Countable Borel Equivalence Relations IV

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An application of Ioana Superrigidity

The Kechris Conjecture

$\equiv_T$ is universal.

Observation

There exists a universal countable Borel equivalence relation E on $\mathcal{P}(\mathbb{N})$ such that $E \subseteq \equiv_T$.

Proof.

Identifying the free group $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, we have that $E_\infty \subseteq \equiv_T$. □

An application of Ioana Superrigidity

Conjecture (Hjorth)

If F is a universal countable Borel equivalence relation on the standard Borel space X and E is a countable Borel equivalence relation such that $F \subseteq E$, then E is also universal.

Theorem (Thomas 2002)

There exists a pair of countable Borel equivalence relations $F \subseteq E$ on a standard Borel space X such that $E <_B F$.

The ring of p-adic integers

- The ring $\mathbb{Z}_p$ of p-adic integers is the inverse limit of the system

$$\cdots \xrightarrow{\varphi_{n+1}} \mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{\varphi_n} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_1} \mathbb{Z}/p\mathbb{Z}.$$

- So we can express each $z \in \mathbb{Z}_p$ as a formal sum

$$z = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots$$

where each $0 \leq a_n < p$.

- The p-adic norm is given by

$$|z|_p = p^{-\operatorname{ord}_p(z)}, \qquad \operatorname{ord}_p(z) = \min\{n \mid a_n \neq 0\};$$

and the p-adic metric is given by

$$d_p(x, y) = |x - y|_p.$$

The ring of p-adic integers

Theorem

- *$\mathbb{Z}_p$ is a compact complete separable metric space.*
- *$\mathbb{Z}$ is a dense subring of $\mathbb{Z}_p$.*

Corollary

- *$SL_n(\mathbb{Z}_p)$ is a compact Polish group.*
- *$SL_n(\mathbb{Z})$ is a dense subgroup of $SL_n(\mathbb{Z}_p)$.*

Note that $SL_n(\mathbb{Z}_p)$ is the inverse limit of the system

$$\cdots \xrightarrow{\theta_{n+1}} SL_n(\mathbb{Z}/p^{n+1}\mathbb{Z}) \xrightarrow{\theta_n} SL_n(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\theta_{n-1}} \cdots \xrightarrow{\theta_1} SL_n(\mathbb{Z}/p\mathbb{Z}).$$

The p-adic probability measure

Theorem

Since $SL_n(\mathbb{Z}_p)$ is a compact group, there exists a unique Haar probability measure on $SL_n(\mathbb{Z}_p)$; i.e. a probability measure μ_p which is invariant under the left translation action.

In fact, μ_p is simply the inverse limit of the counting measures on

$$\cdots \xrightarrow{\theta_{n+1}} SL_n(\mathbb{Z}/p^{n+1}\mathbb{Z}) \xrightarrow{\theta_n} SL_n(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\theta_{n-1}} \cdots \xrightarrow{\theta_1} SL_n(\mathbb{Z}/p\mathbb{Z}).$$

Observation

If $H \leqslant SL_n(\mathbb{Z}_p)$ is an open subgroup, then H has finite index and

$$\mu_p(H) = \frac{1}{[SL_n(\mathbb{Z}_p) : H]}$$

The p-adic probability measure

Theorem

μ_p is the unique $SL_n(\mathbb{Z})$-invariant probability measure on $SL_n(\mathbb{Z}_p)$.

Proof.

- $SL_n(\mathbb{Z}_p)$ acts continuously on the space $\mathcal{M}$ of probability measures on $SL_n(\mathbb{Z}_p)$.
- Hence if ν is any probability measure on $SL_n(\mathbb{Z}_p)$, then

$$S_\nu = \{g \in SL_n(\mathbb{Z}_p) \mid \nu \text{ is } g\text{-invariant } \}$$

  is a closed subgroup of $SL_n(\mathbb{Z}_p)$.
- Thus, by density, any $SL_n(\mathbb{Z})$-invariant probability measure is actually $SL_n(\mathbb{Z}_p)$-invariant and hence must be μ_p.

$\square$

Unique ergodicity

Definition

The action of G on the standard probability space (X, μ) is uniquely ergodic iff μ is the unique G-invariant probability measure on X.

Observation

If the action of G on (X, μ) is uniquely ergodic, then G acts ergodically.

Proof.

- Suppose that there exists a G-invariant Borel subset $A \subseteq X$ with $0 < \mu(A) < 1$. Let $B = X \smallsetminus A$.
- Then we can define distinct G-invariant probability measures by

$$\nu_1(Z) = \mu(Z \cap A)/\mu(A)$$
$$\nu_2(Z) = \mu(Z \cap B)/\mu(B).$$

□

Ergodic Components

Observation

The action of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$ is not strongly mixing.

- Let $\Lambda = \ker \varphi$ and $H = \ker \psi$ be the kernels of the maps

$$\varphi : SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/p\mathbb{Z})$$

and

$$\psi : SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_p/p\mathbb{Z}_p) \cong SL_n(\mathbb{Z}/p\mathbb{Z}).$$

- Then H is the closure of Λ and the ergodic decomposition of the Λ-action is given by

$$SL_n(\mathbb{Z}_p) = Hg_1 \sqcup \cdots \sqcup Hg_d, \qquad d = |SL_n(\mathbb{Z}/p\mathbb{Z})|.$$

- The Hg_i are the ergodic components of the Λ-action.

Ergodic Components

Definition

Let $F \subseteq E$ be the orbit equivalence relations of the actions of Λ and $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$.

Theorem (Thomas 2002)

If $n \geq 3$, then $E <_B F$.

By considering the ergodic decomposition of the Λ-action

$$SL_n(\mathbb{Z}_p) = Hg_1 \sqcup \cdots \sqcup Hg_d, \qquad d = |SL_n(\mathbb{Z}/p\mathbb{Z})|,$$

we see that

$$F = E_1 \oplus \cdots \oplus E_d, \qquad \text{where } E_i = F \restriction Hg_i.$$

$E_i \sim_B E$ for $1 \leq i \leq d$.

- First we claim that the inclusion map $Hg_i \to SL_n(\mathbb{Z}_p)$ is a Borel reduction from E_i to E.
- So suppose that $x, y \in Hg_i$ and that $x \, E \, y$.
- Then there exists $\gamma \in SL_n(\mathbb{Z})$ such that $\gamma x = y$.
- It follows that

$$\emptyset \neq \gamma H g_i \cap H g_i = H \gamma g_i \cap H g_i.$$

- Hence $\gamma \in SL_n(\mathbb{Z}) \cap H = \Lambda$ and so $x \, E_i \, y$.

$E_i \sim_B E$ for $1 \leq i \leq d$.

- In order to show that $E \leq_B E_i$, we choose our coset representatives g_k so that each $g_k \in SL_n(\mathbb{Z})$.
- For each $1 \leq k \leq d$, define $h_k : Hg_k \to Hg_i$ by $h_k(x) = g_i g_k^{-1} x$.
- We claim that $h = h_1 \cup \cdots \cup h_d$ is a Borel reduction from E to E_i.
- If $x, y \in SL_n(\mathbb{Z}_p)$, then

$$\begin{aligned} x\, E\, y \quad & \text{iff} \quad h(x)\, E\, h(y) \\ & \text{iff} \quad h(x)\, E_i\, h(y). \end{aligned}$$

- The last equivalence follows because $h(x), h(y) \in Hg_i$.

The Heart of the Matter

Lemma

If $F \subseteq E$ are the orbit equivalence relations of the actions of Λ and $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$, then

$$F \sim_B \underbrace{E \oplus \cdots \oplus E}_{d \text{ times}}.$$

Hence it is enough to prove ...

Theorem

If $n \geq 3$, then

$$E <_B E \oplus E <_B \cdots <_B \underbrace{E \oplus \cdots \oplus E}_{d \text{ times}} <_B \cdots$$

The Heart of the Matter

Some Notation

- $(K, \mu) = (SL_n(\mathbb{Z}_p), \mu_p)$.
- $\Gamma = SL_n(\mathbb{Z})$.
- E is the orbit equivalence relation of Γ on K.

Then it's enough to prove ...

Theorem

If $f : K \to K$ is a Borel reduction from E to E, then

$$\mu(\Gamma \cdot f[K]) = 1.$$

The proof begins

- Suppose that $f : K \to K$ is a Borel reduction from E to E.
- Then we can define a Borel cocycle $\alpha : \Gamma \times K \to \Gamma$ by

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

- By Ioana Superrigidity, there exists
  - a subgroup $\Delta \leqslant \Gamma$ of finite index
  - an ergodic component $X \subseteq K$ for the Δ-action

  such that $\alpha \restriction (\Delta \times X)$ is equivalent to a group homomorphism

$$\psi : \Delta \to SL_n(\mathbb{Z}).$$

- After slightly adjusting f, we can suppose that $\alpha \restriction (\Delta \times X) = \psi$.

An application of Margulis Superrigidity

- To simplify the presentation, suppose that $n \geq 3$ is odd.

Theorem

Suppose that $\Delta \leqslant SL_n(\mathbb{Z})$ is a subgroup of finite index and that

$$\psi : \Delta \to SL_n(\mathbb{Z})$$

is a group homomorphism. Then either:

- $\psi[\Delta]$ *is finite; or*
- ψ *is an embedding and* $\psi[\Delta]$ *is a subgroup of finite index in* $SL_n(\mathbb{Z})$.

Case 1

- Suppose that $\psi[\Delta]$ is finite.
- Recall that $\psi(g) \cdot f(x) = f(g \cdot x)$ for all $g \in \Delta$ and $x \in X$.
- Thus we can define a Δ-invariant map $\Phi : X \to [K]^{<\omega}$ by

$$\Phi(x) = \{\, f(g \cdot x) \mid g \in \Delta \,\}.$$

- Since Δ acts ergodically on X, it follows that Φ is constant on a μ-conull subset of X, which is a contradiction.

Case 2

- Suppose that ψ is an embedding and that $\psi[\Delta]$ is a subgroup of finite index in $SL_n(\mathbb{Z})$.
- Let $Y_1, \cdots, Y_d$ be the ergodic components for the action of $\psi[\Delta]$ on K.
- Since Δ acts ergodically on X, we can suppose that there exists $Y = Y_i$ such that $f : X \to Y$.
- Since $\psi(g) \cdot f(x) = f(g \cdot x)$, we can define a $\psi[\Delta]$-invariant probability measure ν on Y by

$$\nu(\,Z\,) = \mu(\,f^{-1}(Z)\,)/\mu(X).$$

- Since the action of $\psi[\Delta]$ on Y is uniquely ergodic,

$$\nu(Z) = \mu(Z)/\mu(Y).$$

- Hence $\mu(\,f[X]\,) = \mu(Y) > 0$ and so $\mu(\,\Gamma \cdot f[K]\,) = 1$.

Ioana's Cocycle Superrigidity Theorem

Theorem (Ioana)

- *Let* Γ *be a countably infinite Kazhdan group and let* (X, μ) *be a free ergodic profinite* Γ*-space.*
- *Suppose that* H *is any countable group and that* $\alpha : \Gamma \times X \to H$ *is a Borel cocycle.*
- *Then there exists a subgroup* $\Delta \leqslant \Gamma$ *of finite index and an ergodic component* $Y \subseteq X$ *for the* Δ*-action such that* $\alpha \restriction (\Delta \times Y)$ *is equivalent to a homomorphism* $\psi : \Delta \to H$*.*

Profinite Actions

Definition

- *Let Γ be a countable group.*
- *For each $n \in \mathbb{N}$, let $(\, X_n, \mu_n \,)$ be a finite Γ-space, where $\mu_n(Y) = |Y|/|X_n|$.*
- *Suppose that each $(\, X_n, \mu_n \,)$ is a quotient of $(\, X_{n+1}, \mu_{n+1} \,)$; say,*

$$\cdots \xrightarrow{q_{n+1}} (\, X_{n+1}, \mu_{n+1} \,) \xrightarrow{q_n} (\, X_n, \mu_n \,) \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_0} (\, X_0, \mu_0 \,).$$

- *Then the canonical action of Γ on*

$$(\, X, \mu \,) = \varprojlim (\, X_n, \mu_n \,)$$

is said to be a profinite action.

Examples of Profinite Actions

Definition

- *A countably infinite group* Γ *is* *residually finite* *iff there exists a chain of finite index normal subgroups*

$$\Gamma_0 > \Gamma_1 > \cdots > \Gamma_n > \cdots$$

  such that $\bigcap \Gamma_n = 1$.
- *Then* Γ *is a dense subgroup of the* *profinite group* $\varprojlim \Gamma / \Gamma_n$.

Example

Let K be a profinite group and let Γ be a countable dense subgroup. If L is a closed subgroup of K, then the action of Γ on K/L is profinite.

Example

The action of $SL_n(\mathbb{Z})$ on the projective space $PG(n-1, \mathbb{Q}_p)$ is profinite.

A final application of Ioana Superrigidity

Definition

Fix some $n \geq 3$. Let S be a nonempty set of primes and regard $SL_n(\mathbb{Z})$ as a subgroup of

$$G(S) = \prod_{p \in S} SL_n(\mathbb{Z}_p)$$

via the diagonal embedding. Let E_S be the corresponding orbit equivalence relation.

Theorem (Thomas 2002)

If $S \neq T$, then E_S and E_T are incomparable with respect to Borel bireducibility.

Sketch of Proof

- For simplicity, suppose that $S = \{ p \}$ and $T = \{ q \}$, where $p \neq q$ are distinct primes.
- Suppose that $f : SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_q)$ is a Borel reduction from $E_{\{p\}}$ to $E_{\{q\}}$.
- Then arguing as above, after passing to subgroups of finite index and ergodic components if necessary, we find that

$$(SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p) \cong (SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_q), \mu_q)$$

as measure-preserving permutation groups.

Basic Problem

How can we recognize the prime p in $(SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p)$?

The Automorphism Group

Definition

- Aut($SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p$) *consists of the measure-preserving bijections* $\varphi : SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_p)$ *such that for all* $\gamma \in SL_n(\mathbb{Z})$,

$$\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x) \quad \text{for } \mu_p\text{-a.e. } x.$$

- *As usual, we identify two such maps if they agree* μ_p*-a.e.*

Example

For each $g \in SL_n(\mathbb{Z}_p)$, we can define a corresponding automorphism by $\varphi(x) = x\,g$.

Proposition (Gefter-Golodets 1988)

Aut($SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p$) $= SL_n(\mathbb{Z}_p)$.

Proof of Gefter-Golodets

- Suppose that $\varphi \in \text{Aut}(\, SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p\,)$.
- For each $x \in SL_n(\mathbb{Z}_p)$, let $h(x) \in SL_n(\mathbb{Z}_p)$ be such that

$$\varphi(x) = x\, h(x).$$

- If $\gamma \in SL_n(\mathbb{Z})$, then

$$\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x) = \gamma \cdot x\, h(x)$$

  and so $h(\gamma \cdot x) = h(x)$.
- Since $SL_n(\mathbb{Z})$ acts ergodically on $(\, SL_n(\mathbb{Z}_p), \mu_p\,)$, there exists $g \in SL_n(\mathbb{Z}_p)$ such that

$$h(x) = g \quad \text{for } \mu_p\text{-a.e. } x.$$

The Punchline

Basic Question

How do we recognize the prime p in the topological group $SL_n(\mathbb{Z}_p)$?

Theorem (Folklore)

- *$SL_n(\mathbb{Z}_p)$ is "virtually" a pro-p group.*
- *More precisely, if H is any open subgroup, then*

$$[\, SL_n(\mathbb{Z}_p) : H \,] = b\, p^{\ell}$$

for some $\ell \geq 0$ and some divisor b of $|SL_n(\mathbb{Z}/p\mathbb{Z})|$.

Open Problems

Finally I will mention some long outstanding open problems concerning:

- Hyperfinite relations
- Treeable relations
- Universal relations

Hyperfinite relations

Theorem (Dougherty-Jackson-Kechris)

If E is a countable Borel equivalence relation on a standard Borel space X, then the following are equivalent:

(a) $E \leq_B E_0$.

(b) *E is hyperfinite; i.e. there exists an increasing sequence*

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

of finite Borel equivalence relations on X such that $E = \bigcup_{n \in \mathbb{N}} F_n$.

(c) *There exists a Borel action of $\mathbb{Z}$ on X such that $E = E_{\mathbb{Z}}^X$.*

Hyperfinite relations

Theorem (Jackson-Kechris-Louveau)

If G is a countable nonamenable group, then E_G is not hyperfinite.

Remark

Recall that E_G is the orbit equivalence relation arising from the free action of G on $(\, (2)^G, \mu \,)$.

Question (Weiss)

Suppose that G is a countable amenable group and that X is a standard Borel G-space. Does it follow that E_G^X is hyperfinite?

Hyperfinite relations

Theorem (Connes-Feldman-Weiss)

Suppose that G is a countable amenable group and that X is a standard Borel G-space. If μ is any Borel probability measure on X, then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is hyperfinite.

Theorem (Gao-Jackson)

If G is a countable abelian group and X is a standard Borel G-space, then E_G^X is hyperfinite.

Definition

The countable Borel equivalence relation E on X is said to be treeable iff there is a Borel acyclic graph (X, R) whose connected components are the E-classes.

Example

If the countable free group $\mathbb{F}$ has a free Borel action on X, then the corresponding orbit equivalence relation $E_{\mathbb{F}}^{X}$ is treeable.

Theorem (Jackson-Kechris-Louveau)

If E is treeable, then there exists a free Borel action of a countable free group $\mathbb{F}$ on a standard Borel space Y such that $E \sim_B E_{\mathbb{F}}^{Y}$.

Treeable relations

Definition

Let $E_{\infty T}$ be the orbit equivalence relation arising from the free action of $\mathbb{F}_2$ on $(2)^{\mathbb{F}_2}$.

Theorem (Jackson-Kechris-Louveau)

$E_{\infty T}$ is a universal treeable relation.

Question (Jackson-Kechris-Louveau)

Do there exist infinitely many nonsmooth treeable relations up to Borel bireducibility?

Remark

Currently, only 3 such relations are known; namely: E_0, $E_{\infty T}$ and the other one(s).

Treeable relations

Theorem (Hjorth)

If E is a profinite treeable relation, then $E <_B E_{\infty T}$.

Example (Thomas)

Let S be a nonempty set of primes and regard $SL_2(\mathbb{Z})$ as a subgroup of

$$G(S) = \prod_{p \in S} SL_2(\mathbb{Z}_p)$$

via the diagonal embedding. Then the corresponding orbit equivalence relation E_S is a non-hyperfinite profinite treeable relation.

Treeable relations

Conjecture (Thomas)

If $S \neq T$, then E_S and E_T are incomparable with respect to Borel bireducibility.

Conjecture (Thomas)

If S is any nonempty set of primes, then

$$E_S <_B E_S \oplus E_S <_B \cdots <_B \underbrace{E_S \oplus \cdots \oplus E_S}_{n \text{ times}} <_B \cdots$$

An implausible analog of the von Neumann Conjecture

False Conjecture (Day)

If G is a countable nonamenable group, then G contains a free nonabelian subgroup.

Theorem (Ol'shanskii)

There exists a periodic nonamenable group.

Conjecture (Kechris)

If E is a non-hyperfinite countable Borel equivalence relation, then there exists a non-hyperfinite treeable relation F such that $F \leq_B E$.

Some truly embarassing questions ...

Conjecture (Hjorth)

If E is a universal countable Borel equivalence relation on the standard Borel space X and F is a countable Borel equivalence relation such that $E \subseteq F$, then F is also universal.

Question (Jackson-Kechris-Louveau)

Suppose that E is a universal countable Borel equivalence relation on the standard Borel space X and that $Y \subseteq X$ is an E-invariant Borel subset. Does it follow that either $E \restriction Y$ or $E \restriction (X \smallsetminus Y)$ is universal?

Some truly embarassing questions ...

Definition

If E, E' are countable Borel, then E' is a minimal cover of E iff:

- $E <_B E'$
- *If the countable Borel F satisfies $E \leq_B F \leq_B E'$, then either $E \sim_B F$ or $F \sim_B E'$.*

Open Problem

Find an example of a nonsmooth countable Borel equivalence relation which has a minimal cover.

Open Problem

Find an example of a nonuniversal countable Borel equivalence relation which doesn't have a minimal cover.