Countable Borel Equivalence Relations: The Appendix

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17th November 2007

Popa's Cocycle Superrigidity Theorem

In this lecture, we shall sketch the proof of:

Theorem (Popa)

Let Γ be a countably infinite Kazhdan group and let G be a countable group such that $\Gamma \trianglelefteq G$. If H is any countable group, then every Borel cocycle

$$\alpha : G \times 2^G \to H$$

is equivalent to a group homomorphism of G into H.

Remark

- *Popa's original proof was written in the framework of Operator Algebras.*
- *This presentation is based upon Furman's Ergodic-theoretic account.*

The plan of the talk

Remark

- *More accurately, we shall prove Popa's Theorem for the shift action of G on* $([0,1]^G, \nu)$*, where* ν *is the usual product probability measure.*
- *It is then fairly straightforward to deduce the corresponding result for the quotient G-space* $(2^G, \mu)$*.*

The four steps of the proof

- *Extending homomorphisms*
- *Popa's criterion for untwisting cocycles*
- *Malleability of the action*
- *Local rigidity of cocycles*

Extending homomorphisms

Theorem

- *Suppose that the action of the countable group G on the standard probability space* (X, μ) *is strongly mixing.*
- *Let H be any countable group and let* $\alpha : G \times X \to H$ *be a Borel cocycle.*
- *Suppose that there exists an infinite normal subgroup* $\Gamma \trianglelefteq G$ *such that* $\alpha \restriction \Gamma \times X = \varphi$ *is a group homomorphism.*

Then α *is a group homomorphism.*

Remark

Thus we can focus our attention on the strongly mixing action of the Kazhdan group Γ *on* $([0,1]^G, \nu)$*.*

Exploiting the identity $\alpha(gh, x) = \alpha(g, h \cdot x)\alpha(h, x)$

- Fix some $g \in G$ and define $\theta : X \to H$ by $\theta(x) = \alpha(g, x)$.
- We must show that θ is μ-a.e. constant.
- Let $\gamma \in \Gamma$ and let $\gamma' = g\gamma g^{-1} \in \Gamma$.
- Then for μ-a.e. $x \in X$:

$$\begin{aligned}
\theta(\gamma \cdot x) &= \alpha(g, \gamma \cdot x) \\
&= \alpha(g\gamma, x)\,\alpha(\gamma, x)^{-1} \\
&= \alpha(\gamma' g, x)\,\varphi(\gamma)^{-1} \\
&= \alpha(\gamma', g \cdot x)\,\alpha(g, x)\,\varphi(\gamma)^{-1} \\
&= \varphi(g\gamma g^{-1})\,\theta(x)\,\varphi(\gamma)^{-1}.
\end{aligned}$$

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&= \varphi(g\gamma g^{-1})\, \theta(x)\, \varphi(\gamma)^{-1}.
\end{aligned}$$

Regarding θ as a Borel homomorphism

- Consider the action of Γ on H defined by

$$\gamma * h = \varphi(g\gamma g^{-1})\, h\, \varphi(\gamma)^{-1}.$$

- Then for all $\gamma \in \Gamma$,

$$\gamma * \theta(x) = \theta(\gamma \cdot x) \quad \mu\text{-a.e. } x \in X.$$

- Choose some $a \in H$ such that $Y = \{x \in X \mid \theta(x) = a\}$ has positive μ-measure; and let

$$\Gamma_a = \{\gamma \in \Gamma \mid \gamma * a = a\}.$$

The punchline

Claim

Γ_a is an infinite subgroup of Γ.

- Since the action of G on (X, μ) is strongly mixing, it follows that Γ_a acts ergodically on (X, μ).
- Clearly $Y = \{x \in X \mid \theta(x) = a\}$ is Γ_a-invariant.
- Since $\mu(Y) > 0$, it follows that $\mu(Y) = 1$ and hence θ is μ-a.e. constant, as desired.

Proof of Claim

- If $\gamma \in \Gamma \smallsetminus \Gamma_a$ and $x \in Y$, then

$$\theta(\gamma \cdot x) = \gamma * \theta(x) = \gamma * a \neq a$$

  and hence $\gamma \cdot x \notin Y$. Thus $\gamma(Y) \cap Y = \emptyset$.

- Hence if $\{\gamma_i \mid i \in I\}$ are coset representatives of Γ_a in Γ, then:
  - $\{\gamma_i(Y) \mid i \in I\}$ are pairwise disjoint.
  - $\mu(\gamma_i(Y)) = \mu(Y) > 0$ for all $i \in I$.

- Thus $[\Gamma : \Gamma_a] < \infty$ and so Γ_a is infinite, as desired.

More generally

Arguing inductively, the theorem holds if there exists a smooth chain of subgroups

$$\Gamma = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_\beta \leqslant \cdots \leqslant G_\alpha = G,$$

such that $G_\beta \trianglelefteq G_{\beta+1}$ for all $\beta < \alpha$.

- Or even: $G_{\beta+1}$ is generated by elements g such that

$$|gG_\beta g^{-1} \cap G_\beta| = \infty.$$

Example

The theorem holds when $\Gamma = SL_n(\mathbb{Z})$ and $G = GL_n(\mathbb{Q})$.

Popa's criterion for untwisting cocycles

Theorem

- *Suppose that the action of the countable group* Γ *on the standard probability space* (X, μ) *is strongly mixing.*
- *Let* H *be any countable group and let* $\alpha : \Gamma \times X \to H$ *be a Borel cocycle.*
- *Consider the diagonal action of* Γ *on* $(X \times X, \mu \times \mu)$ *and let* $\alpha_1, \alpha_2 : \Gamma \times (X \times X) \to H$ *be the cocycles defined by*

$$\alpha_1(\gamma, (x_1, x_2)) = \alpha(\gamma, x_1) \quad \text{and} \quad \alpha_2(\gamma, (x_1, x_2)) = \alpha(\gamma, x_2).$$

Then α *is equivalent to a group homomorphism if and only if* α_1 *is equivalent to* α_2*.*

The trivial direction

- Suppose that $\alpha : \Gamma \times X \to H$ is equivalent to the group homomorphism $\varphi : \Gamma \to H$.
- Then there exists a Borel map $b : X \to H$ such that

$$\varphi(\gamma) = b(\gamma \cdot x)\alpha(\gamma, x)b(x)^{-1} \quad \mu\text{-a.e. } x$$

- Hence for $(\mu \times \mu)$-a.e. (x_1, x_2)

$$b(\gamma \cdot x_1)\alpha(\gamma, x_1)b(x_1)^{-1} = b(\gamma \cdot x_2)\alpha(\gamma, x_2)b(x_2)^{-1}$$

and so the Borel map $B : X \times X \to H$ defined by

$$B(x_1, x_2) = b(x_1)^{-1} b(x_2)$$

witnesses that α_1 is equivalent to α_2.

The nontrivial direction

- Suppose that there exists a Borel map $B : X \times X \to H$ such that for $(\mu \times \mu)$-a.e. (x_1, x_2)

$$B(\gamma \cdot x_1, \gamma \cdot x_2)\alpha(\gamma, x_2)B(x_1, x_2)^{-1} = \alpha(\gamma, x_1).$$

- Then with some work, it turns out that there are Borel maps ψ, $\theta : X \to H$ such that for $(\mu \times \mu)$-a.e. (x_1, x_2)

$$B(x_1, x_2) = \psi(x_1)\theta(x_2)$$

  and so

$$\theta(\gamma \cdot x_2)\alpha(\gamma, x_2)\theta(x_2)^{-1} = \psi(\gamma \cdot x_1)^{-1}\alpha(\gamma, x_1)\psi(x_1).$$

- Fixing $\gamma \in \Gamma$, it follows that the left and right sides of the above equation are μ-a.e. constant, say $\varphi(\gamma)$. And this implies that α is equivalent to the group homomorphism $\varphi : \Gamma \to H$.

Some work ...

- For $(\mu \times \mu \times \mu)$-a.e. (x_1, x_2, x_3),

$$\alpha(\gamma, x_1) = B(\gamma \cdot x_1, \gamma \cdot x_2)\alpha(\gamma, x_2)B(x_1, x_2)^{-1}$$

$$\alpha(\gamma, x_3) = B(\gamma \cdot x_3, \gamma \cdot x_2)\alpha(\gamma, x_2)B(x_3, x_2)^{-1}.$$

- Hence substituting the first identity into the second,

$$\alpha(\gamma, x_3) = B(\gamma \cdot x_3, \gamma \cdot x_2)B(\gamma \cdot x_1, \gamma \cdot x_2)^{-1}\alpha(\gamma, x_1)B(x_1, x_2)B(x_3, x_2)^{-1}.$$

Some work ...

- Setting $\Phi(x_1, x_2, x_3) = B(x_1, x_2)B(x_3, x_2)^{-1}$, we obtain
$$\Phi(\gamma \cdot x_1, \gamma \cdot x_2, \gamma \cdot x_3) = \alpha(\gamma, x_1)\Phi(x_1, x_2, x_3)\alpha(\gamma, x_3)^{-1}.$$

- We will prove that for $(\mu \times \mu \times \mu)$-a.e. (x_1, x_2, x_3),
$$\Phi(x_1, x_2, x_3) = f(x_1, x_3).$$

- Substituting a random element $a \in X$ for x_3, we obtain that for $(\mu \times \mu)$-a.e. (x_1, x_2),
$$B(x_1, x_2)B(a, x_2)^{-1} = f(x_1, a)$$
and hence
$$B(x_1, x_2) = f(x_1, a)B(a, x_2) = \psi(x_1)\theta(x_2).$$

Some work ...

- By strong mixing, Γ acts ergodically on

$$Z = \{\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle \mid x_1 = y_1, x_3 = y_3\} \cong X^4.$$

- Letting δ be the Kronecker delta function, for a.e. $\langle \bar{x}, \bar{y} \rangle \in Z$,

$$\begin{aligned} &\delta(\Phi(\gamma \cdot \bar{x}), \Phi(\gamma \cdot \bar{y})) \\ &= \delta(\alpha(\gamma, x_1)\Phi(\bar{x})\alpha(\gamma, x_3)^{-1}, \alpha(\gamma, x_1)\Phi(\bar{y})\alpha(\gamma, x_3)^{-1}) \\ &= \delta(\Phi(\bar{x}), \Phi(\bar{y})). \end{aligned}$$

- By ergodicity and the countability of H, we must have that

$$\Phi(\bar{x}) = \Phi(\bar{y}) \quad \text{for a.e. } \langle \bar{x}, \bar{y} \rangle \in Z.$$

A malleable action

- Let Γ be a countably infinite group and let G be any countable group such that $\Gamma \leqslant G$.
- Consider the diagonal action of Γ on $([0,1]^G \times [0,1]^G, \nu \times \nu)$.
- Let H be any countable group and let Z^1 be the space of all Borel cocycles

$$\alpha : \Gamma \times ([0,1]^G \times [0,1]^G) \to H$$

identified modulo $(\nu \times \nu)$-null sets, equipped with the topology of convergence in measure.

A malleable action

- In other words, for each $F \in [\Gamma]^{<\omega}$, $\alpha \in Z^1$ and $\epsilon > 0$, there is a corresponding basic open subset $V_{F,\alpha,\epsilon}$ consisting of the cocycles $\beta \in Z^1$ such that for all $\gamma \in F$,

$$(\nu \times \nu)(\{\bar{x} \in [0,1]^G \times [0,1]^G \mid \alpha(\gamma, \bar{x}) = \beta(\gamma, \bar{x})\}) > 1 - \epsilon.$$

Theorem

If $\beta \in Z^1$ is any cocycle, then there exists a continuous map $t \mapsto \beta_t$, $t \in [0,1]$ such that

$$\beta_0(x_1, x_2) = \beta(x_1, x_2) \quad \textit{and} \quad \beta_1(x_1, x_2) = \beta(x_2, x_1).$$

A continuous transformation

- For each $t \in [0,1]$, define $T_t \in \mathrm{Aut}([0,1] \times [0,1], m \times m)$ by
$$T_t(x,y) = \begin{cases} (x,y) & \text{if } x, y \in [0, 1-t] \\ (y,x) & \text{otherwise} \end{cases}$$
and let $\Delta_t \in \mathrm{Aut}([0,1]^G \times [0,1]^G, \mu \times \mu)$ be the corresponding "diagonal automorphism".
- Then for all (x_1, x_2), we have that $\Delta_0(x_1, x_2) = (x_1, x_2)$ and $\Delta_1(x_1, x_2) = (x_2, x_1)$.
- Furthermore, each Δ_t commutes with the diagonal action of Γ on $[0,1]^G \times [0,1]^G$.
- Hence if $\beta : \Gamma \times ([0,1]^G \times [0,1]^G) \to H$ is any Borel cocycle, then
$$\beta_t(g, (x_1, x_2)) = \beta(g, \Delta_t(x_1, x_2))$$
is also a cocycle.

Local Rigidity of Cocyles

Theorem

- *Let Γ be a countably infinite Kazhdan group with finite generating set S and let X be a standard Borel Γ-space with invariant ergodic probability measure μ.*
- *Let H be any countable group.*

Then there exists $\epsilon > 0$ such that if $\alpha, \beta : \Gamma \times X \to H$ are Borel cocycles with

$$\mu(\{x \in X \mid \alpha(\gamma, x) = \beta(\gamma, x)\}) > 1 - \epsilon \quad \text{for all } \gamma \in S,$$

then α and β are equivalent.

Remark

Since the shift action of Γ on $([0,1]^G, \nu)$ is strongly mixing, it follows that the diagonal action of Γ on $([0,1]^G \times [0,1]^G, \nu \times \nu)$ is ergodic.

An associated unitary representation

- Let $\epsilon > 0$ be sufficiently small.
- Define a Borel action of Γ on the infinite measure space

$$(\tilde{X}, \tilde{\mu}) = (X \times H, \mu \times m_H),$$

where m_H is the counting measure, by

$$\gamma \cdot (x, h) = (\gamma \cdot x,\ \alpha(\gamma, x)\, h\, \beta(\gamma, x)^{-1}\ ).$$

- Consider the induced unitary action π on the Hilbert space $\mathcal{H} = L^2(\tilde{X}, \tilde{\mu})$, defined by

$$(\pi(\gamma) \cdot F)(x, h) = F(\gamma^{-1} \cdot (x, h)).$$

An invariant vector

- Let $F_0 \in \mathcal{H}$ be the characteristic function of the set $X \times \{1_H\} \subseteq X \times H$.
- Then F_0 is a unit vector such that for all $\gamma \in S$,

$$\langle \pi(\gamma) \cdot F_0, F_0 \rangle = \mu(\{x \in X \mid \alpha(\gamma, x) = \beta(\gamma, x)\}) > 1 - \epsilon.$$

- Since Γ is a Kazhdan group and ϵ is sufficiently small, there exists a Γ-invariant unit vector $F : X \times H \to \mathbb{C}$ such that $||F_0 - F|| < 1/10$.
- By Fubini, the function $F_x : H \to [0, \infty)$, defined by $F_x(h) = |F(x, h)|^2$, is summable for μ-a.e. $x \in X$.
- Furthermore, if $\gamma \in \Gamma$, then for μ-a.e. $x \in X$,

$$\{F_{\gamma \cdot x}(h) \mid h \in H\} = \{F_x(h) \mid h \in H\}.$$

The punchline

- By the ergodicity of the action of Γ on (X, μ), there exist constants w, p, k such that for μ-a.e. $x \in X$, $w = \sum_{h \in H} F_x(h) = 1$, $p = \max\{F_x(h) \mid h \in H\}$ and $k = |\{h \in H \mid F_x(h) = p\}|$.
- Clearly $p \leq 1/k$ and so $F_x(1_H) = |F(x, 1_H)|^2 \leq 1/k$.
- Thus $1 - 1/\sqrt{k} \leq ||F - F_0|| < 1/10$ and so $k = 1$.
- Hence we can define a Borel function $\varphi : X \to H$ such that $F_x(\varphi(x)) = p$ for μ-a.e. $x \in X$.
- Since F is Γ-invariant and

$$\gamma \cdot (x, h) = (\gamma \cdot x,\ \alpha(\gamma, x)\, h\, \beta(\gamma, x)^{-1}\ ),$$

we must have that

$$\varphi(\gamma \cdot x) = \alpha(\gamma, x)\, \varphi(x)\, \beta(\gamma, x)^{-1} \quad \mu\text{-a.e. } x \in X.$$