{S}}_{0,\eta}(\alpha), i^{\mathcal{T}}_{0,\theta}(\alpha)).$$

As $\alpha < \beta$, $\mathcal{Q}$ has the hull property at α, so

$$\wp(\alpha) \cap \mathcal{P} = \wp(\alpha) \cap \mathcal{Q}.$$

Next we use the fact that

$$\Phi = \{\xi < \Omega \mid i^{\mathcal{S}}_{0,\eta}(\xi) = \xi = i^{\mathcal{T}}_{0,\theta}(\xi)\}$$

is thick in both $\mathcal{P}$ and $\mathcal{Q}$ to show that if $X \subseteq \alpha$ with $X \in \mathcal{P}$, then

$$i^{\mathcal{S}}_{0,\eta}(X) \cap \alpha^* = i^{\mathcal{T}}_{0,\theta}(X) \cap \alpha^*$$

First note that $\alpha \subseteq \Phi$. Then, given $X \subseteq \alpha$ with $X \in \mathcal{P}$, choose a Skolem term τ and $c \in \Phi^{<\omega}$ such that

$$X = \tau^{\mathcal{P}}[c] \cap \alpha.$$

Let

$$Y = \tau^{\mathcal{Q}}[c] \cap \alpha.$$

Then

$$\begin{aligned} i^{\mathcal{S}}_{0,\eta}(X) \cap \alpha^* &= i^{\mathcal{S}}_{0,\eta}(\tau^{\mathcal{P}}[c] \cap \alpha) \cap \alpha^* \\ &= \tau^{\mathcal{M}^{\mathcal{S}}_{\eta}}[c] \cap \alpha^* \\ &= \tau^{\mathcal{M}^{\mathcal{T}}_{\theta}}[c] \cap \alpha^* \\ &= i^{\mathcal{T}}_{0,\theta}(\tau^{\mathcal{Q}}[c] \cap \alpha) \cap \alpha^* \\ &= i^{\mathcal{T}}_{0,\theta}(Y) \cap \alpha^*. \end{aligned}$$

Also

$$X = i^{\mathcal{S}}_{0,\eta}(X) \cap \alpha = i^{\mathcal{T}}_{0,\theta}(Y) \cap \alpha = Y.$$

We have seen that the first extenders used along $[0,\eta]_S$ and $[0,\theta]_T$ are comparable, which is impossible in a coiteration. (E.g., see the subclaim in the proof of Theorem 3.11 of [41].)

The same contradiction is obtained similarly by assuming that

$$\mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) < \beta.$$

⊣

2.18 Definition. Let $\mathcal{P}$ be a mouse of height $< \Omega$. Then $\mathcal{P}$ is A_0 *sound* iff there exists a thick weasel $\mathcal{P}^*$ such that $\mathcal{P} \lhd \mathcal{P}^*$ and $\mathcal{P}^*$ has the definability property at all $\alpha \in \mathrm{On} \cap \mathcal{P}$.

The point of isolating A_0 sound mice is that they line up as the next theorem shows.

2.19 Theorem. *Let $\mathcal{P}$ and $\mathcal{Q}$ be A_0 sound mice. Then $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{P} \trianglerighteq \mathcal{Q}$.*

Sketch. Let $\mathcal{P}^*$ and $\mathcal{Q}^*$ be A_0 soundness witnesses for $\mathcal{P}$ and $\mathcal{Q}$ respectively. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}^*, \mathcal{Q}^*)$. Then

$$\mathcal{M}^{\mathcal{S}}_{\eta} = \mathcal{M}^{\mathcal{T}}_{\theta}$$

where $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$, and there is no dropping along $[0,\eta]_S$ and $[0,\theta]_T$. It is enough to see that

$$\mathrm{crit}(i^{\mathcal{S}}_{0,\eta}), \mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) \geq \min(\mathrm{On} \cap \mathcal{P}, \mathrm{On} \cap \mathcal{Q}).$$

This is done by contradiction exactly as in the proof of Theorem 2.17 using the hull property and definability property of $\mathcal{P}^*$ and $\mathcal{Q}^*$ at all

$$\alpha < \min(\mathrm{On} \cap \mathcal{P}, \mathrm{On} \cap \mathcal{Q}).$$

⊣

2.20 Definition. K is the union of all the A_0 sound mice.

By Theorem 2.19, K is a premouse. But it is not immediate that K has height Ω.

2.21 Definition. Let $\mathcal{Q}$ be a thick weasel. Then

$$\operatorname{Def}(\mathcal{Q}) = \bigcap\{\operatorname{Hull}^{\mathcal{Q}}(\Gamma) \mid \Gamma \text{ is thick in } \mathcal{Q}\}$$

The plan for proving that K is a weasel is as follows. First we show that K is the Mostowski collapse of $\operatorname{Def}(K^c)$. Then we establish that K^c has the hull and definability properties at U-almost all $\alpha < \Omega$. The last step is to show that $\operatorname{Def}(K^c)$ is unbounded in Ω. The realization of this plan stretches over several theorems.

2.22 Theorem. *Let $\mathcal{P}$ and $\mathcal{Q}$ be thick weasels. Then* $\operatorname{Def}(\mathcal{P}) \simeq \operatorname{Def}(\mathcal{Q})$.

Sketch. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}, \mathcal{Q})$. Then

$$\mathcal{M}^{\mathcal{S}}_{\eta} = \mathcal{M}^{\mathcal{T}}_{\theta}$$

where $\eta + 1 = \operatorname{lh}(\mathcal{S})$ and $\theta + 1 = \operatorname{lh}(\mathcal{T})$, and there is no dropping along $[0, \eta]_S$ and $[0, \theta]_T$. It is enough to see that

$$i^{\mathcal{S}}_{0,\eta}\text{``}\operatorname{Def}(\mathcal{P}) = \operatorname{Def}(\mathcal{M}^{\mathcal{S}}_{0,\eta})$$

and

$$i^{\mathcal{T}}_{0,\theta}\text{``}\operatorname{Def}(\mathcal{Q}) = \operatorname{Def}(\mathcal{M}^{\mathcal{T}}_{0,\theta}).$$

This is an easy exercise using the basic properties of thick sets. $\dashv$

2.23 Theorem. $K \simeq \operatorname{Def}(K^c)$.

Sketch. Let

$$\pi : K' \simeq \operatorname{Def}(K^c)$$

be the inverse of the Mostowski collapse. We must show that $K' = K$.

First let $\mathcal{P} \triangleleft K$ and $\mathcal{P}^*$ be a witness that $\mathcal{P}$ is A_0 sound. Since $\mathcal{P}^*$ has the definability property at all $\alpha < \mathrm{On} \cap \mathcal{P}$,

$$\mathcal{P} \subseteq \operatorname{Def}(\mathcal{P}^*).$$

But $\operatorname{Def}(K^c) \simeq \operatorname{Def}(\mathcal{P}^*) \simeq K'$ by Theorem 2.22. Therefore $\mathcal{P} \triangleleft K'$.

Now let $\mathcal{P} \triangleleft K'$. Let $\theta = \sup(\pi\text{``}(\mathrm{On} \cap \mathcal{P}))$. For each $\alpha \in \theta - \operatorname{Def}(K^c)$, pick an A_0-thick set Γ_α such that

$$\alpha \notin \operatorname{Hull}^{K^c}(\Gamma_\alpha).$$

Let

$$\Delta = \bigcap\{\Gamma_\alpha \mid \alpha \in \theta - \operatorname{Def}(K^c)\}$$

and $\mathcal{Q}$ be the Mostowski collapse of $\operatorname{Hull}^{K^c}(\Delta)$. It is an easy exercise to see that $\mathcal{Q}$ witnesses that $\mathcal{P}$ is A_0 sound. Therefore $\mathcal{P} \triangleleft K$. $\dashv$

By Theorems 2.22 and 2.23, $K \simeq \mathrm{Def}(\mathcal{P})$ whenever $\mathcal{P}$ is a thick weasel.

2.24 Theorem. *Let $\mathcal{Q}$ be a thick weasel. Then there exists a C club in Ω such that $\mathcal{Q}$ has the hull property at α for all inaccessible $\alpha \in C$. In particular,*

$$\{\alpha < \Omega \mid \mathcal{Q} \textit{ has the hull property at } \alpha\} \in U.$$

Sketch. By recursion, define a continuous decreasing sequence

$$\langle X_\alpha \mid \alpha < \Omega \rangle$$

of thick elementary substructures of $\mathcal{Q}$ and an increasing sequence

$$\langle \lambda_\alpha \mid \alpha < \Omega \rangle$$

of cardinals of $\mathcal{Q}$. For all $\alpha < \Omega$, let $\pi_\alpha : \mathcal{P}_\alpha \simeq X_\alpha$ be the inverse of the Mostowski collapse and $\pi_\alpha(\kappa_\alpha) = \lambda_\alpha$. Arrange the construction so that $\langle \kappa_\alpha \mid \alpha \leq \beta \rangle$ is an initial segment of the infinite cardinals of $\mathcal{P}_\beta$ for all $\beta < \Omega$. Also arrange that for all $\alpha < \beta < \Omega$,

$$\pi_\alpha \restriction (\kappa_\alpha + 1) = \pi_\beta \restriction (\kappa_\alpha + 1)$$

and $\mathcal{P}_\beta$ has the hull property at all $\kappa \leq \kappa_\alpha$.

Start the construction with $\kappa_0 = \lambda_0 = \omega$, $X_0 = \mathcal{Q}$ and $\pi_0 = \mathrm{id} \restriction \mathcal{Q}$. If β is a limit ordinal, then $X_\beta = \bigcap_{\alpha < \beta} X_\alpha$ and this determines $\mathcal{P}_\beta$, π_β, κ_β and λ_β by what we said above. The successor step is more complicated. If $A \in \wp(\kappa_\alpha) \cap \mathcal{P}_\alpha$ and there exists a Γ thick in $\mathcal{P}_\alpha$ such that

$$A \notin \text{Mostowski collapse of } \mathrm{Hull}^{\mathcal{P}_\alpha}(\kappa_\alpha \cup \Gamma),$$

then pick such a Γ and call it Γ_A. Then let

$$\Gamma_\alpha = \bigcap(\{\mathcal{P}_\alpha\} \cup \{\Gamma_A \mid A \in \wp(\kappa_\alpha) \cap \mathcal{P}_\alpha \text{ and } \Gamma_A \text{ is defined}\})$$

and

$$X_{\alpha+1} = \mathrm{Hull}^{\mathcal{P}_\alpha}((\kappa_\alpha + 1) \cup \Gamma_\alpha).$$

This determines $\mathcal{P}_{\alpha+1}$, $\pi_{\alpha+1}$, $\kappa_{\alpha+1}$ and $\lambda_{\alpha+1}$ by what we said at the start.

2.25 Lemma. *If γ is a limit ordinal, then $\mathcal{P}_\gamma = \mathcal{P}_{\gamma+1}$.*

Sketch. Suppose not. Then Γ_A is defined for some $A \in \wp(\kappa_\gamma) \cap \mathcal{P}_\gamma$. Let $\mathcal{P}_A$ be the Mostowski collapse of

$$\mathrm{Hull}^{\mathcal{P}_\gamma}(\kappa_\gamma \cup \Gamma_A).$$

and $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}_A, \mathcal{P}_\gamma)$. Then $\mathcal{M}^{\mathcal{S}}_\eta = \mathcal{M}^{\mathcal{T}}_\theta$ where $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$, and there is no dropping along $[0, \eta]_S$ and $[0, \theta]_T$.

Suppose that $\mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) \geq \kappa_\gamma$. Then

$$A \in \wp(\kappa_\gamma) \cap \mathcal{P}_\gamma = \wp(\kappa_\gamma) \cap \mathcal{M}^{\mathcal{T}}_\theta = \wp(\kappa_\gamma) \cap \mathcal{M}^{\mathcal{S}}_\eta \subseteq \wp(\kappa_\gamma) \cap \mathcal{P}_A$$

since all the extenders used on $\mathcal{S}$ have length at least κ_γ. But $A \notin \mathcal{P}_A$, contradiction!

Therefore, $\mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) < \kappa_\gamma$. Let $\beta < \gamma$ be such that

$$\kappa_\beta = \mathrm{crit}(i^{\mathcal{T}}_{0,\theta}).$$

Then β is equal to the least ordinal $\alpha < \gamma$ such that $\mathcal{M}^{\mathcal{T}}_\theta$ does not have the hull property at $\kappa_{\alpha+1}$. This is not precisely what Theorem 2.16 says about $\mathcal{T}$ but the proof shows it. Applying a similar modification of Theorem 2.16 to $\mathcal{S}$ shows that $\kappa_\beta = \mathrm{crit}(i^{\mathcal{S}}_{0,\eta})$. Finally, use the hull property at κ_β in both $\mathcal{P}_A$ and $\mathcal{P}_\gamma$ to see that the first extenders used on $[0,\eta]_S$ and $[0,\theta]_T$ are compatible. This leads to a standard contradiction. $\dashv$

2.26 Lemma. *Let $X = \bigcap\{X_\alpha \mid \alpha < \Omega\}$. Then X is thick in $\mathcal{Q}$.*

Sketch. For each $\alpha < \Omega$, pick C_α club in Ω witnessing that X_α is thick in $\mathcal{Q}$. Let C be the diagonal intersection of $\langle C_\alpha \mid \alpha < \Omega\rangle$. We show that C witnesses that X is thick in $\mathcal{Q}$. Let $\beta \in A_0 \cap C$. Clearly $(\beta^+)^{\mathcal{Q}} = \beta^+$ and β is not the critical point of a total-on-$\mathcal{Q}$ extender on the $\mathcal{Q}$ sequence. For each $\alpha < \beta$, there exists a β-club $D_\alpha \subseteq X_\alpha \cap \beta^+$. Let $D = \bigcap\{D_\alpha \mid \alpha < \beta\}$. Then D is a β-club subset of

$$\textstyle\bigcap_{\alpha<\beta} X_\alpha \cap \beta^+ = X_\beta \cap \beta^+ = X_{\beta+1} \cap \beta^+ = X \cap \beta^+.$$

The first equation holds by the definition of X_β. The second holds by Lemma 2.25. The third holds because $\beta \leq \lambda_\beta$, $\beta^+ \leq \lambda_{\beta+1}$ and

$$X_{\beta+1} \cap (\lambda_{\beta+1} + 1) = X \cap (\lambda_{\beta+1} + 1).$$

In fact, by taking β closed under $\alpha \mapsto \lambda_\alpha$ we get that $\beta = \kappa_\beta = \lambda_\beta$ and $\beta^+ = \kappa_{\beta+1} = \lambda_{\beta+1}$. $\dashv$

2.27 Lemma. *Let $\mathcal{P}$ be the Mostowski collapse of X. Then $\mathcal{P}$ has the hull property at all $\alpha < \Omega$.*

Sketch. Lemma 2.26 implies that $\mathcal{P}$ is a thick weasel. By construction, $\langle \kappa_\alpha \mid \alpha < \Omega\rangle$ lists the infinite cardinals of $\mathcal{P}$ in increasing order and $\mathcal{P}$ has the hull property at κ_α for all $\alpha < \Omega$. $\dashv$

Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}, \mathcal{Q})$. Consider the case in which $\mathcal{S}$ and $\mathcal{T}$ both have length $\Omega + 1$, the other cases being similar. Then $\mathcal{M}^{\mathcal{S}}_\Omega = \mathcal{M}^{\mathcal{T}}_\Omega$ and there is no dropping along $[0,\Omega]_S$ and $[0,\Omega]_T$. Let C be the set of limit ordinals

$$\theta \in [0,\Omega]_S \cap [0,\Omega]_T$$

such that θ is the supremum

$$\{\mathrm{lh}(E^{\mathcal{S}}_\eta) \mid \eta < \theta\} \cup \{\mathrm{lh}(E^{\mathcal{T}}_\eta) \mid \eta < \theta\}.$$

Then C is club in Ω. Consider an arbitrary $\theta \in C$. Then $\mathcal{M}^{\mathcal{S}}_\theta$ has the hull property at θ and, since

$$\mathrm{crit}(i^{\mathcal{S}}_{\theta,\Omega}) \geq \theta,$$

$\mathcal{M}^{\mathcal{S}}_\Omega$ has the hull property at θ. The fact that $\mathrm{crit}(i^{\mathcal{T}}_{\theta,\Omega}) \geq \theta$ can be used to see that $\mathcal{M}^{\mathcal{T}}_\theta$ has the hull property at θ. Now assume that θ is inaccessible. Then $i^{\mathcal{T}}_{0,\theta}(\theta) = \theta$. To finish the proof of the theorem, we show that $\mathcal{Q}$ has the hull property at θ. Suppose $A \in \wp(\theta) \cap \mathcal{Q}$ and Γ is thick in $\mathcal{Q}$. Let $B = i^{\mathcal{T}}_{0,\theta}(A)$. Then $B \in \wp(\theta) \cap \mathcal{M}^{\mathcal{T}}_\theta$ so there is a Skolem term τ and parameters $c \in \theta^{<\omega}$ and $d \in \Gamma^{<\omega}$ such that $d = i^{\mathcal{T}}_{0,\theta}(d)$ and

$$B = \tau^{\mathcal{M}^{\mathcal{T}}_\theta}[c, d] \cap \theta.$$

By minimizing c in this equation we find $b \in \theta^{<\omega}$ such that $c = i^{\mathcal{T}}_{0,\theta}(b)$. Thus

$$A = \tau^{\mathcal{Q}}[b, d] \cap \theta.$$

$\dashv$

We have used the technical hypothesis that Ω is measurable twice already. First, to see that the set of α such that $(\alpha^+)^{K^c} = \alpha^+$ is stationary in Ω. Second, to show that K^c is $(\omega, \Omega+1)$ iterable starting from the fact that if $\mathcal{P}$ is countable and elementarily embeds into K^c, then $\mathcal{P}$ is $(\omega, \omega_1 + 1)$ iterable. The third and final use of the technical hypothesis comes in the proof of the following theorem.

2.28 Theorem. *$\{\alpha < \Omega \mid K^c$ has the definability property at $\alpha\} \in U$.*

Sketch. Suppose not. Let

$$D = \{\alpha < \Omega \mid K^c \text{ does not have the definability property at } \alpha\}.$$

Then $D \in U$. For each $\alpha \in D$, pick a thick Γ_α such that

$$\alpha \notin \mathrm{Hull}^{K^c}(\alpha \cup \Gamma_\alpha).$$

We may assume $\Gamma_\beta \subseteq \Gamma_\alpha$ whenever $\alpha < \beta$ are elements of D. We write $\Gamma = \langle \Gamma_\alpha \mid \alpha \in D\rangle$. Form the iteration

$$V \xrightarrow[j]{} V' \xrightarrow[k]{} V''$$

with $V' = \mathrm{Ult}(V, U)$, $U' = j(U)$ and $V'' = \mathrm{Ult}(V', U')$. We will use the general fact that

$$j \circ j = k \circ j.$$

This equation holds because

$$j([\alpha \mapsto x]_U^V) = [\alpha \mapsto j(x)]_{U'}^{V'}.$$

Let $W = K^c$, $W' = j(W)$ and $W'' = k(W')$. By what we just said, $W'' = j(W')$. Consider the inverse of the Mostowski collapse

$$\pi : \mathcal{P} \simeq \mathrm{Hull}^{W'}(\Omega \cup \Gamma'_\Omega)$$

where $\Gamma'_\alpha = j(\Gamma)_\alpha$. Also let $\Gamma''_\alpha = k(\Gamma')_\alpha$. Then $\Gamma''_\alpha = j(\Gamma')_\alpha$. Since W' does not have the definability property at Ω, $\mathrm{crit}(\pi) = \Omega$. By Theorem 2.24,

$$(W' \text{ has the hull property at } \Omega)^{V'},$$

so

$$\wp(\Omega) \cap \mathcal{P} = \wp(\Omega) \cap W'.$$

Let $\Omega' = j(\Omega)$. Note that $\pi(\Omega) < \Omega'$ because Γ'_Ω is unbounded in Ω'. Let F be the extender of length $\pi(\Omega)$ derived from π. We claim that

$$\pi(A) = j(A) \cap \pi(\Omega)$$

for all $A \in \wp(\Omega) \cap W'$. From the claim, it follows that F is countably certified in V', which can be used to show that F witnesses that Ω is a superstrong cardinal in W'. To prove the claim, pick a Skolem term τ and parameters $c \in \Omega^{<\omega}$ and $d \in (\Gamma'_\Omega)^{<\omega}$ such that $A = \tau^{W'}[c, d] \cap \Omega$. Then

$$j(A) = \tau^{W''}[c, j(d)] \cap \Omega'$$

and

$$j(d) \in (\Gamma''_{\Omega'})^{<\omega} \subseteq (\Gamma''_\Omega)^{<\omega}$$

because Γ'' is a descending sequence and $\Omega' > \Omega$. In particular,

$$\tau^{W''}[c, j(d)] \in \mathrm{Hull}^{W''}(\Omega \cup \Gamma''_\Omega)$$

and

$$A = \tau^{W''}[c, j(d)] \cap \Omega.$$

By elementarity,

$$k(\pi) : k(\mathcal{P}) \simeq \mathrm{Hull}^{W''}(\Omega \cup \Gamma''_\Omega).$$

Finally, since $\mathrm{crit}(k) = \Omega' > \pi(\Omega)$ and $A \subseteq \Omega$,

$$\begin{aligned}
\pi(A) &= k(\pi(A)) = k(\pi)(k(A)) = k(\pi)(A) \\
&= k(\pi)(\tau^{W''}[c, j(d)] \cap \Omega) \\
&= \tau^{W''}[c, j(d)] \cap k(\pi(\Omega)) \\
&= j(A) \cap \pi(\Omega).
\end{aligned}$$

$\dashv$

2.29 Theorem. *K is a weasel.*

Proof. Consider the following recursive construction. Let $\Gamma_0 = \Omega$. Assuming that Γ_α has been defined, if

$$\mathrm{Hull}^{K^c}(\Gamma_\alpha) = \mathrm{Def}(K^c),$$

then stop the construction. Otherwise, let

$$\gamma_\alpha = \min(\mathrm{Hull}^{K^c}(\Gamma_\alpha) - \mathrm{Def}(K^c))$$

and pick $\Gamma_{\alpha+1} \subseteq \Gamma_\alpha$ so that

$$\gamma_\alpha \notin \mathrm{Hull}^{K^c}(\Gamma_{\alpha+1}).$$

If β is a limit ordinal, then let

$$\Gamma_\beta = \bigcap\{\Gamma_\alpha \mid \alpha < \beta\}.$$

Suppose for contradiction that $\mathrm{Def}(K^c)$ is bounded in Ω. Then γ_α and Γ_α are defined for all $\alpha < \Omega$. And there exists an $\alpha < \Omega$ such that

$$\mathrm{Def}(K^c) \cap \Omega \subseteq \gamma_\alpha.$$

By Theorem 2.28, there exists a $\delta \in (\gamma_\alpha, \Omega)$ such that

$$\delta = \sup(\{\gamma_\beta \mid \beta < \delta\}) \leq \gamma_\delta$$

and K^c has the definability property at δ. This implies that there exist an ordinal $\beta \in (\alpha, \delta)$, parameters $c \in (\gamma_\beta)^{<\omega}$ and $d \in (\Gamma_{\delta+1})^{<\omega}$, and a Skolem term τ such that $\delta = \tau^{K^c}[c, d]$. Then c is a witness to the sentence:

$$\text{there exists a } b \in (\gamma_\beta)^{<\omega} \text{ such that } \gamma_\beta < \tau^{K^c}[b, d] < \gamma_{\delta+1}.$$

Since γ_β and $\gamma_{\delta+1}$ are elements of $\mathrm{Hull}^{K^c}(\Gamma_\beta)$ we may pick a witness b to this sentence with

$$b \in \mathrm{Hull}^{K^c}(\Gamma_\beta).$$

By the minimality of γ_β and the fact that $b \in (\gamma_\beta)^{<\omega}$,

$$b \in \mathrm{Def}(K^c).$$

Hence

$$\tau^{K^c}[b, d] \in \mathrm{Hull}^{K^c}(\Gamma_{\delta+1}).$$

By the choice of γ_α and the fact that $\gamma_\alpha < \gamma_\beta < \tau^{K^c}[b, d]$,

$$\tau^{K^c}[b, d] \notin \mathrm{Def}(K^c).$$

By the minimality of $\gamma_{\delta+1}$,

$$\tau^{K^c}[b,d] \geq \gamma_{\delta+1}.$$

But

$$\tau^{K^c}[b,d] < \gamma_{\delta+1},$$

which is a contradiction. ⊣

At the corresponding point in [42], Steel goes on to prove that

$$\{\alpha < \Omega \mid (\alpha^+)^K = \alpha^+\} \in U.$$

(Cf., Theorem 3.1 below.) The calculations involve combinatorics similar to the proof of Theorem 2.29 but we omit them here. From this and Theorem 2.8, it follows that K is universal. Also at this point, Steel shows that K is absolute under forcing in $H(\Omega)$. (Cf., Theorem 3.4 below.) The proof involves abstracting the properties of A_0 in the arguments we have given so far.

2.2. First-Order Definition of K

Now we head in a slightly different direction. Notice that the definition of K we have given is second-order over $H(\Omega)$. Moreover, there is no obvious sense in which the definition works locally. For example, it is not immediate from what we have said so far that $K \cap HC$ is less complex than K.[6] Our next goal is to find an equivalent first-order definition of K that gives meaningful local bounds on complexity. For example, $K \cap HC$ turns out to be Σ_1 definable over $L_{\omega_1}(\mathbb{R})$. (Cf., Theorem 3.5.) By results of Woodin, this is the best possible upper bound on the complexity of $K \cap HC$. The ideas that go into the first-order definition of K are central to the proof of the weak covering theorem in Section 4.

Before launching into the details, let us motivate what is to come. It is not hard to see that all universal weasels have the same subsets of ω, namely those in

$$\mathcal{J}^K_{(\omega_1)^K} = \bigcup\{\mathcal{Q} \mid \mathcal{Q} \text{ is a sound mouse and } \rho^{\mathcal{Q}}_\omega = 1\}.$$

Nor is it hard to see that all universal weasels have the same subsets of $(\omega_1)^K$, namely those in

$$\mathcal{J}^K_{(\omega_2)^K} = \bigcup\{\mathcal{Q} \mid \mathcal{Q} \text{ is a sound mouse with } \rho^{\mathcal{Q}}_\omega = (\omega_1)^K \text{ and } \mathcal{J}^K_{(\omega_1)^K} \triangleleft \mathcal{Q}\}.$$

[6] By definition, $HC = H(\aleph_1)$. The reader should be attentive here to the difference between

$$J^K_{(\omega_1)^K} = HC^K$$

and

$$J^K_{(\omega_1)^V} = K \cap HC.$$

This points to a simultaneous definition of what it means for α to be a cardinal of K on one hand, and $\mathcal{J}^K_{(\alpha^+)^K}$ on the other, by induction on $\alpha < \Omega$. However, the general pattern is more complicated than we have indicated; it has to be by Woodin's result on the complexity of $K \cap HC$. Instead, Steel wove together three definitions,

- α is a cardinal of K,
- $\mathcal{Q}$ is an α strong mouse, and
- $\mathcal{J}^K_{(\alpha^+)^K}$

by induction on $\alpha < \Omega$, where α *strong* is a natural strengthening of iterable. In the end, if α is a cardinal of K, then

$$\mathcal{J}^K_{(\alpha^+)^K} = \bigcup\{\mathcal{Q} \mid \mathcal{Q} \text{ is a sound } \alpha \text{ strong mouse with } \omega\rho^{\mathcal{Q}}_\omega = \alpha \text{ and } \mathcal{J}^K_\alpha \triangleleft \mathcal{Q}\}.$$

The simpler pattern that leaves out α strong holds for α less than the least measurable cardinal of K, as the reader familiar with the core model theory of Dodd and Jensen would expect. A remarkable fact due to Ralf Schindler is that the simpler pattern holds again if $\alpha \geq \aleph_2$. See Theorem 3.6.

Let us make the convention that if $\mathcal{P}$ is a mouse and $\mathcal{T}$ is an iteration tree on $\mathcal{P}$, then we have equipped $\mathcal{P}$ with an $(\omega, \Omega + 1)$ iteration strategy $\Sigma^{\mathcal{P}}$ and $\mathcal{T}$ is consistent with $\Sigma^{\mathcal{P}}$. Unless, of course, we specify otherwise. This will save us some writing and make the main points clearer.

2.30 Definition. Suppose that $\mathcal{Q}$ is a premouse and $\alpha \leq \mathrm{On} \cap \mathcal{Q} \leq \Omega$. Let $\mathcal{P} = \mathcal{J}^{\mathcal{Q}}_\alpha$. Then $\mathcal{Q}$ is α *strong* iff

1. $\mathcal{P}$ is A_0 sound (i.e., $\mathcal{P} \triangleleft K$) and
2. for each witness $\mathcal{P}^*$ that $\mathcal{P}$ is A_0 sound, there exist
   (a) an iteration tree $\mathcal{T}$ on $\mathcal{P}^*$ of successor length $\theta + 1 \leq \Omega + 1$ such that $\nu(E^{\mathcal{T}}_\eta) \geq \alpha$ for all $\eta < \theta$,
   (b) $\mathcal{R} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$ and
   (c) an elementary embedding $\pi : \mathcal{Q} \to \mathcal{R}$ with $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$.

Definition 2.30 does not have the features advertised before in that it is not first-order over $H(\Omega)$ and it is not a natural strengthening of iterability. But there is a satisfactory equivalent formulation that we get to somewhat later. However, the following connection between K and α strong tells us that we are on the right track.

2.31 Theorem. *Let α be a cardinal of K and $\mathcal{Q}$ be a sound premouse that agrees with K below α. Assume that $\rho^{\mathcal{Q}}_\omega = \alpha$. Then*

$$\mathcal{Q} \triangleleft K \iff \mathcal{Q} \textit{ is } \alpha \textit{ strong.}$$

The proof of this and the following closely related basic result are left as reasonable exercises for the reader.

2.32 Theorem. *Let α be a cardinal of K and $\mathcal{P} = \mathcal{J}^K_\alpha$. Suppose that $\mathcal{P}^*$ is a witness that $\mathcal{P}$ is A_0 sound. Let $\beta = (\alpha^+)^K$. Then*

(1) $\beta = (\alpha^+)^{\mathcal{P}^*}$,

(2) *$\mathcal{P}^*$ and K agree below β and*

(3) *$\mathcal{P}^*$ is α strong.*

Theorem 2.31 tells us how to formulate a recursive definition of K in terms of α strong mice. But the definition of α strong involves quantification over weasels that witness A_0 soundness, hence over subsets of $H(\Omega)$, so we are no better off than we started in terms of complexity. The first-order formulation we have in mind involves a generalization of the notion of an iteration tree on a mouse to an iteration tree on a phalanx, which is defined below. Such iteration trees also generalize the *double-rooted* iteration trees that appear in the proofs of condensation, Theorem 5.1 of [41], and solidity, Theorem 5.3 of [41], the difference being that we allow an arbitrary number of roots. (These condensation and solidity theorems originally appeared in [21] where double-rooted iteration trees are called *pseudo-iteration trees*.)

2.33 Definition. Suppose that $\vec{\lambda} = \langle \lambda_\alpha \mid \alpha < \gamma \rangle$ is an increasing sequence of ordinals, and $\vec{\mathcal{Q}} = \langle \mathcal{Q}_\alpha \mid \alpha \leq \gamma \rangle$ is a sequence of mice. Then $(\vec{\mathcal{Q}}, \vec{\lambda})$ is a *phalanx of length* $\gamma + 1$ iff $\mathcal{Q}_\alpha$ and $\mathcal{Q}_\beta$ agree below λ_α whenever $\alpha < \beta \leq \gamma$.

As an example, observe that if $\mathcal{S}$ is an iteration tree of successor length, then

$$(\langle \mathcal{M}^{\mathcal{S}}_\alpha \mid \alpha < \mathrm{lh}(\mathcal{S}) \rangle, \langle \mathrm{lh}(E^{\mathcal{S}}_\alpha) \mid \alpha < \mathrm{lh}(\mathcal{S}) - 1 \rangle)$$

is a phalanx of length $\mathrm{lh}(\mathcal{S})$. Notice that in passing from $\mathcal{S}$ to this phalanx we retain the models and record the relevant amount of agreement between the models but we lose all information about how the models were created and the tree order. Of course, not every phalanx comes from an iteration tree in this way.

2.34 Definition. Let $(\vec{\mathcal{Q}}, \vec{\lambda})$ be a phalanx of length $\gamma + 1$ and $\theta \geq \gamma + 1$. An *iteration tree $\mathcal{T}$ of length θ* on $(\vec{\mathcal{Q}}, \vec{\lambda})$ consists of

- a tree structure $<_T$ on θ for which each ordinal $\leq \gamma$ is a root,
- the corresponding root operation $\mathrm{root}^T : \theta \to \gamma + 1$,
- the corresponding predecessor operation pred^T that maps successor ordinals in the interval $[\gamma + 1, \theta)$ to ordinals $\leq \theta$,
- premice $\mathcal{M}^{\mathcal{T}}_\eta$ for $\eta < \theta$,

- extenders $E^{\mathcal{T}}_\eta$ whenever $\gamma < \eta + 1 < \theta$,
- a set of successor ordinals $D^{\mathcal{T}} \subseteq [\gamma + 1, \theta)$,
- a commutative system of embeddings

$$i^{\mathcal{T}}_{\zeta,\eta} : \mathcal{M}^{\mathcal{T}}_\zeta \to \mathcal{M}^{\mathcal{T}}_\eta$$

indexed by $\zeta <_T \eta$ for which

$$(\zeta, \eta]_T \cap D^{\mathcal{T}} = \emptyset$$

and

- an operation $\deg^{\mathcal{T}} : [\gamma + 1, \theta) \to \omega + 1$

with the following properties.

- If $\alpha \le \gamma$, then $\mathcal{M}^{\mathcal{T}}_\alpha = \mathcal{Q}_\alpha$ and $\lambda^{\mathcal{T}}_\alpha = \lambda_\alpha$.
- If $\gamma < \eta + 1 < \theta$, then $E^{\mathcal{T}}_\eta$ is an extender from the $\mathcal{M}^{\mathcal{T}}_\eta$ sequence, $\mathrm{pred}^T(\eta + 1)$ is the least $\zeta \le \eta$ such that

$$\mathrm{crit}(E^{\mathcal{T}}_\eta) < \lambda^{\mathcal{T}}_\zeta,$$

and

$$\mathcal{M}^{\mathcal{T}}_{\eta+1} = \mathrm{Ult}(\mathcal{N}, E^{\mathcal{T}}_\eta)$$

where $\mathcal{N}$ is the greatest initial segment of $\mathcal{M}^{\mathcal{T}}_{\mathrm{pred}^T(\eta+1)}$ such that $E^{\mathcal{T}}_\eta$ is an extender over $\mathcal{N}$. And

$$\eta + 1 \in D^{\mathcal{T}} \iff \mathcal{N} \ne \mathcal{M}^{\mathcal{T}}_{\mathrm{pred}^T(\eta+1)}.$$

The degree of this ultrapower is $\deg^{\mathcal{T}}(\eta + 1)$, and this degree equals the largest $n \le \omega$ such that

$$\rho^{\mathcal{N}}_n > \mathrm{crit}(E^{\mathcal{T}}_\eta)$$

If $\eta + 1 \notin D^{\mathcal{T}}$, then

$$i^{\mathcal{T}}_{\mathrm{pred}^T(\eta+1),\eta+1} : \mathcal{M}^{\mathcal{T}}_{\mathrm{pred}^T(\eta+1)} \to \mathcal{M}^{\mathcal{T}}_{\eta+1}$$

is the ultrapower embedding. And

$$\lambda^{\mathcal{T}}_\eta = \mathrm{lh}(E^{\mathcal{T}}_\eta).$$

- If $\gamma < \eta < \theta$ and η is a limit ordinal, then
$$[\mathrm{root}^T(\eta), \eta)_T$$
is a cofinal branch of $T \restriction \eta$. Moreover,
$$D^{\mathcal{T}} \cap [\mathrm{root}^T(\eta), \eta)_T$$
is finite and $\mathcal{M}^{\mathcal{T}}_\eta$ is the direct limit of the models $\mathcal{M}^{\mathcal{T}}_\zeta$ under the embeddings
$$i^{\mathcal{T}}_{\iota,\zeta} : \mathcal{M}^{\mathcal{T}}_\iota \to \mathcal{M}^{\mathcal{T}}_\zeta$$
for $\iota, \zeta \in [\mathrm{root}^T(\eta), \eta)_T - \max(D^{\mathcal{T}})$ with $\iota <_T \zeta$. In addition,
$$\deg^{\mathcal{T}}(\eta) = \liminf_{\zeta <_T \eta} \deg^{\mathcal{T}}(\zeta).$$

Just like with iteration trees on a single mouse, in the literature one sees the ultrapower embedding $\mathcal{N} \to \mathcal{M}^{\mathcal{T}}_{\eta+1}$ above denoted
$$i^*_{\eta+1} : \mathcal{M}^*_{\eta+1} \to \mathcal{M}_{\eta+1}.$$
Adding a superscript $\mathcal{T}$ leads to admittedly unattractive notation but we do not break with tradition.

We remark that in most cases of interest, the degree is non-increasing between drops in model so the lim inf ends up being the eventual value. The phrase *drop in degree* has the obvious meaning.

The notion of an iteration strategy generalizes in the obvious way to phalanxes. An iteration strategy picks cofinal branches at limit stages and is responsible for wellfoundedness in both successor and limit stages. When we speak of an iteration tree on an iterable phalanx, the reader should assume that the iteration tree is compatible with a fixed iteration strategy on the phalanx.

The following theorem is the key step towards a recursive definition of α strong. We write $<\beta$ strong to mean α strong for all $\alpha < \beta$.

2.35 Theorem. *Suppose that α is a cardinal of K and $\mathcal{Q}$ is a premouse of height $\leq \Omega$ that agrees with K below α. Then the following are equivalent.*

(1) *$\mathcal{Q}$ is α strong.*

(2) *For all $<\alpha$ strong premice $\mathcal{P}$,*
$$(\langle \mathcal{P}, \mathcal{Q} \rangle, \langle \alpha \rangle)$$
is an $\Omega + 1$ iterable phalanx.

Proof. That (2) implies (1) is an immediate consequence of Theorem 2.32(3) and the following result.

2.36 Lemma. *Suppose that α is a cardinal of K and $\mathcal{P} = \mathcal{J}^K_\alpha$. Let $\mathcal{P}^*$ be a witness that $\mathcal{P}$ is A_0 sound and $\mathcal{Q}$ be a premouse of height $\leq \Omega$. Suppose that $(\langle \mathcal{P}^*, \mathcal{Q}\rangle, \langle \alpha \rangle)$ is an $\Omega{+}1$ iterable phalanx. Then clause (2) of Definition 2.30 holds for $\mathcal{Q}$ and $\mathcal{P}^*$.*

Sketch. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of the pair

$$((\langle \mathcal{P}^*, \mathcal{Q}\rangle, \langle \alpha \rangle), \mathcal{P}^*).$$

We have not discussed this sort of coiteration before but it is defined in the natural way, using comparison of extender sequences to decide which extenders to apply at successor stages. The proof of the comparison, Theorem 3.11 of [41], generalizes to show that this coiteration is successful, which means that either $\mathcal{M}^{\mathcal{S}}_{1+\eta} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$ or vice-versa where $1 + \eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$. And that $1 + \eta, \theta \leq \Omega$.

2.37 Claim. $\mathrm{root}^S(1 + \eta) = 1$.

Sketch. For contradiction, suppose that $\mathrm{root}^S(1 + \eta) = 0$. As in the proof of universality Theorem 2.8, the fact that $\mathcal{P}^*$ computes κ^+ correctly for stationary many $\kappa < \Omega$ can be used to see that

$$\mathcal{M}^{\mathcal{S}}_{1+\eta} = \mathcal{M}^{\mathcal{T}}_\theta$$

and there is no dropping along $[0, 1 + \eta]_S$ and $[0, \theta]_T$. We have the embeddings

$$i^{\mathcal{S}}_{0,1+\eta} : \mathcal{P}^* \to \mathcal{M}^{\mathcal{S}}_{1+\eta}$$

and

$$i^{\mathcal{T}}_{0,\theta} : \mathcal{P}^* \to \mathcal{M}^{\mathcal{T}}_{1+\eta}$$

with

$$\mathrm{crit}(i^{\mathcal{S}}_{0,1+\eta}) < \alpha.$$

Theorems 2.13 and 2.16 generalize to iteration trees on phalanxes. Thus using the fact that $\mathcal{P}^*$ has the definability and hull properties at all ordinals $< \alpha$, we see that $i^{\mathcal{S}}_{0,1+\eta}$ and $i^{\mathcal{T}}_{0,\theta}$ have the same critical point and move subsets of their critical point the same way. In other words, the first extenders used along $[0, 1 + \eta]_S$ and $[0, \theta]_T$ agree, which leads to a contradiction as in the proof of comparison, Theorem 3.11 of [41]. $\dashv$

Again as in the proof of Theorem 2.8,

$$\mathcal{M}^{\mathcal{S}}_{1+\eta} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$$

and there is no dropping along $[1, 1 + \eta]_S$. So we have the embedding

$$i^{\mathcal{S}}_{1,1+\eta} : \mathcal{Q} \to \mathcal{M}^{\mathcal{S}}_{1+\eta}$$

with $\mathrm{crit}(i^{\mathcal{S}}_{1,1+\eta}) \geq \alpha$. Since $\mathcal{Q}$ and $\mathcal{P}^*$ agree below α, $\mathrm{lh}(E^{\mathcal{T}}_{\zeta}) \geq \alpha$ for all $\zeta < \theta$. Since α is a cardinal of K, it is a cardinal of $\mathcal{P}^*$. This can be used to see that $\nu(E^{\mathcal{T}}_{\zeta}) \geq \alpha$ for all $\zeta < \theta$. Thus $i^{\mathcal{S}}_{1,1+\eta}$ and $\mathcal{T}$ witness that $\mathcal{Q}$ is α strong relative to $\mathcal{P}^*$ as desired. $\dashv$

We have seen that (2) implies (1). Let β be a cardinal of K. We show that (1) for β implies (2) for β. Suppose that $\mathcal{Q}$ is a β strong premouse and $\mathcal{P}$ is a $<\!\beta$ strong premouse. We must show that $(\langle \mathcal{P}, \mathcal{Q}\rangle, \langle \beta \rangle)$ is $\Omega + 1$ iterable. By the proof of Theorem 2.29, there exist a witness W that $\mathcal{J}^K_{\beta}$ is A_0 sound and an elementary embedding

$$\sigma : W \to K^c.$$

We do not have a lower bound on the critical point of σ, nor is it relevant. By Definition 2.30, for each $\alpha \leq \beta$, we have an iteration tree $\mathcal{T}_\alpha$ of length $\theta_\alpha + 1 \leq \Omega + 1$ on W such that $\nu(F^{\mathcal{T}_\alpha}_{\eta}) \geq \alpha$ for all $\eta < \theta_\alpha$,

$$\mathcal{R}_\alpha \trianglelefteq \mathcal{M}^{\mathcal{T}_\alpha}_{\theta_\alpha}$$

and an elementary embedding π_α with $\pi_\alpha \restriction \alpha = \mathrm{id} \restriction \alpha$. If $\alpha < \beta$, then

$$\pi_\alpha : \mathcal{P} \to \mathcal{R}_\alpha$$

whereas

$$\pi_\beta : \mathcal{Q} \to \mathcal{R}_\beta.$$

Use σ to copy each $\mathcal{T}_\alpha$ to an iteration tree $\sigma\mathcal{T}_\alpha$ on K^c and let

$$\tau_\alpha : \mathcal{R}_\alpha \to \mathcal{S}_\alpha$$

be the restriction of the final copying map to $\mathcal{R}_\alpha$. Then

$$(\tau_\alpha \circ \pi_\alpha) \restriction \alpha = \tau_\alpha \restriction \alpha = \sigma \restriction \alpha$$

for all $\alpha \leq \beta$.

We wish to construe

$$(\langle \mathcal{S}_\alpha \mid \alpha \leq \beta \rangle, \langle \sigma(\alpha) \mid \alpha < \beta \rangle)$$

as a phalanx. Formally, for this we let $\langle \alpha_\eta \mid \eta \leq \theta \rangle$ enumerate the cardinals of K up to and including β and set

$$\mathfrak{F} = (\langle \mathcal{S}_{\alpha_\eta} \mid \eta \leq \theta \rangle, \langle \sigma(\alpha_\eta) \mid \eta < \theta \rangle).$$

Then $\mathfrak{F}$ is a phalanx. There are two basic elements to the remainder of the proof. Notice that all the models of $\mathfrak{F}$ are obtained by iterating K^c. We call such phalanxes K^c *based*. Steel proved that all K^c based phalanxes are $\Omega + 1$ iterable. The reader is referred to §9 of [42] for the proof, which builds on Steel's proof that K^c is $(\omega, \Omega+1)$ iterable. The second idea is that the sequence of embeddings

$$\psi_\eta = \tau_{\alpha_\eta} \circ \pi_{\alpha_\eta}$$

for $\eta \leq \theta$ can be used to pull back an iteration strategy on $\mathfrak{F}$ to an iteration strategy on $(\langle \mathcal{P}, \mathcal{Q} \rangle, \langle \beta \rangle)$. For this we use a generalization of the copying construction in §4.1 of [41]. The generalization is routine except for a few technical details. The main wrinkle comes in the case $\beta = (\alpha^+)^K$ when we apply the shift lemma to an ultrapower of $\mathcal{P}$ by an extender with critical point α. The difficulty is that $\psi_{\theta-1} = \tau_\alpha \circ \pi_\alpha$ and $\psi_\theta = \tau_\beta \circ \pi_\beta$ agree to α in this case whereas agreement to β would be needed to quote Lemma 4.2 of [41]. Nevertheless, a version of the shift lemma still goes through. We refer the reader to pp. 49-50 of [42] for the details. This type of copying construction is used repeatedly in the proof of the weak covering property in Section 4 $\dashv$

We are about to arrive at the much promised definition of K that is first-order over $H(\Omega)$. Clause (b) of Theorem 2.35 quantifies over weasels so there is still something to do.

2.38 Definition. If $\mathcal{T}$ is an iteration tree of length θ, then $\mathcal{T}$ is called *bad* if it is a losing position for player II in the iteration game. In other words,

1. if $\theta = \eta + 1$, then there is an extender F on the $\mathcal{M}_\eta^\mathcal{T}$ sequence such that $\mathrm{lh}(F) > \mathrm{lh}(E_\zeta^\mathcal{T})$ for all $\zeta < \eta$ but if $\zeta \leq \eta$ is least such that $\mathrm{crit}(F) \geq \nu(E_\zeta^\mathcal{T})$ and $\mathcal{N}$ is the greatest initial segment of $\mathcal{M}_\zeta^\mathcal{T}$ over which F is an extender, then $\mathrm{Ult}(\mathcal{N}, F)$ is illfounded where the degree of the ultrapower is as large as possible, and
2. if θ is a limit ordinal, then all cofinal branches of $\mathcal{T}$ have infinitely many drops in model or are illfounded.

Because of our technical hypothesis, $\Omega + 1$ iterability is equivalent to Ω iterability. In light of our anti-large cardinal hypothesis, there are many cases in which $\Omega + 1$ iterability reduces further to the non-existence of a countable bad tree. For example, the proof of Theorem 2.3 can be extended to show that if a premouse $\mathcal{P}$ of height Ω is not $\Omega + 1$ iterable, then there is a countable bad tree on $\mathcal{P}$. We give another useful example.

2.39 Definition. A premouse $\mathcal{P}$ is defined to be *properly small* iff $\mathcal{P}$ has no Woodin cardinals and $\mathcal{P}$ has a largest cardinal.

Notice that if $\mathcal{P}$ is a weasel and $\mu < \Omega$, then $\mathcal{J}^\mathcal{P}_{(\mu^+)^\mathcal{P}}$ is properly small. It is also easy to see that the properly small levels of K that project to α are unbounded in $(\alpha^+)^K$. If each premouse of a phalanx is properly small, then the $\Omega + 1$ iterability of the phalanx reduces to the non-existence of a countable bad tree on the phalanx. Arguing along these lines we obtain the following characterization.

2.40 Theorem. *Suppose that α is a cardinal of K and $\mathcal{Q}$ is a properly small premouse of height $< \Omega$ that agrees with K below α. Then the following are equivalent.*

(1) *$\mathcal{Q}$ is not α strong.*

(2) *There is a properly small $<\alpha$ strong premouse $\mathcal{P}$ with the same cardinality as $\mathcal{Q}$ and a countable bad iteration tree on the phalanx*

$$(\langle \mathcal{P}, \mathcal{Q} \rangle , \langle \alpha \rangle) .$$

Sketch. Suppose that $\mathcal{Q}$ is not α strong. Then there exist a weasel $\mathcal{P}^*$ that witnesses $\mathcal{J}^K_\alpha$ is A_0 sound and a bad iteration tree on

$$(\langle \mathcal{P}^*, \mathcal{Q} \rangle, \langle \alpha \rangle).$$

The same bad iteration tree can be construed as a bad iteration tree, call it $\mathcal{U}$, on

$$(\langle \mathcal{P}^{**}, \mathcal{Q} \rangle, \langle \alpha \rangle)$$

for some properly small $\mathcal{P}^{**} \lhd \mathcal{P}^*$. Let $Y \prec H(\Omega)$ with $\mathcal{U} \in Y$ such that Y has the same cardinality as $\mathcal{Q}$. Let $\tau : N \simeq Y$ with N transitive. Then $\tau^{-1}(\mathcal{Q}) = \mathcal{Q}$. Let $\mathcal{P} = \tau^{-1}(\mathcal{P}^{**})$. Then $\mathcal{P}$ is $<\alpha$ strong, properly small and has the same cardinality as $\mathcal{Q}$. Let $X \prec N$ with X countable and $\tau^{-1}(\mathcal{U}) \in X$. Let $\sigma : M \simeq X$ with M transitive. Let $\mathcal{S} = (\tau \circ \sigma)^{-1}(\mathcal{U})$. An absoluteness argument like that used in the proof of Theorem 2.3 shows that $\mathcal{S}$ is bad. (Here is where the hypothesis that $\mathcal{Q}$ is properly small is used.) Let $\mathcal{T} = \sigma\mathcal{S}$. Then $\mathcal{T}$ is a countable bad iteration tree on $(\langle \mathcal{P}, \mathcal{Q} \rangle, \langle \alpha \rangle)$. $\dashv$

Finally, we reach Steel's recursive definition of K.

2.41 Theorem. *Let $\mathcal{M}$ be a premouse of height $< \Omega$. Then $\mathcal{M} \lhd K$ iff there exist $\theta < \Omega$, an increasing continuous sequence of ordinals*

$$\langle \alpha_\eta \mid \eta \leq \theta + 1 \rangle$$

starting with $\alpha_0 = \omega$, an $\lhd$ increasing continuous sequence of premice

$$\langle \mathcal{R}_\eta \mid \eta \leq \theta + 1 \rangle$$

with $\mathcal{M} \lhd \mathcal{R}_{\theta+1}$ and a double-indexed sequence of sets

$$\langle \mathcal{F}_{\zeta,\eta} \mid \zeta \leq \eta \leq \theta \rangle$$

that satisfy the following conditions.

(1) *For all* $\zeta \leq \eta \leq \theta$ *and* $\mathcal{Q}$,
$$\mathcal{Q} \in \mathcal{F}_{\zeta,\eta}$$
iff $\mathcal{Q}$ *is a properly small premouse of cardinality* $|\alpha_\eta|$ *such that*
$$\mathcal{R}_\zeta \lhd \mathcal{Q}$$
and, if
$$\mathcal{P} \in \bigcap\{\mathcal{F}_{\iota,\eta} \mid \iota < \zeta\},$$
then
$$(\langle \mathcal{P}, \mathcal{Q}\rangle, \langle \alpha_\zeta \rangle)$$
is a phalanx on which there is no countable bad iteration tree.

(2) *For all* $\eta \leq \theta$,
$$\{\mathcal{Q} \in \mathcal{F}_{\eta,\eta} \mid \mathcal{Q} \textit{ is sound and } \omega\rho_\omega^{\mathcal{Q}} = \alpha_\eta\}$$
is a family of premice that are pairwise comparable under $\trianglelefteq$. *Moreover, the union of this family is* $\mathcal{R}_{\eta+1}$, *which is a premouse of height* $\alpha_{\eta+1}$.

Sketch. The idea is that, for all $\zeta \leq \eta < \Omega$,
$$\alpha_\eta = (\aleph_\eta)^K$$
$$\mathcal{R}_\eta = \mathcal{J}^K_{(\aleph_\eta)^K}$$
and $\mathcal{F}_{\zeta,\eta}$ is the set of properly small $(\aleph_\zeta)^K$ strong premice of size $|(\aleph_\eta)^K|$. And θ is large enough so that
$$\mathcal{M} \lhd \mathcal{J}^K_{(\aleph_{\theta+1})^K}.$$
$\dashv$

3. Core Model Tools

Throughout this section, we continue to assume the anti-large cardinal hypothesis,

there is no proper class model with a Woodin cardinal

and the technical hypothesis,

U *is a normal measure over* Ω.

Under these hypotheses, in the previous section, a certain transitive model of ZFC of ordinal height Ω is defined and named K. Here, we list properties of K that are useful in applications. For the most part, it is not necessary to read the previous section to make sense of these properties.

3.1. Covering Properties

Jensen showed that if $0^\#$ does not exist and A is an uncountable set of ordinals, then there exists a set $B \in L$ such that $A \subseteq B$ and $|A| = |B|$. Dodd and Jensen proved the same theorem for K under the hypothesis that there is no inner model with a measurable cardinal. If there is a measurable cardinal, then the Jensen covering property for K fails in any Prikry forcing extension. Mitchell proved that if there is no inner model with $o(\kappa) = \kappa^{++}$, then K still satisfies several consequences of the Jensen covering property and that these weak covering properties are still useful in applications. Mitchell's work in this regard and the history behind it is the subject of the Handbook chapter [18].

The first result we list in this subsection, which is due to Steel, says that K computes the successor of almost every cardinal correctly.

3.1 Theorem.

$$\{\kappa < \Omega \mid (\kappa^+)^K = \kappa\} \in U.$$

The reader should cite Theorem 5.18(2) of [42] when applying Theorem 3.1. We mentioned this result in Section 2 just after the proof of Theorem 2.29.

Many people would identify the following result, which is due to Mitchell and the author, as the weak covering theorem for K. It implies that K computes successors of singular cardinals correctly but contains other applicable information.

3.2 Theorem. *Let κ be a cardinal of K such that*

$$\omega_2 \leq \kappa < \Omega.$$

and

$$\lambda = (\kappa^+)^K.$$

Then

$$\mathrm{cf}(\lambda) \geq |\kappa|.$$

Thus either $\lambda = |\kappa|^+$ or $\mathrm{cf}(\lambda) = |\kappa|$.

The reader should cite Theorem 0.1 of [19] when applying Theorem 3.2. The proof builds on that of Theorem 1.1 [20], which is the special case in which $|\kappa|$ is a countably closed cardinal. We outline the proof under this and further simplifying assumptions in Section 4.

The next result, which is due to Steel and the author, says that K computes successors of weakly compact cardinals correctly. The corresponding fact for L under the assumption that $0^\#$ does not exist was observed by Kunen in the 1970s.

3.3 Theorem. *Let κ be a weakly compact cardinal such that $\kappa < \Omega$. Then*

$$(\kappa^+)^K = \kappa^+.$$

The reader should cite Theorem 3.1 of [30] when applying Theorem 3.3.

3.2. Absoluteness, Complexity and Correctness

Steel proved the following theorem, which says that K is forcing absolute.

3.4 Theorem. *Let $\mathbb{P} \in H(\Omega)$ be a poset. Then*

$$\Vdash_{\mathbb{P}} K = K^V.$$

The reader should cite Theorem 5.18(3) of [42] when applying Theorem 3.4. We mentioned this result in Section 2 just after the proof of Theorem 2.29.

Using his first-order definition of K, Steel carried out the first part of the following computation of $K \cap HC$. Think of this as the set of reals that code the countable levels of K, countable in V that is. The second part, a computation done by Schindler, shows that the complexity drops if only finitely many countable ordinals are strong cardinals in K.

3.5 Theorem. *There is a Σ_1 formula $\varphi(x)$ such that for all $a \in HC$,*

$$a \in K \iff L_{\omega_1}(\mathbb{R}) \models \varphi[a].$$

Moreover, if $K \cap HC$ has at most finitely many strong cardinals, then there is a formula $\psi(x)$ such that for all $a \in HC$,

$$a \in K \Longleftrightarrow HC \models \psi[a].$$

The reader should cite Theorem 6.15 of [42] when applying the first part of Theorem 3.5. We mentioned this result in Section 2; it is a corollary to Theorem 2.41. The reader should cite Theorems 3.4 and 3.6 of [11] when applying the *moreover* part of Theorem 3.5.

Steel defined the levels of K by recursion on their ordinal height $< \Omega$. It turns out that iterability alone is not enough to guarantee that a mouse with all the right first-order properties to be a level of K is actually a level of K. So, simultaneous with his recursive definition of the levels of K, Steel defined increasingly strong forms of iterability. This is explained in detail in Section 2.2. The following theorem of Schindler shows that there is a tremendous simplification in the recursive definition for levels of K of height $\geq \aleph_2$.

3.6 Theorem. *Let κ be a cardinal of K such that $\aleph_2 \leq \kappa < \Omega$. Suppose that $\mathcal{M}$ is a mouse such that*

(1) *$\mathcal{M}$ and K agree below κ,*

(2) *$\rho_\omega^{\mathcal{M}} \leq \kappa$ and*

(3) *$\mathcal{M}$ is sound above κ.*

Then $\mathcal{M}$ is an initial segment of K.

One says that above $\aleph_2$, K is obtained by stacking mice. The reader should cite Lemma 3.5 of [10] when using Theorem 3.6 and should consult Lemma 2.2 of [34] as well. The proof of Theorem 3.6 builds on the proof of Theorem 3.2.

By definition, a class M is Σ^1_n correct iff $M \prec_{\Sigma^1_n} V$. In other words, for each Σ^1_n formula $\psi(x)$ and $a \in \mathbb{R} \cap M$,

$$\psi[a]^M \iff \psi[a].$$

Jensen proved that if $x^\#$ exists for all $x \subseteq \omega$ but there is no inner model with a measurable cardinal, then K is Σ^1_3 correct. The following result is due to Steel.

3.7 Theorem. *Suppose that there exists a measurable cardinal $< \Omega$. Then K is Σ^1_3 correct.*

The reader should cite Theorem 7.9 of [42] when applying Theorem 3.7. It is not known if the existence of a measurable cardinal $< \Omega$ is needed. There is also an attractive conjecture regarding Σ^1_4 correctness that has been open for about a decade.[7]

3.3. Embeddings of K

The first result in this subsection, which is due to Steel, says that K is rigid.

3.8 Theorem. *If $j : K \to K$ is an elementary embedding, then j is the identity.*

The reader should cite Theorem 8.8 of [42] when applying Theorem 3.8.

Steel proved the following result, which says that K is universal.

3.9 Theorem. *K is the unique universal weasel that elementarily embeds into all other universal weasels.*

The reader should cite Theorem 8.10 of [42] when applying Theorem 3.9. Universal weasels were defined in Section 2. See Definitions 2.1 and 2.7.

Now we turn to external embeddings and their actions on K. The question is whether the restriction to K of an embedding from an iteration of V is the embedding from an iteration of K.

[7] Assume that $M_1(x)$ exists for all sets x but that there is no model with two Woodin cardinals. Show that K is Σ^1_4 correct.

3.10 Theorem. *Suppose that $\mathcal{T}$ is an iteration tree on V with final model N and branch embedding*

$$\pi : V \to N.$$

Assume that

(1) *$\mathcal{T}$ is finite and ${}^{\omega}N \subseteq N$, or*

(2) *$\mathcal{T}$ is countable and ρ-maximal in the sense of Neeman [24].*

Then there is an iteration tree on K whose last model is K^N and whose branch embedding is $\pi \restriction K$.

Keep in mind that even if the external iteration tree $\mathcal{T}$ consists of a single ultrapower by a normal measure, the corresponding iteration tree on K may be infinite and quite complicated. Schindler proved Theorem 3.10 under assumption (1). The author observed that Schindler's proof goes through with assumption (2). The reader should cite Corollary 3.1 of [36] in case (1) and Corollary 3.2 [36] in case (2) when applying Theorem 3.10.

3.4. Maximality

Steel proved that K is maximal in the following sense.

3.11 Theorem. *Let F be an extender that coheres with the extender sequence of K. Suppose that (K, F) is countably certified. Then F is on the extender sequence of K.*

The reader should cite Theorem 8.6 of [42] when applying Theorem 3.11. This can be used to see that certain large cardinals reflect to K. For example, if $\kappa < \Omega$ and κ is a λ strong cardinal for all $\lambda < \Omega$, then κ has the same property in K. The proof of a theorem slightly more general than Theorem 8.6 of [42], applications of maximality and other results along these lines by Steel and the author can be found in [30]. For example, Theorem 3.4 of [30] says that if κ is a cardinal such that $\aleph_2 \leq \kappa < \Omega$, then $H(\kappa) \cap K$ is universal for mice in $H(\kappa)$.

3.5. Combinatorial Principles

Jensen's results on the fine structure of L generalize to models of the form $L[E]$.

3.12 Theorem. *Let $\mathcal{Q}$ be a weasel. Then $\mathcal{Q}$ satisfies the following statements.*

(1) *If κ is a cardinal, then $\diamondsuit^{+}_{\kappa^+}$ holds.*

(2) *If κ is an inaccessible cardinal, then*

$$\diamondsuit^+_\kappa \text{ holds} \iff \kappa \text{ is not ineffable.}$$

(3) *If κ is a cardinal, then*

$$\Box_\kappa \text{ holds} \iff \kappa \text{ is not subcompact.}$$

(4) *If κ is a regular cardinal, then there is a κ^+ morass.*

When applying Theorem 3.12, the reader should cite Theorem 1.2 of [28] for the clauses on diamond, which are due to the author. The reader should cite Theorem 2 of [32] for the existence of a $\Box_\kappa$ sequence. (It is a theorem of ZFC due to Burke [4] that if κ is a subcompact cardinal, then $\Box_\kappa$ fails.) Zeman and the author [31] proved the clause on morass.

Even though $\mathcal{Q} = K$ is its most interesting instance, Theorem 3.12 holds in situations in which we do not know how to define K. Neither the anti-large cardinal hypothesis nor the technical hypothesis is used in the proof of Theorem 3.12. This explains why we bothered to mention subcompact cardinals in the clause on square since subcompact cardinals are themselves Woodin cardinals, which do not exist under our anti-large cardinal hypothesis. We should add that only a weak form of iterability is needed for the proof of Theorem 3.12, much less than is assumed in the definition of weasel.

The next result gives conditions under which the $\Box_\kappa$ sequence in K cannot be threaded in V. It is a result of the author.

3.13 Theorem. *Let κ be a cardinal such that*

$$\aleph_2 \leq \kappa < \Omega.$$

Suppose that κ is a limit cardinal of K. Let $\lambda = (\kappa^+)^K$. Then there exists a

$$\langle C_\alpha \mid \alpha < \lambda \rangle \in K$$

such that $\langle C_\alpha \mid \alpha < \lambda \rangle$ is a $\Box_\kappa$ sequence in K and $\langle C_\alpha \mid \alpha < \lambda \rangle$ has no thread. That is, there is no club D in λ such that

$$D \cap \alpha = C_\alpha$$

for all $\alpha \in \lim(D)$.

The reader should cite [27] when applying Theorem 3.13.

3.6. On the Technical Hypothesis

Schindler proved that below “zero hand grenade”, the technical hypothesis can be avoided:

3.14 Theorem. *If there is no proper class model with a proper class of strong cardinals, then the technical hypothesis is not needed for the results in this paper.*

The reader should consult [33] before applying Theorem 3.14. Not only is Theorem 3.14 loosely worded, it does not make sense, at least not literally, since there are results in this paper that explicitly refer to the normal measure U over Ω. For these, Ω should be replaced by On and statements about sets in U should be read as statements about stationary classes of ordinals.

Without going to a more restrictive anti-large cardinal hypothesis, it is not known how to get away without a technical hypothesis. But technical hypotheses weaker than a measurable cardinal are known to suffice. For example, Steel showed that the existence of $X^{\#}$ for all $X \in H(\Omega)$ is enough. Also, Steel and the author showed in Theorem 5.1 of [42] that an ω Erdos cardinal is enough.

4. Proof of Weak Covering

In this section, we discuss elements of the proof of Theorem 3.2 under some simplifying assumptions. Earlier versions of this theorem due to Jensen, Dodd and Jensen, and Mitchell had no technical hypothesis and much stronger anti-large cardinal hypotheses. In particular, their proofs involved linear iterations at most whereas we deal with iteration trees and even some generalizations of iteration trees. To make our task manageable we assume that the reader is familiar with at least one of these earlier proofs, such as any proof in the Handbook chapter [18] or just the proof for L as presented in [29] or in the Handbook chapter [38]. Our emphasis here is on the new complications and how to overcome them, really just a segue into [20] for the reader.

4.1 Definition. A cardinal κ is *countably closed* iff $\mu^{\aleph_0} < \kappa$ for all cardinals $\mu < \kappa$.

For example, if $2^{\aleph_0} < \aleph_\omega$ then $\aleph_\omega$ is countably closed. The following special case of Theorem 3.2 was proved in [20]. We continue to assume the same anti-large cardinal hypothesis and technical hypothesis as in all earlier sections, therefore K exists.

4.2 Theorem. *Let κ be a cardinal of K such that $|\kappa|$ is countably closed and $\lambda = (\kappa^+)^K$. Then*

$$\mathrm{cf}(\lambda) \geq |\kappa|.$$

Thus either $\lambda = |\kappa|^+$ or $\mathrm{cf}(\lambda) = |\kappa|$.

For example, if $2^{\aleph_0} < \aleph_\omega$ and $\kappa = \aleph_\omega$, then $(\kappa^+)^K = \aleph_{\omega+1}$.

Outline. The proof begins pretty much as do earlier proofs of weak covering under stronger anti-large cardinal hypotheses. Let $\lambda = (\kappa^+)^K$ and assume for contradiction that

$$\mathrm{cf}(\lambda) < |\kappa|.$$

By taking the union of an elementary chain of length ω_1, we find

$$X \prec (V_{\Omega+1}, \in, U)$$

with

$$\sup(X \cap \lambda) = \lambda$$

and

$${}^{\omega}X \subseteq X$$

such that

$$|X| < \mathrm{cf}(\lambda)^{\aleph_0} < |\kappa|.$$

Let $\pi : N \simeq X$ with N transitive and $\delta = \mathrm{crit}(\pi)$. Note that $\pi(\delta) \leq \kappa$. Let

$$\overline{\kappa} = \pi^{-1}(\kappa)$$

$$\overline{\lambda} = \pi^{-1}(\lambda)$$

and

$$\overline{\Omega} = \pi^{-1}(\Omega).$$

Consider an arbitrary $\mu \leq \Omega$. Let $E_\pi{\restriction}\mu$ be the extender of length μ derived from π. This means the following. For each $a \in [\mu]^{<\omega}$, let

$$\delta_a = \min(\{\gamma \in \overline{\Omega} \cap N \mid a \in [\pi(\gamma)]^{<\omega}\}).$$

Then let

$$(E_\pi)_a = \{X \subseteq [\delta_a]^{|a|} \mid a \in \pi(X)\}.$$

Notice that $(E_\pi)_a$ is an ultrafilter over

$$\wp([\delta_a]^{|a|}) \cap N.$$

And

$$E_\pi{\restriction}\mu = \{(a, X) \mid a \in [\mu]^{<\omega} \text{ and } X \in (E_\pi)_a\}.$$

The point of this extender is that if M is a transitive model and

$$\wp(\delta_a) \cap M \subseteq N$$

for all $a \in [\mu]^{<\omega}$, then it makes sense to talk about the ultrapower map

$$i^M_E : M \to \mathrm{Ult}(M, F)$$

where
$$F = E_\pi \cap ([\mu]^{<\omega} \times M).$$
Put another way, we may apply F to M iff
$$\wp(\gamma) \cap M \subseteq N$$
for all γ such that $\pi(\gamma) \geq \mu$. Define
$$\mathrm{Ult}(M, \pi, \mu) = \mathrm{Ult}(M, E_\pi \restriction \mu).$$
Here are a few more general remarks. If $a \in [\delta]^{<\omega}$, then $(E_\pi)_a$ is principal and therefore $E_\pi \restriction \delta$ is trivial in the sense that it gives rise to the identity embedding. Observe that $(E_\pi)_{\{\delta\}}$ is equivalent to the normal measure derived from π,
$$\{X \subseteq \delta \mid \delta \in \pi(X)\},$$
in the sense that they determine the same ultrapower. We call
$$E_\pi \restriction \pi(\delta)$$
the superstrong extender derived from π. And we call $E_\pi \restriction \mu$ a *long extender* whenever $\pi(\delta) < \mu$ or, equivalently, whenever $\delta_a > \delta$ for some $a \in [\mu]^{<\omega}$. Long extenders come up in the covering theorem for L in exactly the same way although the terminology had not been established when Jensen discovered the proof. The reader may refer to [29] for an account of Jensen's proof in these terms.

Instead of K we work with an A_0 soundness witness for a large enough initial segment of K. Large enough for us means height Ω_0 where
$$\Omega > \Omega_0 \geq |\lambda|^+ = |\kappa|^+.$$
But for convenience we assume that Ω_0 is an inaccessible cardinal. Let W be the witness that $\mathcal{J}^K_{\Omega_0}$ is A_0 sound that comes out of the proof of Theorem 2.24. There is an elementary embedding $\sigma : W \to K^c$ that is relevant later in the current proof. Let
$$\overline{W} = \pi^{-1}(W)$$
and $(\overline{\mathcal{T}}, \mathcal{T})$ be the coiteration of $(\overline{W}, W)$. Say
$$\theta + 1 = \mathrm{lh}(\mathcal{T})$$
and
$$\overline{\theta} + 1 = \mathrm{lh}(\overline{\mathcal{T}}).$$
Simplify the notation by setting
$$W_\eta = \mathcal{M}^{\mathcal{T}}_\eta$$

for $\eta \leq \theta$ and

$$\overline{W}_\eta = \mathcal{M}_\eta^{\overline{\mathcal{T}}}$$

for $\eta \leq \overline{\theta}$. Because W is universal,

$$D^{\overline{\mathcal{T}}} \cap [0, \overline{\theta}]_{\overline{T}} = \emptyset$$

and

$$\overline{W}_{\overline{\theta}} \triangleleft W_\theta.$$

Nothing we have said so far differs significantly from earlier proofs of weak covering under stronger anti-large cardinal hypotheses except possibly that we are using W instead of K. Before continuing, let us review the main points of the earlier proofs and compare and contrast them with the current proof. In the earlier proofs, it is shown that $\overline{K} = \pi^{-1}(K)$ does not move in its coiteration with K. The current proof shows this too but in an indirect way.[8] Now let $\eta \leq \theta$ be least such that

$$\nu(E_\eta^{\mathcal{T}}) > \overline{\kappa}$$

if there is such an η; otherwise let $\eta = 0$. In the earlier proofs, it is shown that there exist $\mathcal{P} \trianglelefteq W_\eta$ and $n < \omega$ such that

$$\rho_n^{\mathcal{P}} \geq \overline{\lambda} = (\overline{\kappa}^+)^{\mathcal{P}}$$

and

$$\mathcal{P} = \mathrm{Hull}_{n+1}^{\mathcal{P}}(\overline{\kappa} \cup p_{n+1}^{\mathcal{P}}).$$

People refer to $\mathcal{P}$ as the *least mouse missing from N at $\overline{\kappa}$*. The current proof is different in that W_η might be a weasel and

$$(\overline{\kappa}^+)^{W_\eta} = \overline{\lambda}.$$

In this case, we set $\mathcal{P} = W_\eta$. Then $\mathcal{P}$ is a thick weasel. Moreover, because $\nu(E_\zeta^{\mathcal{T}}) \leq \overline{\kappa}$ for all $\zeta < \eta$, we conclude that $\mathcal{P}$ has the hull and definability properties at μ whenever $\overline{\kappa} \leq \mu < \Omega_0$. This collection of facts about $\mathcal{P}$ turns out to be an adequate substitute if $\mathcal{P}$ happens to be a weasel instead of a premouse of height $< \Omega$. Moving on with our discussion, in the earlier proofs, $E_\pi \restriction \lambda$ is an extender over $\mathcal{P}$ and one sets

$$\mathcal{R} = \mathrm{Ult}(\mathcal{P}, \pi, \lambda).$$

People refer to $\mathcal{R}$ as the *lift up of* $\mathcal{P}$. In the current proof, because iteration trees need not be linear, something along the lines of $\overline{W}$ not moving is needed just to make sense of the definition of $\mathcal{R}$. In the earlier proofs, a

[8] The current proof is a complicated induction that shows no extender of length $< \overline{\Omega}_0$ is used on $\overline{\mathcal{T}}$.

standard argument using the fact that ${}^{\omega}X \subseteq X$ shows that $\mathcal{R}$ is an iterable premouse. The current proof is different on this point. For although $\mathcal{R}$ is wellfounded, it can fail to be a premouse! This happens exactly when $\rho_1(\mathcal{P}) \leq \overline{\kappa}$ (that is, $n = 0$), $\mathcal{P}$ has a top extender with critical point $\mu < \overline{\kappa}$ and π is discontinuous at $(\mu^+)^{\mathcal{P}}$. For then the top extender of $\mathcal{R}$ is not total on $\mathcal{R}$ since its critical point is $\pi(\mu)$ but it only measures sets in

$$J^{\mathcal{R}}_{\sup(\pi``(\mu^+)^{\mathcal{P}})} \lhd J^{\mathcal{R}}_{(\pi(\mu)^+)^{\mathcal{R}}}.$$

We call $\mathcal{R}$ a *protomouse* and its top predicate $F^{\mathcal{R}}$ an *extender fragment.* Vaguely put, our answer to the possibility that $\mathcal{R}$ is not a premouse is to find an actual premouse that corresponds to $\mathcal{R}$. But let us set aside this complication until later and assume that $\mathcal{R}$ is a premouse. In the discussion so far, we have implicitly used some basic facts about the ultrapower embedding $\tilde{\pi} : \mathcal{P} \to \mathcal{R}$, mainly that

$$\tilde{\pi} \restriction \overline{\lambda} = \pi \restriction \overline{\lambda}.$$

It is also easy to see that

$$\mathcal{R} = \mathrm{Ult}_n(\mathcal{P}, \pi, \kappa).$$

And that, if $\mathcal{P}$ is not a weasel, then

$$\mathcal{R} = \mathrm{Hull}^{\mathcal{R}}_{n+1}(\kappa \cup \tilde{\pi}(p^{\mathcal{P}}_{n+1})) = \mathrm{Hull}^{\mathcal{R}}_{n+1}(\kappa \cup p^{\mathcal{R}}_{n+1}),$$

whereas if $\mathcal{P}$ is a weasel, then $\mathcal{R}$ is a thick weasel with the hull and definability properties at μ whenever $\kappa \leq \mu < \Omega_0$. The last step in the earlier proofs is to analyze the coiteration of $\mathcal{R}$ versus W to obtain the contradiction

$$\lambda = (\kappa^+)^K = (\kappa^+)^W > (\kappa^+)^{\mathcal{R}} = \sup(\tilde{\pi}``\overline{\lambda}) = \sup(\pi``\overline{\lambda}) = \lambda.$$

At the analogous step in the current proof, we coiterate $(\langle W, \mathcal{R}\rangle, \langle \kappa \rangle)$ versus W. For this we need that the phalanx is iterable. Basically, we need to know that $\mathcal{R}$ is κ strong whereas in the earlier proofs, iterability was enough. Our solution, which we make precise soon, is to work up to this phalanx by an induction that involves other phalanxes. In summary, the new complications are:

- how to show that $\overline{W}$ does not move,
- $\mathcal{P}$ and $\mathcal{R}$ could be weasels,
- $\mathcal{R}$ might be a protomouse but not a premouse and
- how to show that $\mathcal{R}$ is κ strong.

We need more definitions to explain our strategy for dealing with these new complications. Let $\vec{\kappa}$ enumerate the infinite cardinals of $\overline{W}_{\overline{\theta}}$ up to $\overline{\Omega}_0 = \pi^{-1}(\Omega_0)$. Thus

$$\kappa_\alpha = (\aleph_\alpha)^{\overline{W}_{\overline{\theta}}}$$

for all $\alpha < \overline{\Omega}_0$. Also let $\vec{\lambda}$ enumerate the infinite successor cardinals of $\overline{W}_{\overline{\theta}}$. Thus $\lambda_\alpha = \kappa_{\alpha+1}$ for all $\alpha < \overline{\Omega}_0$. The main idea for dealing with the first and last new complications involves an induction on $\gamma < \overline{\Omega}_0$ with six induction hypotheses. As they are introduced, we assume $(1)_\alpha$ through $(6)_\alpha$ for all $\alpha < \gamma$. Our obligation is to prove $(1)_\gamma$ through $(6)_\gamma$. The first induction hypothesis tells us that $\overline{W}$ has not moved yet. We use the notation $\overline{E}_\eta = E_\eta^{\overline{\mathcal{T}}}$.

$(1)_\alpha$ *For all $\eta \leq \overline{\theta}$, if $\overline{E}_\eta \neq \emptyset$, then $\mathrm{lh}(\overline{E}_\eta) > \lambda_\alpha$.*

The next step is to derive a phalanx from $\mathcal{T}$. Let $\eta(\alpha)$ be the least $\eta \leq \theta$ such that

$$\nu(E_\eta^{\mathcal{T}}) > \kappa_\alpha$$

if there is such an η; otherwise, let $\eta(\alpha) = 0$. Then let $\mathcal{P}_\alpha$ be the unique $\mathcal{P} \trianglelefteq W_{\eta(\alpha)}$ such that for some $n < \omega$

$$\rho_n^{\mathcal{P}} \geq \lambda_\alpha = (\kappa_\alpha^+)^{\mathcal{P}}$$

and $\rho_{n+1}^{\mathcal{P}} \leq \kappa_\alpha$ if it exists. In this case,

$$\mathcal{P} = \mathrm{Hull}_{n+1}^{\mathcal{P}}(\kappa_\alpha \cup p_{n+1}^{\mathcal{P}}).$$

Otherwise, let $\mathcal{P}_\alpha = W_{\eta(\alpha)}$. In this case, $\mathcal{P}_\alpha$ is a thick weasel with the hull and definability properties at μ whenever $\kappa_\alpha \leq \mu < \Omega_0$.

4.3 Lemma. *The phalanx $(\vec{\mathcal{P}}{\restriction}(\gamma+1), \vec{\lambda}{\restriction}\gamma)$ is iterable.*

Idea. We may construe an iteration tree on this phalanx as an iteration tree extending $\mathcal{T}{\restriction}(\eta(\gamma)+1)$. But W is iterable. $\dashv$

By our induction hypothesis $(1)_\alpha$, $E_\pi{\restriction}\pi(\kappa_\alpha)$ is an extender over $\mathcal{P}_\alpha$ for each $\alpha < \gamma$.[9] This allows us to define

$$\mathcal{R}_\alpha = \mathrm{Ult}(\mathcal{P}_\alpha, \pi, \pi(\kappa_\alpha))$$

and

$$\Lambda_\alpha = \sup(\pi``\lambda_\alpha) = (\pi(\kappa_\alpha)^+)^{\mathcal{R}_\alpha}.$$

[9]Models on a non-linear iteration tree are not necessarily contained in the starting model. In order to form $\mathrm{Ult}(\mathcal{P}_\alpha, \pi, \pi(\kappa_\alpha))$ we must know that $E_\pi{\restriction}\pi(\kappa_\alpha)$ measures all sets in $\mathcal{P}_\alpha$. The proof presented in [20] overlooks this detail but can can be straightened out easily using the approach shown here.

A standard application of the fact that ${}^{\omega}X \subseteq X$ shows that $\mathcal{R}_\alpha$ is a transitive structure. Let $\pi_\alpha : \mathcal{P}_\alpha \to \mathcal{R}_\alpha$ be the ultrapower map. More standard calculations show that

$$\pi_\alpha \restriction \lambda_\alpha = \pi \restriction \lambda_\alpha$$

and

$$\pi_\alpha(\lambda_\alpha) = \Lambda_\alpha \leq \pi(\lambda_\alpha).$$

Also that

$$\mathcal{R}_\alpha = \mathrm{Ult}(\mathcal{P}_\alpha, \pi, \Lambda_\alpha).$$

And, if $\mathcal{P}_\alpha$ is not a weasel, then

$$\mathcal{R}_\alpha = \mathrm{Hull}_{n+1}^{\mathcal{R}_\alpha}(\pi(\kappa_\alpha) \cup \pi_\alpha(p_{n+1}^{\mathcal{P}_\alpha})) = \mathrm{Hull}_{n+1}^{\mathcal{R}_\alpha}(\pi(\kappa_\alpha) \cup p_{n+1}^{\mathcal{R}_\alpha})$$

for some $n < \omega$, whereas if $\mathcal{P}_\alpha$ is a weasel, then $\mathcal{R}_\alpha$ is a thick weasel with the hull and definability properties at μ whenever $\pi(\kappa_\alpha) \leq \mu < \Omega_0$. But notice that if $\rho_1(\mathcal{P}_\alpha) \leq \kappa_\alpha$, $\mathcal{P}_\alpha$ is an active premouse and π is discontinuous at the cardinal successor of $\mathrm{crit}(F^{\mathcal{P}_\alpha})$ in $\mathcal{P}_\alpha$, then $\mathcal{R}_\alpha$ is not a premouse.

Observe that

$$(\vec{\mathcal{R}} \restriction (\gamma + 1), \vec{\Lambda} \restriction \gamma)$$

satisfies the agreement condition for being a phalanx. We call it a *phalanx of protomice*. Let us examine the situation in which $\beta \leq \gamma$ and $\mathcal{R}_\beta$ is not a premouse. Equivalently, there exist $\alpha < \beta$ with

$$\mathrm{crit}(F^{\mathcal{P}_\beta}) = \kappa_\alpha$$

and

$$\Lambda_\alpha < \pi(\lambda_\alpha).$$

In this case,

$$\mathrm{crit}(F^{\mathcal{R}_\beta}) = \pi(\kappa_\alpha).$$

And, although $F^{\mathcal{R}_\beta}$ is an extender fragment but not an extender over $\mathcal{R}_\beta$, it is an extender over $\mathcal{R}_\alpha$. More generally, if $\mathcal{U}$ is what would naturally be called an iteration tree on

$$(\vec{\mathcal{R}} \restriction (\gamma + 1), \vec{\Lambda} \restriction \gamma)$$

and $\gamma < \beta' < \mathrm{lh}(\mathcal{U})$ with

$$\mathrm{root}^U(\beta') = \beta$$

and

$$D^{\mathcal{U}} \cap (\beta, \beta']_U = \emptyset,$$

then

$$\mathrm{crit}(F^{\mathcal{M}_{\beta'}^{\mathcal{U}}}) = \pi(\kappa_\alpha) = \mathrm{crit}(F^{\mathcal{R}_\beta})$$

and the two extender fragments are total over $\mathcal{R}_\alpha$. Thus $F^{\mathcal{M}_{\beta'}^{\mathcal{U}}}$ could legitimately be applied to $\mathcal{R}_\alpha$ to form an extension of $\mathcal{U}$. While the following result is not used in the current proof, others like it are.

4.4 Lemma. *The phalanx of protomice*

$$(\vec{\mathcal{R}}{\restriction}(\gamma+1), \vec{\Lambda}{\restriction}\gamma)$$

is iterable.

Idea. In the standard way, use the fact that ${}^{\omega}X \subseteq X$ to reduce the iterability of the above phalanx to that of

$$(\vec{\mathcal{P}}{\restriction}(\gamma+1), \vec{\lambda}{\restriction}\gamma).$$

The latter phalanx is iterable by Lemma 4.3. ⊣

Based on our discussion of earlier proofs of weak covering, we would expect to want to iterate

$$(\langle W, \mathcal{R}_\gamma\rangle, \langle \pi(\kappa_\gamma)\rangle).$$

We can make sense of what we mean by this even if $\mathcal{R}_\gamma$ is not a premouse, but iterating this phalanx of protomice does not seem to accomplish much in this case. Our solution to this problem is complicated. For each $\alpha \leq \gamma$, if $\mathcal{R}_\alpha$ is not a premouse, then we define a certain premouse $\mathcal{S}_\alpha$ that agrees with $\mathcal{R}_\alpha$ below Λ_α. We also find a premouse $\mathcal{Q}_\alpha$ that agrees with $\mathcal{P}_\alpha$ below λ_α such that

$$\mathcal{S}_\alpha = \mathrm{Ult}(\mathcal{Q}_\alpha, \pi, \pi(\kappa_\alpha))$$

Only near the end of the current proof will we say exactly what $\mathcal{Q}_\alpha$ and $\mathcal{S}_\alpha$ are in this case. On the other hand, if $\mathcal{R}_\alpha$ is a premouse, then $\mathcal{Q}_\alpha = \mathcal{P}_\alpha$ and $\mathcal{S}_\alpha = \mathcal{R}_\alpha$. The reader is asked to consider this case only for the moment.

As we just indicated, the main thing we want to know besides $(1)_\gamma$ is that $\mathcal{S}_\gamma$ is $\pi(\kappa_\gamma)$ strong, so we make it an induction hypothesis in the following way.

(2)$_\alpha$ $(\langle W, \mathcal{S}_\alpha\rangle, \langle \pi(\kappa_\alpha)\rangle)$ *is an iterable phalanx.*

(3)$_\alpha$ $(\langle \overline{W}, \mathcal{Q}_\alpha\rangle, \langle \kappa_\alpha\rangle)$ *is an iterable phalanx.*

4.5 Lemma. $(3)_\gamma$ *implies* $(2)_\gamma$*.*

Idea. The proof uses the fact that $\mathcal{S}_\gamma = \mathrm{Ult}(\mathcal{Q}_\gamma, \pi, \lambda_\gamma)$ together with countable closure ${}^{\omega}X \subseteq X$. It is not as routine as Lemma 4.4 though. ⊣

The next hypothesis is the key to showing that $\overline{W}$ does not move. It also represents an interesting switch in that $\overline{W}$ appears as the starting model instead of the back-up model.

(4)$_\alpha$ $((\vec{\mathcal{P}}{\restriction}\alpha)^\frown\langle \overline{W}\rangle, \vec{\lambda}{\restriction}\alpha)$ *is an iterable phalanx.*

4.6 Lemma. $(4)_\gamma$ *implies* $(1)_\gamma$.

Idea. Let $(\mathcal{U}, \mathcal{V})$ be the coiteration of the phalanxes

$$((\vec{\mathcal{P}}\restriction\gamma)^\frown\langle\overline{W}\rangle, \vec{\lambda}\restriction\gamma)$$

and

$$((\vec{\mathcal{P}}\restriction\gamma)^\frown\langle\mathcal{P}_\gamma\rangle, \vec{\lambda}\restriction\gamma)$$

The former phalanx is iterable by $(4)_\gamma$. The latter phalanx is iterable by Lemma 4.3. In particular, $\mathcal{V}$ can be construed as an extension of $\mathcal{T}\restriction\eta(\gamma)+1$. Let $\zeta + 1 = \mathrm{lh}(\mathcal{U})$. Standard arguments can be used to see that

$$\gamma = \mathrm{root}^U(\zeta)$$

and

$$D^{\mathcal{U}} \cap (\gamma, \zeta]_U = \emptyset.$$

These arguments use the hull and definability properties at κ_α when $\mathcal{P}_\alpha$ is a thick weasel and soundness at κ_α otherwise. Suppose for contradiction that $(1)_\gamma$ fails. Since $(1)_\alpha$ holds for all $\alpha < \gamma$,

$$\mathrm{lh}(\overline{E}_0) = \lambda_\gamma.$$

Consequently, the first extenders used on $\mathcal{U}$ and $\overline{\mathcal{T}}$ are the same, i.e.,

$$E^{\mathcal{U}}_\gamma = \overline{E}_0.$$

Hence

$$\mathrm{lh}(E^{\mathcal{U}}_\gamma) = \lambda_\gamma < ((\kappa_\gamma)^+)^{\overline{W}}.$$

This can be used to see that if

$$\gamma = \mathrm{pred}^U(\iota + 1)$$

then

$$\kappa_\gamma \leq \mathrm{crit}(E^{\mathcal{U}}_\iota)$$

so

$$\iota + 1 \in D^{\mathcal{U}},$$

which is a contradiction. ⊣

Here is a fact whose proof is like that of Lemma 4.6. Hypothesis $(4)_\alpha$ implies that there is an iteration tree $\mathcal{V}_\alpha$ on W that extends $\mathcal{T}\restriction(\eta(\alpha)+1)$, an initial segment $\mathcal{N}_\alpha$ of the last model of $\mathcal{V}_\alpha$ with

$$\wp(\kappa_\alpha) \cap \overline{W} = \wp(\kappa_\alpha) \cap \mathcal{N}_\alpha$$

and an elementary embedding $k_\alpha : \overline{W} \to \mathcal{N}_\alpha$ with $k_\alpha\restriction\kappa_\alpha = \mathrm{id}\restriction\kappa_\alpha$. This fact and the notation just established comes up again when we prove $(3)_\gamma$.

(5)$_\alpha$ $((\vec{\mathcal{R}}\restriction\alpha)^\frown\langle W\rangle, \vec{\Lambda}\restriction\alpha)$ *is an iterable phalanx of protomice.*

It makes sense to iterate this phalanx of protomice for reasons like those we gave before Lemma 4.4. The difference is that W is the starting model instead of $\mathcal{R}_\gamma$.

4.7 Lemma. $(5)_\gamma$ *implies* $(4)_\gamma$*.*

Idea. Consider the sequence of embeddings

$$\langle \pi_\alpha \mid \alpha < \gamma\rangle^\frown\langle\pi\rangle.$$

Since $\pi_\alpha\restriction\lambda_\alpha = \pi\restriction\lambda_\alpha$ for all $\alpha < \gamma$, this sequence can be used to reduce the iterability of

$$((\vec{\mathcal{P}}\restriction\gamma)^\frown\langle\overline{W}\rangle, \vec{\lambda}\restriction\gamma)$$

to that of

$$((\vec{\mathcal{R}}\restriction\gamma)^\frown\langle W\rangle, \vec{\Lambda}\restriction\gamma).$$

The latter phalanx is iterable by $(5)_\gamma$. ⊣

It is worth noticing that the iteration trees on $((\vec{\mathcal{R}}\restriction\gamma)^\frown\langle W\rangle, \vec{\Lambda}\restriction\gamma)$ that are relevant to the proof of Lemma 4.7 have a special form: whenever $\alpha < \gamma$ and an extender is applied to $\mathcal{R}_\alpha$, the critical point of the extender is exactly $\pi(\kappa_\alpha)$. Similarly, only special iteration trees are relevant to the proof of Lemma 4.4.

(6)$_\alpha$ $((\vec{\mathcal{S}}\restriction\alpha)^\frown\langle W\rangle, \vec{\Lambda}\restriction\alpha)$ *is an iterable phalanx.*

4.8 Lemma. $(6)_\gamma$ *implies* $(5)_\gamma$*.*

Since we have not defined $\vec{S}\restriction\gamma$ it would be meaningless to sketch the proof of Lemma 4.8, which is not easy. It is interesting, though, that the proof involves a variant of the usual copying constructions in which the tree structure changes. And an ultrapower by an extender fragment in an iteration tree on $((\vec{\mathcal{R}}\restriction\gamma)^\frown\langle W\rangle, \vec{\Lambda}\restriction\gamma)$ corresponds to something like padding in the copied iteration tree on $((\vec{\mathcal{S}}\restriction\gamma)^\frown\langle W\rangle, \vec{\Lambda}\restriction\gamma)$.

Having seen that

$$(6)_\gamma \implies (5)_\gamma \implies (4)_\gamma \implies (1)_\gamma$$

and

$$(3)_\gamma \implies (2)_\gamma$$

it remains to prove $(3)_\gamma$ and $(6)_\gamma$, which we do next.

4.9 Lemma. $(3)_\gamma$ *holds.*

Idea. We must see that $(\langle \overline{W}, \mathcal{Q}_\gamma \rangle, \langle \kappa_\gamma \rangle)$ is iterable. Consider the sequence of embeddings

$$\langle k_\alpha \mid \alpha < \gamma \rangle^\frown \langle \mathrm{id} \restriction \mathcal{Q}_\gamma \rangle$$

where $k_\alpha : \overline{W} \to \mathcal{N}_\alpha$ was defined just after the proof of Lemma 4.6. Since $k_\alpha \restriction \kappa_\alpha = \mathrm{id} \restriction \kappa_\alpha$, this sequence of embeddings can be used to reduce the iterability of

$$(\langle \overline{W}, \mathcal{Q}_\gamma \rangle, \langle \kappa_\gamma \rangle)$$

to that of

$$(\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle^\frown \langle \mathcal{Q}_\gamma \rangle, \vec{\kappa} \restriction \gamma).$$

There is a subtlety in the copying construction that also came up at the end of the proof of Theorem 2.35 but once again we omit this detail. The phalanx $(\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle^\frown \langle \mathcal{Q}_\gamma \rangle, \vec{\kappa} \restriction \gamma)$ is what we call W *based* because each of its models appears on an iteration tree on W. For $\alpha < \gamma$, the iteration tree is $\mathcal{V}_\alpha$. And for $\mathcal{Q}_\gamma$ the iteration tree is $\mathcal{T} \restriction (\eta(\gamma) + 1)$ because we are still assuming for simplicity that $\mathcal{Q}_\gamma = \mathcal{P}_\gamma$. (Otherwise a generalized notion of W based is used.) We chose W so that there is an elementary embedding from $\sigma : W \to K^c$. Copy each $\mathcal{V}_\alpha$ to $\sigma \mathcal{V}_\alpha$ and let

$$\sigma^*_\alpha : \mathcal{N}_\alpha \to \mathcal{N}^*_\alpha$$

be the final copy embedding restricted to $\mathcal{N}_\alpha$. Copy $\mathcal{T} \restriction (\eta(\gamma) + 1)$ to $\sigma \mathcal{T} \restriction (\eta(\gamma) + 1)$ and let

$$\sigma^*_\gamma : \mathcal{Q}_\gamma \to \mathcal{Q}^*_\gamma$$

be the final copy embedding restricted to $\mathcal{Q}_\gamma$. The sequence of embeddings

$$\langle \sigma^*_\alpha \mid \alpha \leq \gamma \rangle$$

can be used to reduce the iterability of $(\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle^\frown \langle \mathcal{Q}_\gamma \rangle, \vec{\kappa} \restriction \gamma)$ to that of

$$(\langle \mathcal{N}^*_\alpha \mid \alpha < \gamma \rangle^\frown \langle \mathcal{Q}^*_\gamma \rangle, \langle \sigma(\kappa_\alpha) \mid \alpha < \gamma \rangle).$$

The latter phalanx is K^c based and hence iterable by §9 of [42]. ⊣

The following lemma is the last step in our induction.

4.10 Lemma. $(6)_\gamma$ *holds.*

Idea. Consider an arbitrary $\alpha < \gamma$. Freeing up earlier notation, let $(\mathcal{U}, \mathcal{V})$ be the coiteration of W versus $(\langle W, \mathcal{S}_\alpha \rangle, \langle \pi(\kappa_\alpha) \rangle)$. The latter phalanx is iterable by $(2)_\alpha$. Say $\mathrm{lh}(\mathcal{U}) = \zeta + 1$ and $\mathrm{lh}(\mathcal{V}) = \eta + 1$. Standard arguments as in Section 2 show that

$$\mathcal{M}^{\mathcal{U}}_\zeta \trianglerighteq \mathcal{M}^{\mathcal{V}}_\eta,$$

$1 = \mathrm{root}^V(\eta)$ and $D^{\mathcal{V}} \cap (1, \eta]_V = \emptyset$. Since $\mathcal{S}_\alpha$ and W agree below Λ_α, the extenders used on $\mathcal{U}$ and $\mathcal{V}$ all have length $\geq \Lambda_\alpha$. In the remainder of this proof, we refer to

$$j_\alpha = i^{\mathcal{V}}_{1,\eta}$$

and

$$\mathcal{M}_\alpha = \mathcal{M}^{\mathcal{V}}_\eta.$$

Note that $j_\alpha : \mathcal{S}_\alpha \to \mathcal{M}_\alpha$ is an elementary embedding and

$$j_\alpha \restriction \pi(\kappa_\alpha) = \mathrm{id} \restriction \pi(\kappa_\alpha).$$

To show that $(6)_\gamma$ holds we must see that the phalanx

$$((\vec{\mathcal{S}} \restriction \gamma)^\frown \langle W \rangle, \vec{\Lambda} \restriction \gamma)$$

is iterable. Using the sequence of embeddings

$$\langle j_\alpha \mid \alpha < \gamma \rangle^\frown \langle \mathrm{id} \restriction W \rangle$$

this reduces to seeing that the phalanx

$$(\langle \mathcal{M}_\alpha \mid \alpha < \gamma \rangle^\frown \langle W \rangle, \vec{\Lambda} \restriction \gamma)$$

is iterable. But the latter phalanx is W based. Since W embeds into K^c and all K^c based phalanxes are iterable, all W based phalanxes are iterable. $\dashv$

This concludes the proof by induction that $(1)_\alpha$ through $(6)_\alpha$ hold for all $\alpha < \overline{\Omega}_0$. Now fix $\overline{\alpha}$ so that $\kappa_{\overline{\alpha}} = \overline{\kappa}$. Then $\lambda_{\overline{\alpha}} = \overline{\lambda}$ and $\Lambda_{\overline{\alpha}} = \lambda$. Write $\mathcal{P} = \mathcal{P}_{\overline{\alpha}}$, $\mathcal{R} = \mathcal{R}_{\overline{\alpha}}$, $\mathcal{Q} = \mathcal{Q}_{\overline{\alpha}}$ and $\mathcal{S} = \mathcal{S}_{\overline{\alpha}}$. Also let $\widetilde{\pi} : \mathcal{Q} \to \mathcal{S}$ be the ultrapower embedding.

4.11 Lemma. *It is not the case that $\mathcal{S} = \mathcal{R}$ and $\mathcal{S}$ is not a weasel.*

Idea. Assume otherwise. We build on the facts from the proof of Lemma 4.10 about the coiteration $(\mathcal{U}, \mathcal{V})$ of W versus $(\langle W, \mathcal{S} \rangle, \langle \kappa \rangle)$. We have that

$$\mathcal{S} = \mathrm{Hull}^{\mathcal{S}}_{n+1}(\kappa \cup p^{\mathcal{S}}_{n+1}).$$

Standard arguments show that either

$$\mathcal{S} \lhd \mathcal{M}^{\mathcal{V}}_0 = W$$

or

$$E^{\mathcal{U}}_0 = E^W_\lambda$$

and

$$\mathcal{S} = \mathcal{M}^{\mathcal{V}}_1 = \mathrm{Ult}((\mathcal{M}^*_1)^{\mathcal{V}}, E^{\mathcal{V}}_0).$$

Either way, we get the contradiction

$$\lambda = (\kappa^+)^{\mathcal{S}} < (\kappa^+)^W = (\kappa^+)^K.$$

$\dashv$

4.12 Lemma. *It is not the case that $\mathcal{S} = \mathcal{R}$ and $\mathcal{S}$ is a weasel.*

Idea. Assume otherwise. We build on facts about the coiteration $(\mathcal{U}, \mathcal{V})$ of W versus $(\langle W, \mathcal{S}\rangle, \langle\kappa\rangle)$ from the proof of Lemma 4.10. We have that $\mathcal{S}$ is a thick weasel with the hull and definability properties at μ whenever $\kappa \leq \mu < \Omega_0$. By universality,

$$\mathcal{M}^{\mathcal{U}}_{\zeta} = \mathcal{M}^{\mathcal{V}}_{\eta}.$$

$\mathcal{M}^{\mathcal{V}}_{\eta}$ has the hull property at κ because $\mathrm{crit}(i^{\mathcal{V}}_{1,\eta}) \geq \kappa$. On the other hand, since $\mathcal{M}^{\mathcal{V}}_{\eta}$ also results from the iteration

$$W \xrightarrow[i^{\mathcal{T}}_{0,\eta(\overline{\alpha})}]{} \mathcal{Q} \xrightarrow[\widetilde{\pi}]{} \mathcal{S} \xrightarrow[i^{\mathcal{V}}_{1,\eta}]{} \mathcal{M}^{\mathcal{V}}_{\eta}$$

and $\mathrm{crit}(\widetilde{\pi}) = \mathrm{crit}(\pi) = \delta$, we conclude that $\mathcal{M}^{\mathcal{V}}_{\eta}$ does not have the definability property at δ. Here we are using that $W_{\eta(\overline{\alpha})} = \mathcal{P} = \mathcal{Q}$. This implies that $\mathcal{U}$ is not trivial. Let $E^{\mathcal{U}}_{\iota}$ be the first extender used along $[0, \zeta]_U$. That is,

$$0 = \mathrm{pred}^U(\iota + 1) \leq_U \zeta.$$

Since $\mathcal{M}^{\mathcal{U}}_{\zeta}$ does not have the definability property at δ,

$$\mathrm{crit}(E^{\mathcal{U}}_{\iota}) \leq \delta.$$

Recall that $\mathcal{S}$ and W agree below λ. But λ is a cardinal in both hence not the index of an extender on the sequence of either. Thus,

$$\mathrm{lh}(E^{\mathcal{U}}_{\iota}) > \lambda.$$

This implies that the generators of $E^{\mathcal{U}}_{\iota}$ are unbounded in λ. But then $\mathcal{M}^{\mathcal{U}}_{\zeta}$ does not have the hull property at κ. This is a contradiction. $\dashv$

To wrap things up for this section we give the definitions of $\mathcal{Q}_\beta$ and $\mathcal{S}_\beta$ and discuss how they fit with the outline of the proof of Theorem 4.2 given so far. Recall that if $\mathcal{R}_\beta$ is a premouse, then we already defined $\mathcal{Q}_\beta = \mathcal{P}_\beta$ and $\mathcal{S}_\beta = \mathcal{R}_\beta$. Suppose that $\mathcal{R}_\beta$ is not a premouse. Say $\alpha < \beta$ and $\mathrm{crit}(F^{\mathcal{R}}_{\beta}) = \pi(\kappa_\alpha)$. Recall that $F^{\mathcal{R}}_{\beta}$ is an extender over $\mathcal{R}_\alpha$. Suppose for the moment that $\mathcal{R}_\alpha$ is a premouse. Then what we would do is set

$$\mathcal{S}_\beta = \mathrm{Ult}(\mathcal{R}_\alpha, F^{\mathcal{R}_\beta}) = \mathrm{Ult}(\mathcal{S}_\alpha, F^{\mathcal{R}_\beta})$$

and

$$\mathcal{Q}_\beta = \mathrm{Ult}(\mathcal{P}_\alpha, F^{\mathcal{P}_\beta}) = \mathrm{Ult}(\mathcal{Q}_\alpha, F^{\mathcal{P}_\beta}).$$

It is easy to see that, in this case, $\mathcal{S}_\beta$ is a premouse and

$$\begin{aligned}\mathcal{S}_\beta \text{ is a weasel} &\iff \mathcal{R}_\alpha \text{ is a weasel}\\ &\iff \mathcal{P}_\alpha \text{ is a weasel}\\ &\iff \mathcal{Q}_\beta \text{ is a weasel.}\end{aligned}$$

Of course, $\mathcal{R}_\beta$ is not a weasel. With a little more work, one sees that

$$\mathcal{S}_\beta = \mathrm{Ult}(\mathcal{Q}_\beta, \pi, \pi(\kappa_\alpha)) = \mathrm{Ult}(\mathcal{Q}_\beta, \pi, \Lambda_\alpha).$$

As for $\mathcal{Q}_\alpha$, it is a model on a finite extension of $\mathcal{T}\restriction(\eta(\alpha)+1)$ in this case but not so literally in others.

The general definition of $\mathcal{S}_\beta$ and $\mathcal{Q}_\beta$ is by induction. We set

$$\mathcal{S}_\beta = \mathrm{Ult}(\mathcal{S}_\alpha, F^{\mathcal{R}_\beta})$$

and

$$\mathcal{Q}_\beta = \mathrm{Ult}(\mathcal{Q}_\alpha, F^{\mathcal{P}_\beta})$$

whenever $\alpha < \beta$,

$$\mathrm{crit}(F^{\mathcal{R}}_\beta) = \pi(\kappa_\alpha)$$

and $\mathcal{R}_\beta$ is not a premouse. For example, we could have $\alpha < \beta < \gamma$,

$$\begin{aligned}\mathcal{S}_\alpha &= \mathcal{R}_\alpha\\ \mathcal{S}_\beta &= \mathrm{Ult}(\mathcal{S}_\alpha, F^{\mathcal{R}_\beta})\\ \mathcal{S}_\gamma &= \mathrm{Ult}(\mathcal{S}_\beta, F^{\mathcal{R}_\gamma})\end{aligned}$$

and the analogous equations for $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$ and $\mathcal{Q}_\gamma$. Note that $\mathcal{Q}_\gamma$ is a model on a finite extension of $\mathcal{T}\restriction(\eta(\alpha)+1)$ but not in the conventional sense. What we mean by W based phalanxes and the theorems about them can be generalized accordingly though. This is needed to complete the proof of Lemma 4.9.

Beyond this, we do not attempt to explain how to incorporate this definition of $\mathcal{S}$ and $\mathcal{Q}$ into the proof by induction of $(1)_\alpha$ through $(6)_\alpha$. In particular, the proof of Lemma 4.8 is beyond the scope of this exposition. Instead, we finish this section by showing that it is still possible to obtain a contradiction assuming $(1)_\alpha$ through $(6)_\alpha$ hold for all $\alpha < \overline{\Omega}_0$ without assuming that $\mathcal{S} = \mathcal{R}$. The argument uses two additional concepts: the Dodd decomposition of an extender and fine structure for thick weasels. The simplest case in which $\mathcal{S} \neq \mathcal{R}$ already illustrates the main new ideas. First we look at the non-weasel subcase.

4.13 Lemma. *Let* $\alpha < \beta$ *and* $F = F^{\mathcal{P}_\beta}$. *Suppose that*

$$\rho_{n+1}^{\mathcal{P}_\alpha} \leq \kappa_\alpha < \lambda_\alpha \leq \rho_n^{\mathcal{P}_\alpha}$$

and $\mathcal{Q}_\beta = \mathrm{Ult}(\mathcal{P}_\alpha, F)$. *Then*

$$\mathcal{Q}_\beta = \mathrm{Hull}_{n+1}^{\mathcal{Q}_\beta}(\kappa_\beta \cup p_{n+1}^{\mathcal{Q}_\beta}).$$

Idea. For simplicity, assume $n = 0$. (Otherwise, use the Σ_n mastercode structure for $\mathcal{P}_\alpha$.) Let $i : \mathcal{P}_\alpha \to \mathcal{Q}_\beta$ be the ultrapower map. The lemma is relatively easy to see if $\nu(F) = \kappa_\beta$ because then

$$p_1^{\mathcal{Q}_\beta} = i(p_1^{\mathcal{P}_\alpha}).$$

More generally, we show that

$$p_1^{\mathcal{Q}_\beta} - \kappa_\beta = i(p_1^{\mathcal{P}_\alpha}) \cup (s - \kappa_\beta)$$

for a certain $s \in [\mathrm{lh}(F)]^{<\omega}$ whose identity we are about to reveal.

The *Dodd projectum of* $\mathcal{P}_\beta$, written $\tau^{\mathcal{P}_\beta}$, is the least ordinal τ such that

$$\lambda_\alpha = (\mathrm{crit}(F)^+)^{\mathcal{P}_\beta} \leq \tau \leq \nu(F)$$

and there exists an $s \in [\nu(F)]^{<\omega}$ such that F and $F{\restriction}(\tau \cup s)$ have the same ultrapower. The *Dodd parameter of* $\mathcal{P}_\beta$, written $s^{\mathcal{P}_\beta}$, is the least parameter $s \in [\nu(F)]^{<\omega}$ such that F and $F{\restriction}(\tau^{\mathcal{P}_\beta} \cup s)$ have the same ultrapower.[10] In fact, $\tau = \max(\rho_1^{\mathcal{P}_\beta}, \lambda_\alpha)$. There is a relationship between the $s^{\mathcal{P}_\beta}$ and $p_1^{\mathcal{P}_\beta}$ that is slightly more complicated but not needed here. By a result of Steel in [37], if $\mathcal{P}_\beta$ is 1-sound, then for all $i < |s^{\mathcal{P}_\beta}|$,

$$F{\restriction}(s_i^{\mathcal{P}_\beta} \cup (s^{\mathcal{P}_\beta}{\restriction}i)) \in \mathcal{P}_\beta$$

and for all $\xi < \tau^{\mathcal{P}_\beta}$,

$$F{\restriction}(\xi \cup s^{\mathcal{P}_\beta}) \in \mathcal{P}_\beta.$$

These properties are known as *Dodd solidity* and *Dodd amenability* respectively. Counterexamples for mice that are not 1-sound can be found in [28].

If $\mathcal{P}_\beta \lhd W_{\eta(\beta)}$, then certainly $\mathcal{P}_\beta$ is 1-sound and therefore Dodd solid and Dodd amenable. The fact that F and $F{\restriction}(\kappa_\beta \cup s^{\mathcal{P}_\beta})$ have the same ultrapower translates into

$$\mathcal{Q}_\beta = \mathrm{Hull}_1^{\mathcal{Q}_\beta}(\kappa_\beta \cup i(p_1^{\mathcal{P}_\alpha}) \cup s^{\mathcal{P}_\beta}).$$

The fact that

$$F{\restriction}(s_i^{\mathcal{P}_\beta} \cup (s^{\mathcal{P}_\beta}{\restriction}i)) \in \mathcal{P}_\beta$$

[10] Recall that parameters, i.e., finite sets of ordinals, are often identified with descending sequences of ordinals, and that the ordering on parameters is lexicographic.

for all $i < |s^{\mathcal{P}_\beta} - \kappa_\beta|$ translates into

$$p_1^{\mathcal{Q}_\beta} - \kappa_\beta = i(p_1^{\mathcal{P}_\alpha}) \cup (s^{\mathcal{P}_\beta} - \kappa_\beta).$$

Suppose instead that $\mathcal{P}_\beta = W_{\eta(\beta)}$. Then $\mathcal{P}_\beta$ is not 1-sound. Let $\iota + 1$ be the last drop in model or degree along $[0, \eta(\beta)]_T$ and let

$$W^*_{\iota+1} \trianglelefteq W_{\mathrm{pred}^T(\iota+1)}$$

be the level to which we drop. Also let

$$i^*_{\iota+1} : W^*_{\iota+1} \to W_{\iota+1} = \mathrm{Ult}(W^*_{\iota+1}, F^{\mathcal{T}}_\iota)$$

be the ultrapower embedding. Since $W^*_{\iota+1}$ is 1-sound, it is Dodd solid and Dodd amenable by [37]. Now by induction on ζ such that

$$\iota + 1 \leq_T \zeta \leq_T \eta(\beta)$$

it is possible to show that if

$$s = i^{\mathcal{T}}_{\iota+1,\zeta}(i^*_{\iota+1}(s^{W^*_{\iota+1}})),$$

then F^{W_ζ} and $F^{W_\zeta} \restriction (\nu^{\mathcal{T}}_\zeta \cup s)$ have the same ultrapower,

$$F^{W_\zeta} \restriction (s_i \cup (s \restriction i)) \in W_\zeta$$

for all $i < |s - \nu^{\mathcal{T}}_\zeta|$ and

$$s - \nu^{\mathcal{T}}_\zeta = s^{W_\zeta} - \nu^{\mathcal{T}}_\zeta.$$

By definition, $\kappa_\beta \geq \nu^{\mathcal{T}}_{\eta(\beta)}$. So at the end of this induction we see that if $s = s^{\mathcal{P}_\beta}$, then F and $F \restriction (\kappa_\beta \cup s)$ have the same ultrapower and

$$F \restriction (s_i \cup (s \restriction i)) \in \mathcal{P}_\beta$$

for all $i < |s - \kappa_\beta|$. As before, these facts translate into the desired result. $\dashv$

The facts about Dodd solidity in the proof of Lemma 4.13 can be used to avoid a convoluted argument in [20].[11]

4.14 Lemma. *It is not the case that $\mathcal{S} \neq \mathcal{R}$ and $\mathcal{S}$ is not a weasel.*

Idea. In the simplest case, which is the only one we discuss here, $\overline{\kappa} = \kappa_\beta$ satisfies the hypothesis of Lemma 4.13. Then

$$\mathcal{S} = \mathrm{Hull}^{\mathcal{S}}_{n+1}(\kappa \cup p^{\mathcal{S}}_{n+1}).$$

Now repeat the proof of Lemma 4.11 to obtain a contradiction. $\dashv$

[11] Avoid Lemma 2.1.2 and Corollaries 2.1.3 and 2.1.6 of [20].

4.15 Lemma. *It is not the case that $\mathcal{S} \neq \mathcal{R}$ and $\mathcal{S}$ is a weasel.*

Idea. There is an analog of Lemma 4.13 that is valid when $\mathcal{Q}_\beta$ is a weasel. With this analog, the proof of Lemma 4.12 can be adapted to give the proof of Lemma 4.15. The basic idea behind this analog is as follows.

Once again, we look only at the simplest instance of $\mathcal{Q}_\beta \neq \mathcal{P}_\beta$. That is, $\alpha < \beta$ and $F = F^{\mathcal{P}_\beta}$ and $\mathcal{Q}_\beta = \mathrm{Ult}(\mathcal{P}_\alpha, F)$. But this time suppose that $\mathcal{P}_\alpha = W_{\eta(\alpha)}$ is a thick weasel with the hull and definability properties at μ whenever $\kappa_\alpha \leq \mu < \Omega_0$. Then $\mathcal{Q}_\beta$ is also a thick weasel. If $\nu(F) = \kappa_\beta$, then we can show that $\mathcal{Q}_\beta$ has the hull and definability properties at μ whenever $\kappa_\beta \leq \mu < \Omega_0$, which is just what is needed to run the proof of Lemma 4.12 when $\kappa_\beta = \overline{\kappa}$. More generally, consider again the fact that F and its Dodd decomposition $F\restriction(\tau^{\mathcal{P}_\beta} \cup s^{\mathcal{P}_\beta})$ have the same ultrapower and $\tau^{\mathcal{P}_\beta} \leq \kappa_\beta$. There is a natural sense in which $\mathcal{Q}_\beta$ has the $s^{\mathcal{P}_\beta}$ definability property at μ whenever $\kappa_\beta \leq \mu < \Omega_0$. This fact motivates defining $\kappa^{\mathcal{Q}_\beta}$ to be the least ordinal μ_0 such that there exists a $c \in [\Omega_0]^{<\omega}$ such that $\mathcal{Q}_\beta$ has the c definability property at μ whenever $\mu_0 \leq \mu < \Omega_0$. This is the *class projectum*. We have that

$$\kappa^{\mathcal{Q}_\beta} \leq \kappa_\beta$$

as witnessed by $s^{\mathcal{P}_\beta}$. We also define $c^{\mathcal{Q}_\beta}$ to be the least parameter $c \in [\Omega_0]^{<\omega}$ such that $\mathcal{Q}_\beta$ has the c definability property at μ whenever $\kappa^{\mathcal{Q}_\beta} \leq \mu < \Omega_0$. This is the *class parameter*. The proof of Lemma 4.13 shows that F is Dodd solid above κ_β. This fact translates into

$$c^{\mathcal{Q}_\beta} - \kappa_\beta = s^{\mathcal{P}_\beta}.$$

The two displayed facts above are our version of Lemma 4.13 when $\mathcal{Q}_\beta$ is a weasel. They translate into

$$\kappa^{\mathcal{S}} \leq \kappa$$

and

$$c^{\mathcal{S}} - \kappa = s^{\mathcal{R}}$$

when $\kappa_\beta = \overline{\kappa}$. With some additional work we can adapt the proof of Lemma 4.12 to finish the proof of Lemma 4.15. ⊣

This concludes our outline of the proof of Theorem 4.2. ⊣

5. Applications of Core Models

In this section, we list some results whose proofs use core model theory at a level that involves iteration trees. These are stated in a way that minimizes core model prerequisites. We have also tried to avoid overly technical hypotheses. For example, in some theorems, the hypothesis that Ω is a measurable cardinal can be reduced to the existence of sharps for elements of H_Ω or even less.

5.1. Determinacy

Some of the results in Section 5 are stated in terms of determinacy instead of large cardinals. Often it is easier to phrase things one way or the other but there are reasons to think that there is more to it than that. We begin this subsection by recalling some of the known equiconsistencies between large cardinals and determinacy.

5.1 Theorem. *The following are equiconsistent.*

(1) *There exists a Woodin cardinal.*

(2) Δ^1_2 *determinacy.*

5.2 Theorem. *The following are equiconsistent.*

(1) *There exist infinitely many Woodin cardinals.*

(2) $L(\mathbb{R})$ *determinacy.*

Theorems 5.1 and 5.2 are due to Woodin. A proof that (2) is consistent relative to (1) in Theorem 5.2 is given in the Handbook chapter [22]. The consistency of (1) relative to (2) in the two theorems is given in the Handbook chapter [14]. It would be reasonable for the reader to suspect that these parts of the proofs use core model theory. However, Woodin obtained these results in the 1980's before Steel developed the theory of K at the level of one Woodin cardinal. Woodin used HOD instead of K. In the proof of Theorem 5.1, Woodin showed that if Δ^1_2 determinacy holds, then there exists a real x such that $\omega_2^{L[y]}$ is a Woodin cardinal in $\mathrm{HOD}^{L[y]}$ whenever $x \in L[y]$. And his proof of Theorem 5.2 built on that of Theorem 5.1. More recently, Steel discovered alternate proofs that use core models.

Theorems 5.1 and 5.2 are equiconsistencies between determinacy and the existence of large cardinals. This is a good place to recall some of the known equivalences between determinacy and the existence of mice. For this, we must recall the definition of $M_n^\#(x)$, which can also be found in §7 of [41].

The theory of mice generalizes to a theory of mice built over a real. If $n \leq \omega$ and $x \subseteq \omega$, then there is at most one structure

$$\mathcal{M} = \langle J_\beta^{E,x}, \in, E, F \rangle$$

such that $\mathcal{M}$ is a $\omega_1 + 1$ iterable sound premouse built over x,

$$J^{E,x}_{\mathrm{crit}(F)} \models \text{the number of Woodin cardinals} = n$$

and for all $\alpha < \beta$, if $E_\alpha \neq \emptyset$, then

$$J^{E,x}_{\mathrm{crit}(E_\alpha)} \models \text{the number of Woodin cardinals} < n.$$

If it exists, then this unique mouse built over x is called $M_n^\#(x)$. For $n = 0$, we have that $M_0^\#(x)$ is Turing equivalent to $x^\#$.

Let us point out some features of $M_n^\#$. Recall that the empty extender codes the identity embedding. The next weakest possibility is that the critical point of an extender is the only generator of the extender, in which case the extender codes the embedding from a normal measure. It follows from the definition that $F^{M_n^\#(x)}$ is a measure in this sense and that if δ is the supremum of the Woodin cardinals of $M_n^\#(x)$, then

$$\delta < \mathrm{crit}(F^{M_n^\#(x)})$$

and

$$E_\alpha^{M_n^\#(x)} = \emptyset$$

whenever $\delta \leq \alpha < \mathrm{crit}(F^{M_n^\#(x)})$. Regarding the projectum and standard parameter, it is easy to see that

$$\rho_1^{M_n^\#(x)} = 1$$

and

$$p_1^{M_n^\#(x)} = \emptyset.$$

In particular, $M_n^\#(x)$ is countable. We have enough iterability to guarantee that all (not just the first ω_1 many) iterates of $M_n^\#(x)$ by images of its top extender are wellfounded. By iterating away the top extender of $M_n^\#(x)$ in this way we obtain a proper class model that goes by the name $M_n(x)$. For $n = 0$ we have that $M_0(x) = L[x]$. Observe that $M_n(x)$ has the same Woodin cardinals as $M_n^\#(x)$ and that $M_n(x)$ is $\omega_1 + 1$ iterable. Moreover, the critical points of extenders used on this linear iteration form a club class of $M_n(x)$ indiscernibles. In the case $n = 0$, these are the $L[x]$ indiscernibles.

Let us call a structure that satisfies the first-order properties in the definition of $M_n^\#(x)$ but is λ iterable instead of $\omega_1 + 1$ iterable a λ *iterable* $M_n^\#(x)$.

5.3 Theorem. *Let $n < \omega$ and assume $\mathbf{\Pi^1_n}$ determinacy. Then the following are equivalent.*

(1) $\mathbf{\Pi^1_{n+1}}$ *determinacy.*

(2) *For every $x \in \mathbb{R}$, there is an ω_1 iterable $M_n^\#(x)$.*

(3) *For every $x \in \mathbb{R}$, there is a unique ω_1 iterable $M_n^\#(x)$.*

The case $n = 0$ boils down to the fact that

$$\mathbf{\Pi^1_1} \text{ determinacy} \iff \forall x \in \mathbb{R}\ (x^\# \text{ exists})$$

where the forward implication is due to Martin and the reverse is due to Harrington. The proof that (1) implies (3) is due to Woodin and uses core models. Parts of the proof can be found in the Handbook chapter [14] and Theorem 7.7 of [42]. The proof that (2) implies (1) is due to Woodin for odd n and Neeman for even n. See the Handbook chapter [22].

5.4 Corollary. *The following are equivalent.*

(1) PD.

(2) *For all $n < \omega$ and $x \subseteq \omega$, there is an ω_1 iterable $M_n^\#(x)$.*

(3) *For all $n < \omega$ and $x \subseteq \omega$, there is a unique ω_1 iterable $M_n^\#(x)$.*

(4) *For all $n < \omega$ and $x \subseteq \omega$, there exists a Σ^1_n correct model M with n Woodin cardinals and $x \in M$.*

This equivalence combines results of Martin, Steel and Woodin.

Woodin proved that if $M_\omega^\#(x)$ exists for all $x \subseteq \omega$, then $L(\mathbb{R})$ determinacy holds. See the Handbook chapter [22] for a proof due to Neeman. Steel and Woodin obtained the following optimal result.

5.5 Theorem. *The following are equivalent.*

(1) *$L(\mathbb{R})$ determinacy.*

(2) *For all $x \subseteq \omega$ and every Σ_1 formula φ, if $\varphi[x, \mathbb{R}]$ holds in $L(\mathbb{R})$, then there is a countable, ω_1 iterable model M satisfying* ZF$^-$ *plus there are ω Woodin cardinals such that $x \in M$, and $\varphi[x, \mathbb{R}^*]$ holds in the derived model of M.*

Next we state several theorems which show that some well-known consequences of determinacy are equivalent to determinacy.

5.6 Theorem. *Assume that for all $x \subseteq \omega$, $x^\#$ exists and the $\Sigma^1_3(x)$ separation property holds for subsets of ω. Then $\boldsymbol{\Delta^1_2}$ determinacy holds.*

Steel proved Theorem 5.6 by combining the Σ^1_3 correctness of K, Theorem 3.7, with ideas due to Kechris. See Corollary 7.14 of [42].

Recall that if $A, B \subseteq {}^\omega\omega$, then $A \leq_W B$ iff there is a continuous function $f : {}^\omega\omega \to {}^\omega\omega$ such that $A = f^{-1}[B]$. This is Wadge reducibility, which can also be expressed in terms of games and winning strategies. By Γ Wadge determinacy we mean that for all $A, B \in \Gamma$, either $A \leq_W B$ or $B \leq_W {}^\omega\omega - A$. Under mild assumptions, Γ determinacy implies Γ Wadge determinacy. In the other direction, Harrington showed that $\Pi^1_1(x)$ Wadge determinacy implies $x^\#$ exists, hence $\Pi^1_1(x)$ determinacy by the result due to Martin mentioned earlier. One level up, Hjorth proved the following.

5.7 Theorem. $\boldsymbol{\Pi^1_2}$ *Wadge determinacy implies* $\boldsymbol{\Pi^1_2}$ *determinacy.*

Theorem 5.7 is Theorem 3.15 of [12]. The proof uses the Σ^1_3 correctness of K, Theorem 3.7.

Projective determinacy has the following well-known consequences: every projective subset of $\mathbb{R}$ is Lebesgue measurable and has the property of Baire, and every projective binary relation on $\mathbb{R}$ has a projective uniformization. Woodin once conjectured that the conjunction of these three consequences of PD implies PD, and he proved several theorems that provided evidence in favor of his conjecture. Eventually, Steel disproved Woodin's conjecture by showing that these three consequences of PD hold in $V^{\mathrm{Col}(\omega,\kappa)}$ if V is the minimal extender model with a cardinal λ such that the set of $\kappa < \lambda$ that are $<\lambda$ strong is unbounded in λ. This large cardinal axiom is weaker than the existence of a Woodin cardinal, hence weaker than the consistency strength of PD by Theorem 5.1. The reader is referred to Hauser and Schindler [11] where the history is reviewed more completely than here and Steel's theorem is reversed. While these consequences of determinacy do not match up with determinacy at the projective level, it turns out that they do match up at other levels. For example, Woodin proved the following theorem using his core model induction technique.

5.8 Theorem. *The following statements are equivalent.*

(1) *$L(\mathbb{R})$ determinacy.*

(2) *For every $A \in L(\mathbb{R})$ such that $A \subseteq \mathbb{R} \times \mathbb{R}$ and A is Δ^2_1 definable in $L(\mathbb{R})$ from real parameters,*

  (a) *A is Lebesgue measurable,*

  (b) *A has the property of Baire and*

  (c) *A can be uniformized by a function $f \in L(\mathbb{R})$. (By reflection, f can be chosen to be Δ^2_1 definable in $L(\mathbb{R})$ from real parameters.)*

(3) *Same as* (2) *except instead of* (c) *we have*

  (c′) *A can be uniformized by a function f such that every $B \subseteq \mathbb{R}$, if B is projective in f, then B is are Lebesgue measurable and has the property of Baire. (Note that f is not required to be in $L(\mathbb{R})$.)*

5.2. Tree Representations and Absoluteness

Shoenfield showed that all transitive proper class models of ZFC are Σ^1_2 correct. The proof involves a canonical recursive tree T that projects to a complete $\mathbf{\Sigma}^1_1$ subset of ${}^\omega\omega$ and a tree T^* on $\omega \times \mathrm{On}$ such that

$$\mathrm{proj}([T^*]) = {}^\omega\omega - \mathrm{proj}([T^*])$$

holds in every uncountable transitive model of ZFC. Forcing and Shoenfield absoluteness can be used to reprove the classical theorem that $\boldsymbol{\Sigma^1_1}$ sets are Lebesgue measurable; the argument is due to Solovay.

Suppose that κ be a measurable cardinal. Martin showed that all $\boldsymbol{\Pi^1_1}$ sets are κ-homogeneous and all κ-homogeneous sets are determined.[12] The projection of a κ-homogeneous set is called κ weakly homogeneous and there is a corresponding notion of a κ-weakly homogeneous tree. Martin and Solovay showed that if T is a κ-weakly homogeneous tree, then there is a tree T^* on $\omega \times \mathrm{On}$ such that

$$\mathrm{proj}([T^*]) = {}^\omega\omega - \mathrm{proj}([T^*])$$

in $V^{\mathbb{P}}$ whenever $\mathbb{P} \in V_\kappa$. We say that T and T^* are $<\kappa$ absolutely complemented and that their projections are $<\kappa$ absolutely Suslin. (This property of the projections is also called $<\kappa$ universally Baire.) Martin and Solovay used this to show that $V^{\mathbb{P}}$ is Σ^1_3 correct in $V^{\mathbb{P}*\mathbb{Q}}$ for all $\mathbb{P} * \mathbb{Q} \in V_\kappa$. Forcing and Martin-Solovay absoluteness can be used to see that $\boldsymbol{\Sigma^1_2}$ sets are Lebesgue measurable.

The main theorem of Martin and Steel [16] is that if δ is a Woodin cardinal and $A \subseteq {}^\omega\omega \times {}^\omega\omega$ is δ^+ homogeneous, then ${}^\omega\omega - \mathrm{proj}(A)$ is $<\delta$ homogeneous. This has many important corollaries. For example, suppose that $\delta < \kappa$ where δ is a Woodin cardinal and κ is a measurable cardinal. Then $\boldsymbol{\Pi^1_2}$ sets are $<\delta$ homogeneous and $\boldsymbol{\Sigma^1_3}$ sets are $<\delta$ weakly homogeneous. (The latter was proved by Woodin before Martin and Steel obtained their result.) By Martin, $\boldsymbol{\Pi^1_2}$ sets are determined. By Martin-Solovay, $\boldsymbol{\Sigma^1_3}$ sets are $<\delta$ absolutely Suslin and $V^{\mathbb{P}}$ is Σ^1_4 correct in $V^{\mathbb{P}*\mathbb{Q}}$ whenever $\mathbb{P}*\mathbb{Q} \in V_\delta$. Forcing and Σ^1_4 absoluteness can be used to see that $\boldsymbol{\Sigma^1_3}$ sets are Lebesgue measurable.

Martin and Steel also combined their main theorem with an earlier theorem of Woodin to see that if there are $\delta < \kappa$ such that δ is a limit of Woodin cardinals and κ is a measurable cardinal, then all sets of reals in $L(\mathbb{R})$ are $<\delta$ homogeneous and hence determined. (Note that we are no longer assuming that δ is a Woodin cardinal.) From this it follows that all sets of reals in $L(\mathbb{R})$ are $< \delta$ weakly homogeneous, that the theory of $L(\mathbb{R})$ cannot be changed by forcing and that all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable. (These last three consequences were proved before Martin and Steel obtained their result; see Woodin-Shelah [40] and Woodin [50].)

Now we turn to lower bounds on the large cardinal consistency strength of the properties discussed above.

5.9 Theorem. *Let Ω be a measurable cardinal. Suppose that for all posets $\mathbb{P} \in H(\Omega)$,*

$$(L_{\omega_1}(\mathbb{R}))^{V^{\mathbb{P}}} \equiv (L_{\omega_1}(\mathbb{R}))^V.$$

[12] For these and other notions discussed below, we refer to the Handbook chapter [22].

Then

$$K^c \models \textit{there is a Woodin cardinal.}$$

Theorem 5.9 is due to Woodin and appears as Theorem 7.4 of [42]. The proof uses Theorem 3.5, that $K \cap HC$ is Σ_1 definable over $L_{\omega_1}(\mathbb{R})$. It also uses almost everywhere weak covering, Theorem 3.1, which allows us to use forcing to change the truth value of the statement that ω_1 of the universe is a successor cardinal of K.

5.10 Theorem. *Let Ω be a measurable cardinal. Then the following are equivalent.*

(1) *For all posets $\mathbb{P} \in V_\Omega$,*

$$(L(\mathbb{R}))^{V^{\mathbb{P}}} \equiv (L(\mathbb{R}))^{V}.$$

(2) *For all posets $\mathbb{P} \in V_\Omega$,*

$$(L(\mathbb{R}) \textit{ determinacy})^{V^{\mathbb{P}}}.$$

(3) *For all posets $\mathbb{P} \in V_\Omega$,*

$$(L(\mathbb{R}) \textit{ Lebesgue measurability})^{V^{\mathbb{P}}}.$$

(4) *For all posets $\mathbb{P} \in V_\Omega$,*

$$(\textit{there is no } \omega_1 \textit{ sequence of distinct reals in } L(\mathbb{R}))^{V^{\mathbb{P}}}.$$

(5) *There exists an $\Omega + 1$ iterable model of height Ω with infinitely many Woodin cardinals.*

Theorem 5.10 is due independently to Steel and Woodin and appears as Theorem 3.1 in [43]. The proof that the failure of (5) implies the failure of (4) uses core model theory. Instead of K^c and K, an "excellent" premouse $\mathcal{P}$ is found so that the maximal countably complete construction above $\mathcal{P}$ yields a relativized weasel $K^c(\mathcal{P})$ such that all the Woodin cardinals of $K^c(\mathcal{P})$ are in $\mathcal{P}$ and $K^c(\mathcal{P})$ is $\Omega + 1$ iterable above $\mathcal{P}$. The relativized core model $K(\mathcal{P})$ is extracted from $K^c(\mathcal{P})$ as in Section 2. One of the main tools is a version of the recursive definition of K, Theorem 3.5, that shows $K(\mathcal{P}) \cap HC \in L(\mathbb{R})$ in this more general context.

5.3. Ideals and Generic Embeddings

Let κ be an uncountable cardinal and let I be a κ-complete ideal on $\mathcal{P}(\kappa)$. Assume that I is κ^+-saturated. In other words, $\mathcal{P}(\kappa)/I$ has the κ^+-chain condition. Suppose that G is V generic over $\mathcal{P}(\kappa)/I$. Let

$$j : V \to M = \mathrm{Ult}(V, G)$$

be the ultrapower map computed in $V[G]$. Then M is transitive, $\mathrm{crit}(j) = \kappa$ and

$${}^{<j(\kappa)}M \subseteq M.$$

Such a j is called a *generic almost huge embedding.* The story of saturated ideals from the forcing side is far too rich to tell here but we do mention a couple of results. Shelah showed that if δ is a Woodin cardinal, then there is a semiproper poset $\mathbb{P}$ with the δ chain condition such that the nonstationary ideal over ω_1 is $\aleph_2$ saturated in $V^{\mathbb{P}}$. The following result in this subsection comes close to showing that one Woodin cardinal is the exact consistency strength.

5.11 Theorem. *Assume that Ω is a measurable cardinal and $\kappa < \Omega$. Let $\mathbb{P} \in V_\Omega$ be a poset. Suppose that forcing with $\mathbb{P}$ produces a generic almost huge embedding. Then there is a model of height Ω that satisfies "there is a Woodin cardinal".*

Theorem 5.11 is due to Steel and appears as Theorem 7.1 of [42]. The proof uses core model theory. In particular, it uses forcing absoluteness, Theorem 3.4 and the recursive definition of K, Theorem 3.5.

If I is a countably complete non-trivial ideal on $\mathcal{P}(\omega_1)$, then I is $\aleph_1$-dense iff $\mathcal{P}(\omega_1)/I$ has a dense subset of cardinality $\aleph_1$. This implies that forcing with $\mathcal{P}(\omega_1)/I$ is equivalent to forcing with $\mathrm{Col}(\omega, \omega_1)$, which in turn implies that I is $\aleph_2$ saturated. It also implies that $\mathcal{P}(\omega_1)/I$ is weakly homogeneous in the sense of forcing; we just say that I is homogeneous in this case. Woodin showed that the existence of a $\aleph_1$ dense ideal is consistent relative to $L(\mathbb{R})$ determinacy in [49]. Using core models, Steel proved that if there is a homogeneous ideal on ω_1 and CH holds, then PD holds. Building on this, Woodin showed his hypothesis was optimal.

5.12 Theorem. *The following are equiconsistent over* ZFC*.*

(1) *There is an $\aleph_1$ dense ideal over ω_1.*

(2) *$L(\mathbb{R})$ determinacy.*

The passage from (1) to (2) uses core models. In particular, it uses K and a method due to Woodin known as the *core model induction.* Woodin proves that if $A \subseteq \mathbb{R}$ and $A \in L(\mathbb{R})$, then A is determined. One could say that his proof is by induction on the least $(\alpha, n) \in \mathrm{On} \times \omega$ such that $A \in \Sigma_{n+1}(J_\alpha(\mathbb{R}))$. Steel uses a version of the core model induction in [44].

5.4. Square and Aronszajn Trees

This section is actually on the failure of square and the non-existence of Aronszajn trees, i.e., the tree property.

If λ is an ordinal and $C = \langle C_\alpha \mid \alpha < \lambda \rangle$, then C is a *coherent sequence* iff for all limit $\beta < \lambda$,

- C_β is club in β and
- if $\alpha \in \lim(C_\beta)$, then $C_\alpha = \alpha \cap C_\beta$.

If C is a coherent sequence, then D is a *thread* of C iff D is club in λ and $C_\alpha = \alpha \cap D$ for all $\alpha \in \lim(D)$. The principle $\square(\lambda)$ says that there is a coherent sequence of length λ with no thread. The principle $\square_\kappa$ says that there is a coherent sequence C of length κ^+ such that $\mathrm{type}(C_\alpha) \leq \kappa$ for all limit $\alpha < \lambda$. Coherent sequences are the topic of the Handbook chapter [46]. In this and the next section, it is convenient to set $\mathfrak{c} = 2^{\aleph_0}$.

5.13 Theorem. *Let $\kappa \geq \max(\aleph_2, \mathfrak{c})$. Suppose that both $\square_\kappa$ and $\square(\kappa)$ fail. Then $L(\mathbb{R})$ determinacy holds.*

See [27], which explains credit for Theorem 5.13 and related results, and has a proper introduction. Two basic elements of the proof are generalizations of Theorems 3.2 and 3.13. Theorem 3.6 is also used. The author derived PD from the hypothesis of Theorem 5.13. In fact, he showed that $M_n(X)$ exists for all $n < \omega$ and bounded $X \subseteq \kappa^+$. Steel observed that the author's proof meshed with techniques from [44] to give the result as stated.

Todorcevic proved that if if $\square(\kappa)$ holds then there is an Aronszajn tree on κ. See [46]. From this and Theorem 5.13, one may conclude, for example, that if $\mathfrak{c} \leq \aleph_2$ and the tree property holds at $\aleph_2$ and $\aleph_3$, then $L(\mathbb{R})$ determinacy holds. Related theorems about the tree property were proved earlier without going through square; see [6] and its bibliography.

5.14 Theorem. *Suppose that κ is a singular strong limit cardinal and $\square_\kappa$ fails. Then $L(\mathbb{R})$ determinacy holds.*

Theorem 5.14 is Theorem 0.1 of [44], which includes an explanation of credit and related results. The proof uses Theorem 3.12, a generalization of Theorem 3.2 and a version of Woodin's core model induction due to Steel.

5.15 Theorem. *Suppose that κ is a weakly compact cardinal and $\square_\kappa$ fails. Then $L(\mathbb{R})$ determinacy holds.*

Theorem 5.15 is Corollary 8 of [32], which includes an explanation of credit. Two basic elements of the proof are generalizations of Theorems 3.3 and 3.12.

5.16 Theorem. *Suppose that κ is a measurable cardinal and $\square_\kappa$ fails. Then there is a model of height κ that satisfies "there is a proper class of strong cardinals" and "there is a proper class of Woodin cardinals".*

See [2], which includes an explanation of credit. The proof uses a generalization of Theorem 3.12 for K^c and nothing about K. The hypothesis of Theorem 5.16 holds if κ is strongly compact by a well-known theorem of Solovay. Woodin has shown that the conclusion of Theorem 5.16 implies the consistency of $\mathrm{ZF} + \mathrm{AD}_\mathbb{R}$ where $\mathrm{AD}_\mathbb{R}$ asserts that all real games of length ω are determined.

The following is a very recent theorem due to Jensen, Schimmerling, Schindler and Steel [13].

5.17 Theorem. *Let $\kappa \geq \max(\aleph_3, \mathfrak{c})$. Suppose that both $\square_\kappa$ and $\square(\kappa)$ fail. Then there is a proper class model that satisfies "there is a proper class of strong cardinals" and "there is a proper class of Woodin cardinals".*

5.5. Forcing Axioms

If $\mathcal{C}$ is a class of posets, then $\mathrm{FA}(\mathcal{C})$ says that for all $\mathbb{P} \in \mathcal{C}$ and $\mathcal{D}$ with $|\mathcal{D}| = \aleph_1$, there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$. By definition,

$$\mathrm{PFA} \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ is proper}\}).$$

This is the Proper Forcing Axiom. For any cardinal λ, we set

$$\mathrm{PFA}(\lambda) = \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ is proper and } |\mathbb{P}| = \lambda\}).$$

Todorcevic and Velickovic showed that $\mathrm{PFA}(\mathfrak{c})$ implies that $\mathfrak{c} = \aleph_2$. See Theorem 1.8 of [48] and Theorem 3.16 of [3]. Todorcevic [47] showed that if λ is an ordinal such that $\mathrm{cf}(\lambda) \geq \aleph_2$, then $\mathrm{PFA}(\lambda^{\aleph_0})$ implies the failure of $\square(\lambda)$. Therefore $\mathrm{PFA}(\mathfrak{c}^+)$ implies the the hypothesis of Theorem 5.13.

5.18 Corollary. $\mathrm{PFA}(\mathfrak{c}^+)$ *implies $L(\mathbb{R})$ determinacy.*

Note too that $\mathrm{PFA}(\mathfrak{c}^{++})$ implies the hypothesis of Theorem 5.17.

5.19 Corollary. $\mathrm{PFA}(\mathfrak{c}^{++})$ *implies that there is a proper class model that satisfies "there is a proper class of strong cardinals" and "there is a proper class of Woodin cardinals".*

Baumgartner and Shelah showed that PFA is consistent relative to the existence of a supercompact cardinal. The levels of the PFA hierarchy described above do not require a supercompact cardinal. For example, Neeman and Schimmerling [25] showed that the consistency strength of $\mathrm{PFA}(\mathfrak{c}^+)$ is strictly less than the existence of a cardinal κ that is κ^+-supercompact. More about this shortly.

By definition,

$$\mathrm{SPFA} \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ is semi-proper}\})$$

and

$$\mathrm{MM} \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ preserves stationary subsets of } \omega_1\}).$$

These are the Semi-proper Forcing Axiom and Martin's Maximum respectively. It is straightforward to see that MM implies SPFA, which in turn implies PFA. Foreman, Magidor and Shelah showed that MM is consistent relative to a supercompact cardinal; see Theorem 5 of [7]. Their proof used Shelah's revised countable support iteration. (Donder and Fuchs [5] is a good source for this.) Later, Shelah [39] proved that SPFA and MM are equivalent.

Recall that a poset $\mathbb{P} = (P, <_P)$ is λ-*linked* iff there is a function $\ell : P \to \lambda$ such that for all $p, q \in P$, if $\ell(p) = \ell(q)$, then p and q are compatible in $\mathbb{P}$. Here are two obvious comments. If $|P| = \lambda$, then $\mathbb{P}$ is λ-linked. If $\mathbb{P}$ is λ-linked, then $\mathbb{P}$ has the λ^+-chain condition. For any cardinal λ, we define

$$\mathrm{SPFA}(\lambda\text{-linked}) \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ is semi-proper and } \lambda\text{-linked}\})$$

and

$$\mathrm{MM}(\lambda) \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ preserves stationary subsets of } \omega_1 \text{ and } |P| = \lambda\}).$$

Shelah [39] showed that SPFA implies MM. In [25], this theorem is refined to SPFA($\mathfrak{c}^+$-linked) implies MM($\mathfrak{c}$). This is useful because Neeman and Schimmerling also show in [25] that SPFA($\mathfrak{c}^+$-linked) is consistent relative to the existence of a cardinal λ that is (λ, Σ^2_1)-subcompact. Without reproducing the definition, we remark that a witness that λ is (λ, Σ^2_1)-subcompact is a certain family of elementary embeddings of the form

$$\pi : H(\kappa^+) \to H(\lambda^+)$$

with $\mathrm{crit}(\pi) = \kappa$ and $\pi(\kappa) = \lambda$. Our point here is that each embedding of this sort comes from a superstrong extender. Consequently, (λ, Σ^2_1)-subcompactness is strictly weaker than κ^+-supercompactness in the large cardinal hierarchy. The consistency proof in [25] of SPFA($\mathfrak{c}^+$-linked) uses a revised countable support iteration of semi-proper posets of length λ as did Shelah's consistency proof of SPFA. Not surprisingly, if countable supports and proper posets are used instead, then one obtains a model of PFA($\mathfrak{c}^+$-linked) starting from the same large cardinal in the ground model. The theory of extender models can accommodate (λ, Σ^2_1)-subcompactness but core model techniques are not sufficiently developed to measure the consistency strength of PFA($\mathfrak{c}^+$-linked). However, there is evidence towards an equiconsistency: Neeman [23] showed that in order to force PFA($\mathfrak{c}^+$-linked) by proper forcing over an extender model, if λ is $\aleph_2$ of the generic extension, then λ is (λ, Σ^2_1)-subcompact in the ground model.

Taking a fundamentally different approach, Woodin [51] showed that $\mathrm{MM}(\mathfrak{c})$ is consistent relative to the theory

$$\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}} + \Theta \text{ is regular}$$

where

$$\Theta = \sup(\{\alpha \in \mathrm{On} \mid \text{ there is a surjection } f : \mathbb{R} \to \alpha\}).$$

The proof uses Woodin's $\mathbb{P}_{\max}$ theory; see the Handbook chapter [15] for an introduction to this technique.

Todorcevic showed that $\mathrm{MM}(\mathfrak{c})$ implies the Stationary Reflection Principle $\mathrm{SRP}(\omega_2)$, which says that for every stationary $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$, if for all stationary $T \subseteq \omega_1$, $\{X \in S \mid X \cap \omega_1 \in T\}$ is stationary in $\mathcal{P}_{\omega_1}(\omega_2)$, then there exists an α such that $\omega_1 < \alpha < \omega_2$ and $S \cap \mathcal{P}_{\omega_1}(\alpha)$ contains a club in $\mathcal{P}_{\omega_1}(\alpha)$. This version comes from Definition 9.74(3) and Lemma 9.75(1) of Woodin [51]. The reason we bring this up here is the following core model result of Steel and Zoble [45].

5.20 Theorem. $\mathrm{SRP}(\omega_2)$ *implies* $L(\mathbb{R})$ *determinacy.*

The proof builds on that of Corollary 9.86 of Woodin [51], which says that $\mathrm{SRP}(\omega_2)$ implies PD.

Another well-known variant of MM is Bounded Martin's Maximum or BMM, which says that if $\mathbb{P}$ preserves stationary subsets of ω_1, then

$$(H(\omega_2))^V \prec_{\Sigma_1} (H(\omega_2))^{V^{\mathbb{P}}}.$$

Woodin has shown that BMM is consistent relative to the existence of $\omega+1$ many Woodin cardinals; see Theorem 10.99 of [51]. The following lower bound by Schindler [35] uses core models.

5.21 Theorem. BMM *implies that for every set* X *there is a model with a strong cardinal containing* X.

5.6. The Failure of UBH

The theory of iteration trees was initiated by Martin and Steel in the context of inner models in [17] and determinacy in [16]. Three Hypotheses are isolated in §5 of the former paper: UBH (Unique Branches), CBH (Cofinal Branches) and SBH (Strategic Branches). These hypotheses have to do with iteration trees on V but their motivation is the construction of inner models with large cardinals. Results, both positive and negative, about the three hypotheses and their variants give useful information towards a solution to the inner model problem.

Woodin showed that UBH and CBH are false assuming sufficient large cardinals. This lead to the question of consistency strength and the following core model result of Steel [43].

5.22 Theorem. *Suppose that there is a non-overlapping iteration tree $\mathcal{T}$ on V with cofinal wellfounded branches $b \neq c$. Then there is an inner model with infinitely many Woodin cardinals. If, in addition,*

$$\delta(\mathcal{T}) \in \operatorname{ran}(i^{\mathcal{T}}_{0,b}) \cap \operatorname{ran}(i^{\mathcal{T}}_{0,c}),$$

then there is an inner model with a strong cardinal that is a limit of Woodin cardinals.

Woodin eventually reduced the large cardinal assumption in his refutations of UBH and CBH to a supercompact cardinal. Motivated by this, Neeman and Steel [26] constructed counterexamples starting from much less in the way of large cardinals. For example, under a large cardinal assumption slightly stronger than the one mentioned in Theorem 5.22, they constructed an iteration tree on V with distinct cofinal wellfounded branches. See [26] for a discussion on additional results on UBH and CBH and their failure.

5.7. Cardinality and Cofinality

Shelah famously showed that if $\aleph_\omega$ is a strong limit cardinal, then $(\aleph_\omega)^{\aleph_0} < \aleph_{\omega_4}$. See the Handbook chapter [1]. An important conjecture is that the actual bound is $\aleph_{\omega_1}$. The following theorem appears as Theorem 1.1 of [10]. It provides valuable information about what it would take to obtain a counterexample to the conjecture.

5.23 Theorem. *Let α be a limit ordinal. Suppose that $2^{|\alpha|} < \aleph_\alpha$ and $2^{|\alpha|^+} < \aleph_{|\alpha|^+}$ but $(\aleph_\alpha)^{|\alpha|} > \aleph_{|\alpha|^+}$. Then $M_n(X)$ exists for all $n < \omega$ and bounded $X \subseteq \aleph_{|\alpha|^+}$.*

The following theorem appears as Theorem 1.4 of [10]. Recently, Gitik showed that its hypothesis is consistent relative to the existence of a supercompact cardinal. See [8].

5.24 Theorem. *Let λ be a cardinal such that $\omega < \mathrm{cf}(\lambda) < \lambda$. Suppose that $\{\kappa < \lambda \mid 2^\kappa = \kappa^+\}$ is stationary and co-stationary in λ. Then $M_n^\#(X)$ exists for all $X \subseteq \lambda$.*

In [9], Gitik showed that if there is a proper class of strongly compact cardinals, then there is a model of ZF in which all uncountable cardinals are singular. Towards measuring the consistency strength of this statement, Daniel Busche showed the following, which will appear in his Ph.D. thesis.

5.25 Theorem. *Suppose that all uncountable cardinals are singular. Then* AD *holds in $L(\mathbb{R})^{\mathrm{HOD}^{\mathbb{P}}}$ for some $\mathbb{P} \in \mathrm{HOD}$.*

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Index