Countable Borel Equivalence Relations IV

Simon Thomas

Rutgers University

17th November 2007
An application of Ioana Superrigidity

The Kechris Conjecture

$\equiv_T$ is universal.

Observation

There exists a universal countable Borel equivalence relation $E$ on $\mathcal{P}(\mathbb{N})$ such that $E \subseteq \equiv_T$.

Proof.

Identifying the free group $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, we have that $E_\infty \subseteq \equiv_T$. 

Simon Thomas (Rutgers University)
Appalachian Set Theory Workshop
17th November 2007
An application of Ioana Superrigidity

Conjecture (Hjorth)

*If* $F$ *is a universal countable Borel equivalence relation on the standard Borel space* $X$ *and* $E$ *is a countable Borel equivalence relation such that* $F \subseteq E$, *then* $E$ *is also universal.*

Theorem (Thomas 2002)

*There exists a pair of countable Borel equivalence relations* $F \subseteq E$ *on a standard Borel space* $X$ *such that* $E \prec_B F$. 
The ring of $p$-adic integers

- The ring $\mathbb{Z}_p$ of $p$-adic integers is the inverse limit of the system

  $\cdots \xrightarrow{\varphi_{n+1}} \mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{\varphi_n} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_1} \mathbb{Z}/p\mathbb{Z}.$

- So we can express each $z \in \mathbb{Z}_p$ as a formal sum

  $$z = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots$$

  where each $0 \leq a_n < p$.

- The $p$-adic norm is given by

  $$|z|_p = p^{-\text{ord}_p(z)}, \quad \text{ord}_p(z) = \min\{n \mid a_n \neq 0\};$$

  and the $p$-adic metric is given by

  $$d_p(x, y) = |x - y|_p.$$
The ring of $p$-adic integers

**Theorem**

- $\mathbb{Z}_p$ is a compact complete separable metric space.
- $\mathbb{Z}$ is a dense subring of $\mathbb{Z}_p$.

**Corollary**

- $SL_n(\mathbb{Z}_p)$ is a compact Polish group.
- $SL_n(\mathbb{Z})$ is a dense subgroup of $SL_n(\mathbb{Z}_p)$.

Note that $SL_n(\mathbb{Z}_p)$ is the inverse limit of the system

$$\cdots \xrightarrow{\theta_{n+1}} SL_n(\mathbb{Z}/p^{n+1}\mathbb{Z}) \xrightarrow{\theta_n} SL_n(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\theta_{n-1}} \cdots \xrightarrow{\theta_1} SL_n(\mathbb{Z}/p\mathbb{Z}).$$
The $p$-adic probability measure

**Theorem**

Since $\text{SL}_n(\mathbb{Z}_p)$ is a compact group, there exists a unique **Haar probability measure** on $\text{SL}_n(\mathbb{Z}_p)$; i.e. a probability measure $\mu_p$ which is invariant under the left translation action.

In fact, $\mu_p$ is simply the inverse limit of the counting measures on

$$\ldots \xrightarrow{\theta_{n+1}} \text{SL}_n(\mathbb{Z}/p^{n+1}\mathbb{Z}) \xrightarrow{\theta_n} \text{SL}_n(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\theta_{n-1}} \ldots \xrightarrow{\theta_1} \text{SL}_n(\mathbb{Z}/p\mathbb{Z}).$$

**Observation**

If $H \leq \text{SL}_n(\mathbb{Z}_p)$ is an open subgroup, then $H$ has finite index and

$$\mu_p(H) = \frac{1}{[\text{SL}_n(\mathbb{Z}_p) : H]}.$$
The $p$-adic probability measure

Theorem

$\mu_p$ is the unique $SL_n(\mathbb{Z})$-invariant probability measure on $SL_n(\mathbb{Z}_p)$.

Proof.

- $SL_n(\mathbb{Z}_p)$ acts continuously on the space $\mathcal{M}$ of probability measures on $SL_n(\mathbb{Z}_p)$.
- Hence if $\nu$ is any probability measure on $SL_n(\mathbb{Z}_p)$, then

$$S_\nu = \{ g \in SL_n(\mathbb{Z}_p) \mid \nu \text{ is } g\text{-invariant} \}$$

is a closed subgroup of $SL_n(\mathbb{Z}_p)$.

- Thus, by density, any $SL_n(\mathbb{Z})$-invariant probability measure is actually $SL_n(\mathbb{Z}_p)$-invariant and hence must be $\mu_p$. 

Simon Thomas (Rutgers University) Appalachian Set Theory Workshop 17th November 2007
Unique ergodicity

Definition

The action of $G$ on the standard probability space $(X, \mu)$ is uniquely ergodic iff $\mu$ is the unique $G$-invariant probability measure on $X$.

Observation

If the action of $G$ on $(X, \mu)$ is uniquely ergodic, then $G$ acts ergodically.

Proof.

Suppose that there exists a $G$-invariant Borel subset $A \subseteq X$ with $0 < \mu(A) < 1$. Let $B = X \setminus A$.

Then we can define distinct $G$-invariant probability measures by

\[
\nu_1(Z) = \frac{\mu(Z \cap A)}{\mu(A)} \\
\nu_2(Z) = \frac{\mu(Z \cap B)}{\mu(B)}.
\]
Ergodic Components

Observation

The action of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$ is not strongly mixing.

- Let $\Lambda = \ker \varphi$ and $H = \ker \psi$ be the kernels of the maps
  
  $$\varphi : SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/p\mathbb{Z})$$

  and

  $$\psi : SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_p/p\mathbb{Z}_p) \cong SL_n(\mathbb{Z}/p\mathbb{Z}).$$

- Then $H$ is the closure of $\Lambda$ and the ergodic decomposition of the $\Lambda$-action is given by

  $$SL_n(\mathbb{Z}_p) = Hg_1 \sqcup \cdots \sqcup Hg_d, \quad d = |SL_n(\mathbb{Z}/p\mathbb{Z})|.$$ 

- The $Hg_i$ are the ergodic components of the $\Lambda$-action.
Ergodic Components

**Definition**

Let $F \subseteq E$ be the orbit equivalence relations of the actions of $\Lambda$ and $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$.

**Theorem (Thomas 2002)**

If $n \geq 3$, then $E <_B F$.

By considering the ergodic decomposition of the $\Lambda$-action

$$SL_n(\mathbb{Z}_p) = Hg_1 \sqcup \cdots \sqcup Hg_d, \quad d = |SL_n(\mathbb{Z}/p\mathbb{Z})|,$$

we see that

$$F = E_1 \oplus \cdots \oplus E_d, \quad \text{where } E_i = F \upharpoonright Hg_i.$$
$E_i \sim_B E$ for $1 \leq i \leq d$.

- First we claim that the inclusion map $Hg_i \to SL_n(\mathbb{Z}_p)$ is a Borel reduction from $E_i$ to $E$.
- So suppose that $x, y \in Hg_i$ and that $x E y$.
- Then there exists $\gamma \in SL_n(\mathbb{Z})$ such that $\gamma x = y$.
- It follows that
  \[
  \emptyset \neq \gamma Hg_i \cap Hg_i = H \gamma g_i \cap Hg_i.
  \]
- Hence $\gamma \in SL_n(\mathbb{Z}) \cap H = \Lambda$ and so $x E_i y$. 
In order to show that $E \leq_B E_i$, we choose our coset representatives $g_k$ so that each $g_k \in SL_n(\mathbb{Z})$.

For each $1 \leq k \leq d$, define $h_k : Hg_k \rightarrow Hg_i$ by $h_k(x) = g_i g_k^{-1} x$.

We claim that $h = h_1 \cup \cdots \cup h_d$ is a Borel reduction from $E$ to $E_i$.

If $x, y \in SL_n(\mathbb{Z}_p)$, then

$$x E y \iff h(x) E h(y) \iff h(x) E_i h(y).$$

The last equivalence follows because $h(x), h(y) \in Hg_i$. 

$E_i \sim_B E$ for $1 \leq i \leq d$. 

Lemma

If $F \subseteq E$ are the orbit equivalence relations of the actions of $\Lambda$ and $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$, then

$$F \sim_B E \oplus \cdots \oplus E.$$ 

d $d$ times

Hence it is enough to prove ...

Theorem

If $n \geq 3$, then

$$E <_B E \oplus E <_B \cdots <_B E \oplus \cdots \oplus E <_B \cdots$$ 

d $d$ times
Some Notation

- \((K, \mu) = (\text{SL}_n(\mathbb{Z}_p), \mu_p)\).
- \(\Gamma = \text{SL}_n(\mathbb{Z})\).
- \(E\) is the orbit equivalence relation of \(\Gamma\) on \(K\).

Then it’s enough to prove ... 

Theorem

*If* \(f : K \rightarrow K\) *is a Borel reduction from* \(E\) *to* \(E\), *then*

\[
\mu(\Gamma \cdot f[K]) = 1.
\]
The proof begins

- Suppose that \( f : K \to K \) is a Borel reduction from \( E \) to \( E \).
- Then we can define a Borel cocycle \( \alpha : \Gamma \times K \to \Gamma \) by
  \[
  \alpha(g, x) = \text{the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).
  \]

- By Ioana Superrigidity, there exists
  - a subgroup \( \Delta \leq \Gamma \) of finite index
  - an ergodic component \( X \subseteq K \) for the \( \Delta \)-action
  such that \( \alpha \restriction (\Delta \times X) \) is equivalent to a group homomorphism
    \[
    \psi : \Delta \to \text{SL}_n(\mathbb{Z}).
    \]

- After slightly adjusting \( f \), we can suppose that \( \alpha \restriction (\Delta \times X) = \psi \).
An application of Margulis Superrigidity

To simplify the presentation, suppose that \( n \geq 3 \) is odd.

**Theorem**

Suppose that \( \Delta \leq SL_n(\mathbb{Z}) \) is a subgroup of finite index and that \( \psi : \Delta \rightarrow SL_n(\mathbb{Z}) \) is a group homomorphism. Then either:

- \( \psi[\Delta] \) is finite; or
- \( \psi \) is an embedding and \( \psi[\Delta] \) is a subgroup of finite index in \( SL_n(\mathbb{Z}) \).
Suppose that $\psi[\Delta]$ is finite.
Recall that $\psi(g) \cdot f(x) = f(g \cdot x)$ for all $g \in \Delta$ and $x \in X$.
Thus we can define a $\Delta$-invariant map $\Phi : X \to [K]^{<\omega}$ by
\[ \Phi(x) = \{ f(g \cdot x) \mid g \in \Delta \}. \]
Since $\Delta$ acts ergodically on $X$, it follows that $\Phi$ is constant on a $\mu$-conull subset of $X$, which is a contradiction.
Case 2

- Suppose that $\psi$ is an embedding and that $\psi[\Delta]$ is a subgroup of finite index in $SL_n(\mathbb{Z})$.
- Let $Y_1, \cdots, Y_d$ be the ergodic components for the action of $\psi[\Delta]$ on $K$.
- Since $\Delta$ acts ergodically on $X$, we can suppose that there exists $Y = Y_i$ such that $f : X \to Y$.
- Since $\psi(g) \cdot f(x) = f(g \cdot x)$, we can define a $\psi[\Delta]$-invariant probability measure $\nu$ on $Y$ by
  \[ \nu(Z) = \frac{\mu(f^{-1}(Z))}{\mu(X)}. \]
- Since the action of $\psi[\Delta]$ on $Y$ is uniquely ergodic,
  \[ \nu(Z) = \frac{\mu(Z)}{\mu(Y)}. \]
- Hence $\mu(f[X]) = \mu(Y) > 0$ and so $\mu(\Gamma \cdot f[K]) = 1$. 
Theorem (Ioana)

Let $\Gamma$ be a countably infinite Kazhdan group and let $(X, \mu)$ be a free ergodic profinite $\Gamma$-space.

Suppose that $H$ is any countable group and that $\alpha : \Gamma \times X \to H$ is a Borel cocycle.

Then there exists a subgroup $\Delta \leq \Gamma$ of finite index and an ergodic component $Y \subseteq X$ for the $\Delta$-action such that $\alpha \upharpoonright (\Delta \times Y)$ is equivalent to a homomorphism $\psi : \Delta \to H$. 
Profinite Actions

Definition

Let $\Gamma$ be a countable group.

For each $n \in \mathbb{N}$, let $(X_n, \mu_n)$ be a finite $\Gamma$-space, where $\mu_n(Y) = |Y|/|X_n|$. Suppose that each $(X_n, \mu_n)$ is a quotient of $(X_{n+1}, \mu_{n+1})$; say,

$$
\cdots \xrightarrow{q_{n+1}} (X_{n+1}, \mu_{n+1}) \xrightarrow{q_n} (X_n, \mu_n) \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_0} (X_0, \mu_0).
$$

Then the canonical action of $\Gamma$ on

$$(X, \mu) = \lim_{\leftarrow} (X_n, \mu_n)$$

is said to be a profinite action.
Examples of Profinite Actions

**Definition**

A countably infinite group $\Gamma$ is *residually finite* iff there exists a chain of finite index normal subgroups

$$\Gamma_0 > \Gamma_1 > \cdots > \Gamma_n > \cdots$$

such that $\bigcap \Gamma_n = 1$.

Then $\Gamma$ is a dense subgroup of the profinite group $\lim \leftarrow \Gamma / \Gamma_n$.

**Example**

Let $K$ be a profinite group and let $\Gamma$ be a countable dense subgroup. If $L$ is a closed subgroup of $K$, then the action of $\Gamma$ on $K/L$ is profinite.

**Example**

The action of $SL_n(\mathbb{Z})$ on the projective space $PG(n-1, \mathbb{Q}_p)$ is profinite.
A final application of Ioana Superrigidity

**Definition**

Fix some $n \geq 3$. Let $S$ be a nonempty set of primes and regard $SL_n(\mathbb{Z})$ as a subgroup of $G(S) = \prod_{p \in S} SL_n(\mathbb{Z}_p)$ via the diagonal embedding. Let $E_S$ be the corresponding orbit equivalence relation.

**Theorem (Thomas 2002)**

If $S \neq T$, then $E_S$ and $E_T$ are incomparable with respect to Borel bireducibility.
Sketch of Proof

- For simplicity, suppose that $S = \{ p \}$ and $T = \{ q \}$, where $p \neq q$ are distinct primes.
- Suppose that $f : SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_q)$ is a Borel reduction from $E\{p\}$ to $E\{q\}$.
- Then arguing as above, after passing to subgroups of finite index and ergodic components if necessary, we find that
  \[ ( SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p ) \cong ( SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_q), \mu_q ) \]
  as measure-preserving permutation groups.

Basic Problem

*How can we recognize the prime $p$ in $( SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p )$?*
The Automorphism Group

Definition

- \( \text{Aut}( SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p ) \) consists of the measure-preserving bijections \( \varphi : SL_n(\mathbb{Z}_p) \rightarrow SL_n(\mathbb{Z}_p) \) such that for all \( \gamma \in SL_n(\mathbb{Z}) \),

\[
\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x) \quad \text{for } \mu_p\text{-a.e. } x.
\]

- As usual, we identify two such maps if they agree \( \mu_p\)-a.e.

Example

For each \( g \in SL_n(\mathbb{Z}_p) \), we can define a corresponding automorphism by \( \varphi(x) = x \cdot g \).

Proposition (Gefter-Golodets 1988)

\( \text{Aut}( SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p ) = SL_n(\mathbb{Z}_p) \).
Proof of Gefter-Golodets

- Suppose that $\varphi \in \text{Aut}(\text{SL}_n(\mathbb{Z}), \text{SL}_n(\mathbb{Z}_p), \mu_p)$.
- For each $x \in \text{SL}_n(\mathbb{Z}_p)$, let $h(x) \in \text{SL}_n(\mathbb{Z}_p)$ be such that
  $$\varphi(x) = x \cdot h(x).$$

- If $\gamma \in \text{SL}_n(\mathbb{Z})$, then
  $$\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x) = \gamma \cdot x \cdot h(x)$$
  and so $h(\gamma \cdot x) = h(x)$.

- Since $\text{SL}_n(\mathbb{Z})$ acts ergodically on $(\text{SL}_n(\mathbb{Z}_p), \mu_p)$, there exists $g \in \text{SL}_n(\mathbb{Z}_p)$ such that
  $$h(x) = g \quad \text{for } \mu_p\text{-a.e. } x.$$
The Punchline

Basic Question

How do we recognize the prime $p$ in the topological group $SL_n(\mathbb{Z}_p)$?

Theorem (Folklore)

- $SL_n(\mathbb{Z}_p)$ is “virtually” a pro-$p$ group.
- More precisely, if $H$ is any open subgroup, then

$$[ SL_n(\mathbb{Z}_p) : H ] = b \, p^\ell$$

for some $\ell \geq 0$ and some divisor $b$ of $|SL_n(\mathbb{Z}/p\mathbb{Z})|$. 
Finally I will mention some long outstanding open problems concerning:

- Hyperfinite relations
- Treeable relations
- Universal relations
Hyperfinite relations

**Theorem (Dougherty-Jackson-Kechris)**

If $E$ is a countable Borel equivalence relation on a standard Borel space $X$, then the following are equivalent:

(a) $E \leq_{B} E_0$.

(b) $E$ is **hyperfinite**; i.e. there exists an increasing sequence

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

of finite Borel equivalence relations on $X$ such that $E = \bigcup_{n \in \mathbb{N}} F_n$.

(c) There exists a Borel action of $\mathbb{Z}$ on $X$ such that $E = E^X_{\mathbb{Z}}$. 

Simon Thomas (Rutgers University)  Appalachian Set Theory Workshop  17th November 2007
Theorem (Jackson-Kechris-Louveau)

If $G$ is a countable nonamenable group, then $E_G$ is not hyperfinite.

Remark

Recall that $E_G$ is the orbit equivalence relation arising from the free action of $G$ on $(\mathbb{2}^G, \mu)$.

Question (Weiss)

Suppose that $G$ is a countable amenable group and that $X$ is a standard Borel $G$-space. Does it follow that $E^X_G$ is hyperfinite?
Hyperfinite relations

Theorem (Connes-Feldman-Weiss)
Suppose that $G$ is a countable amenable group and that $X$ is a standard Borel $G$-space. If $\mu$ is any Borel probability measure on $X$, then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is hyperfinite.

Theorem (Gao-Jackson)
If $G$ is a countable abelian group and $X$ is a standard Borel $G$-space, then $E^X_G$ is hyperfinite.
**Definition**

The countable Borel equivalence relation $E$ on $X$ is said to be **treeable** iff there is a Borel acyclic graph $(X, R)$ whose connected components are the $E$-classes.

**Example**

If the countable free group $F$ has a free Borel action on $X$, then the corresponding orbit equivalence relation $E^X_F$ is treeable.

**Theorem (Jackson-Kechris-Louveau)**

If $E$ is treeable, then there exists a free Borel action of a countable free group $F$ on a standard Borel space $Y$ such that $E \sim_B E^Y_F$. 

Simon Thomas (Rutgers University)  
Appalachian Set Theory Workshop  
17th November 2007
Treeable relations

**Definition**

Let $E_{\infty T}$ be the orbit equivalence relation arising from the free action of $F_2$ on $(2)^{F_2}$.

**Theorem (Jackson-Kechris-Louveau)**

$E_{\infty T}$ is a universal treeable relation.

**Question (Jackson-Kechris-Louveau)**

Do there exist infinitely many nonsmooth treeable relations up to Borel bireducibility?

**Remark**

Currently, only 3 such relations are known; namely: $E_0$, $E_{\infty T}$ and the other one(s).
Theorem (Hjorth)

If $E$ is a profinite treeable relation, then $E \prec_B E_\infty^T$. 

Example (Thomas)

Let $S$ be a nonempty set of primes and regard $SL_2(\mathbb{Z})$ as a subgroup of $G (S) = \prod_{p \in S} SL_2(\mathbb{Z}_p)$ via the diagonal embedding. Then the corresponding orbit equivalence relation $E_S$ is a non-hyperfinite profinite treeable relation.
Conjecture (Thomas)

If $S \neq T$, then $E_S$ and $E_T$ are incomparable with respect to Borel bireducibility.

Conjecture (Thomas)

If $S$ is any nonempty set of primes, then

$$E_S <_B E_S \oplus E_S <_B \cdots <_B E_S \oplus \cdots \oplus E_S <_B \cdots$$

$n$ times
An implausible analog of the von Neumann Conjecture

False Conjecture (Day)

*If G is a countable nonamenable group, then G contains a free nonabelian subgroup.*

Theorem (Ol’shanskii)

*There exists a periodic nonamenable group.*

Conjecture (Kechris)

*If E is a non-hyperfinite countable Borel equivalence relation, then there exists a non-hyperfinite treeable relation F such that \( F \leq_B E \).*
Conjecture (Hjorth)

If $E$ is a universal countable Borel equivalence relation on the standard Borel space $X$ and $F$ is a countable Borel equivalence relation such that $E \subseteq F$, then $F$ is also universal.

Question (Jackson-Kechris-Louveau)

Suppose that $E$ is a universal countable Borel equivalence relation on the standard Borel space $X$ and that $Y \subseteq X$ is an $E$-invariant Borel subset. Does it follow that either $E \upharpoonright Y$ or $E \upharpoonright (X \setminus Y)$ is universal?
Some truly embarassing questions ...

**Definition**

If $E, E'$ are countable Borel, then $E'$ is a **minimal cover** of $E$ iff:

- $E <_B E'$
- If the countable Borel $F$ satisfies $E \leq_B F \leq_B E'$, then either $E \sim_B F$ or $F \sim_B E'$.

**Open Problem**

Find an example of a nonsmooth countable Borel equivalence relation which **has** a minimal cover.

**Open Problem**

Find an example of a nonuniversal countable Borel equivalence relation which **doesn’t have** a minimal cover.