Countable Borel Equivalence Relations III

Simon Thomas

Rutgers University

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A quick recap

The Fundamental Question

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space. To what extent does the data $(X, E^X_G)$ “remember” the group $G$ and its action on $X$?

More accurately, to what extent does the data $(C, E^X_G \upharpoonright C)$ “remember” the group $G$ and its action on $X$, where $C$ is an arbitrary Borel complete section?

Further Hypotheses

We shall usually also assume that:

- $G$ acts freely on $X$; i.e. $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$.
- There exists a $G$-invariant probability measure $\mu$ on $X$. 
Definition

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space. Then the $G$-invariant probability measure $\mu$ is said to be ergodic iff $\mu(A) = 0, 1$ for every $G$-invariant Borel subset $A \subseteq X$.

Theorem

If $\mu$ is a $G$-invariant probability measure on the standard Borel $G$-space $X$, then the following statements are equivalent.

- The action of $G$ on $(X, \mu)$ is ergodic.
- If $Y$ is a standard Borel space and $f : X \to Y$ is a $G$-invariant Borel function, then there exists a $G$-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \upharpoonright M$ is a constant function.
Strong mixing

Definition

The action of $G$ on the standard probability space $(X, \mu)$ is **strongly mixing** iff for any Borel subsets $A, B \subseteq X$, we have that

$$\mu(g(A) \cap B) \to \mu(A) \cdot \mu(B) \quad \text{as } g \to \infty.$$ 

In other words, if $\langle g_n \mid n \in \mathbb{N} \rangle$ is any sequence of distinct elements of $G$, then

$$\lim_{n \to \infty} \mu(g_n(A) \cap B) = \mu(A) \cdot \mu(B).$$

Observation

If $H \leq G$ is an infinite subgroup of $G$, then the action of $H$ on $(X, \mu)$ is also strongly mixing.
Strong mixing continued

**Observation**

*If the action of $G$ on $(X, \mu)$ is strongly mixing, then $G$ acts ergodically on $(X, \mu)$.***

**Proof.**

If $A \subseteq X$ is a $G$-invariant Borel subset, then

$$\mu(A)^2 = \lim_{g \to \infty} \mu(g(A) \cap A) = \lim_{g \to \infty} \mu(A) = \mu(A).$$

Hence $\mu(A) = 0, 1$.

**Remark**

With more effort, it can be shown that for each $n \geq 2$, the diagonal action of $G$ on $(X^n, \mu^n)$ is also ergodic.
Bernoulli actions are strongly mixing

**Theorem**

The action of $G$ on $(2^G, \mu)$ is strongly mixing.

- Consider the case when there exist finite subsets $S, T \subset G$ and subsets $\mathcal{F} \subseteq 2^S$, $\mathcal{G} \subseteq 2^T$ such that $A = \{ f \in (2)^G \mid f \upharpoonright S \in \mathcal{F} \}$ and $B = \{ f \in (2)^G \mid f \upharpoonright T \in \mathcal{G} \}$.
- If $\langle g_n \mid n \in \mathbb{N} \rangle$ is a sequence of distinct elements of $G$, then
  
  $g_n(S) \cap T = \emptyset$

  for all but finitely many $n$.
- This means that $g_n(A), B$ are independent events and so
  
  $\mu(g_n(A) \cap B) = \mu(g_n(A)) \cdot \mu(B) = \mu(A) \cdot \mu(B)$. 
Borel cocycles

- Let $G$ be a countable group and let $X$ be a standard Borel $G$-space with invariant ergodic probability measure $\mu$.
- Suppose that the countable group $H$ has a free Borel action on $Y$ and that
  
  $$f : X \to Y$$
  
  is a Borel homomorphism between the corresponding orbit equivalence relations.
- Then we can define a Borel cocycle

  $$\alpha : G \times X \to H$$

  by setting

  $$\alpha(g, x) = \text{the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$
The cocycle identity

Note that

\[ f(x) \overset{\alpha(g,x)}{\longrightarrow} f(g \cdot x) \overset{\alpha(h,g \cdot x)}{\longrightarrow} f(hg \cdot x) \]

and hence we have the identity:

\[ \alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x) \quad \mu\text{-a.e } x \]

In particular, \( f \) is a permutation group homomorphism iff

\[ \alpha(g, x) = \alpha(g) \]

is a group homomorphism.
Cocycle equivalence

\[
\beta(g, x) = b(g \cdot x) \alpha(g, x) b(x)^{-1} \quad \mu\text{-a.e } x
\]
Popa’s Cocycle Superrigidity Theorem

Theorem (Popa)

Let $\Gamma$ be a countably infinite Kazhdan group and let $G$ be a countable group such that $\Gamma \leq G$. If $H$ is any countable group, then every Borel cocycle

$$\alpha : G \times (2)^G \to H$$

is equivalent to a group homomorphism of $G$ into $H$.

Remarks

- For example, we let $\Gamma = \text{SL}_n(\mathbb{Z})$ for any $n \geq 3$.
- For example, we can let $G = \Gamma \times S$, where $S$ is any countable group.
Definition

$E_G$ denotes the orbit equivalence relation of the Bernoulli action of the countable group on $(2^G, \mu)$.

Theorem

- Let $G = SL_3(\mathbb{Z}) \times S$, where $S$ is any countable group.
- Let $H$ be any countable group and let $Y$ be a free standard Borel $H$-space.

If there exists a $\mu$-nontrivial Borel homomorphism from $E_G$ to $E^Y_H$, then there exists a virtual embedding $\pi : G \to H$. 
Proof of Theorem

- Suppose that $f : (2)^G \to Y$ is a $\mu$-nontrivial Borel homomorphism from $E_G$ to $E_Y^Y$.
- Then we can define a Borel cocycle $\alpha : G \times (2)^G \to H$ by
  \[
  \alpha(g, x) = \text{the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).
  \]
- By Popa, after deleting a nullset and slightly adjusting $f$, we can suppose that $\alpha : G \to H$ is a group homomorphism.
- Suppose that $K = \ker \alpha$ is infinite.
- First note that if $k \in K$, then $f(k \cdot x) = \alpha(k) \cdot f(x) = f(x)$ and so $f : (2)^G \to X$ is $K$-invariant.
- Next note that since the action of $G$ is strongly mixing, it follows that $K$ acts ergodically on $( (2)^G, \mu )$.
- But then the $K$-invariant function $f : (2)^G \to X$ is $\mu$-a.e. constant and so $f$ is $\mu$-trivial, which is a contradiction!
Torsion-free abelian groups of finite rank

**Definition**

An additive subgroup $G \leq \mathbb{Q}^n$ has rank $n$ iff $G$ contains $n$ linearly independent elements.

**Definition**

Let $\cong_n$ denote the isomorphism relation on the standard Borel space $R(\mathbb{Q}^n)$ of torsion-free abelian groups of rank $n$.

Recall that if $A, B \in R(\mathbb{Q}^n)$, then

$$A \cong B \quad \text{iff} \quad \exists g \in GL_n(\mathbb{Q}) \quad g(A) = B.$$ 

In other words, $\cong_n$ is the orbit equivalence relation for the action of $GL_n(\mathbb{Q})$ on the space $R(\mathbb{Q}^n)$. 
Some History

- In 1937, Baer gave a satisfactory classification of the rank 1 groups. (In fact, the isomorphism relation is hyperfinite.)
- In 1938, Kurosh and Malcev independently gave an unsatisfactory classification of the higher rank groups.

Problem (Fuchs 1973)

*Characterize the torsion-free abelian groups of rank 2 by invariants.*

Conjecture (Hjorth-Kechris 1996)

*The isomorphism relation for the torsion-free abelian groups of rank 2 is countable universal.*

- In 1998, Hjorth proved that the classification problem for the rank 2 groups was strictly harder than that for the rank 1 groups.
An application of Superrigidity

Theorem (Thomas 2000)

The complexity of the classification problems for the torsion-free abelian groups of rank \( n \) increases strictly with the rank \( n \).

Corollary

For each \( n \geq 1 \), the isomorphism relation for the torsion-free abelian groups of rank \( n \) is not countable universal.

A slightly embarrassing question

Is the isomorphism relation on the space of torsion-free abelian groups of finite rank countable universal?

Theorem (Thomas 2006)

The isomorphism relation on the space of torsion-free abelian groups of finite rank is not countable universal.
**E₀-ergodicity**

**Definition**

- Let $E$, $F$ be countable Borel equivalence relations on $X$, $Y$ and let $\mu$ be an $E$-invariant probability measure on $X$.
- Then $E$ is $F$-ergodic iff every Borel homomorphism $f : X \rightarrow Y$ from $E$ to $F$ is $\mu$-trivial.

**Remark**

- Thus $id_\mathbb{R}$-ergodicity coincides with the usual ergodicity.
- Clearly if $E$ is $F$-ergodic and $F' \leq_B F$, then $E$ is also $F'$-ergodic.
**Theorem (Jones-Schmidt)**

\( E \) is \( E_0 \)-ergodic iff \( E \) has no “nontrivial almost invariant subsets”.

**Definition**

Let \( E = E_X^X \) be a countable Borel equivalence relation and let \( \mu \) be an \( E \)-invariant probability measure on \( X \). Then \( E \) has nontrivial almost invariant subsets iff there exists a sequence of Borel subsets

\[
\langle A_n \subseteq X \mid n \in \mathbb{N} \rangle
\]

satisfying the following conditions:

- \( \mu(g \cdot A_n \triangle A_n) \to 0 \) for all \( g \in G \).
- There exists \( \delta > 0 \) such that \( \delta < \mu(A_n) < 1 - \delta \) for all \( n \in \mathbb{N} \).
**Theorem (Jones-Schmidt)**

Let $G$ be a countable group and let $H \leq G$ be a nonamenable subgroup. Then the shift action of $H$ on $(2^G, \mu)$ is $E_0$-ergodic.

**Remark**

The proof makes use of the associated unitary representation of $H$ on the Hilbert space $L^2(2^G, \mu)$.

**Remark**

For later use, note that if $E$ is $E_0$-ergodic and $F$ is hyperfinite, then $E$ is $F$-ergodic.
The non-universality proof begins

- Let $S$ be a suitably chosen countable group and let $G = SL_3(\mathbb{Z}) \times S$.
- Let $E = E_G$ be the orbit equivalence relation of the action of $G$ on $((2)^G, \mu)$.
- Suppose that $f : (2)^G \rightarrow \bigsqcup_{n \geq 1} R(\mathbb{Q}^n)$ is a Borel reduction from $E$ to the isomorphism relation.
- After deleting a nullset, we can suppose that $f : (2)^G \rightarrow R(\mathbb{Q}^n)$ for some fixed $n \geq 1$. 
The non-universality proof begins

- At this point, we would like to define a corresponding Borel cocycle

\[ \alpha : G \times (2)^G \to GL_n(\mathbb{Q}). \]

- Then we could choose \( S \) to be a group which doesn’t embed into \( GL_n(\mathbb{Q}) \) for all \( n \).
- Unfortunately \( GL_n(\mathbb{Q}) \) doesn’t act freely on \( R(\mathbb{Q}^n) \).
- In fact, if \( B \in R(\mathbb{Q}^n) \), then the stabilizer of \( B \) in \( GL_n(\mathbb{Q}) \) is precisely its automorphism group \( \text{Aut}(B) \).
- What to do? Change the category!
The quasi-equality relation

**Definition**

If $A, B \in R(\mathbb{Q}^n)$, then $A$ and $B$ are said to be **quasi-equal**, written $A \approx_n B$, iff $A \cap B$ has finite index in both $A$ and $B$.

**Theorem (Thomas)**

The quasi-equality relation $\approx_n$ is hyperfinite.

For each $A \in R(\mathbb{Q}^n)$, let $[A]$ be the $\approx_n$-class containing $A$. We shall consider the induced action of $GL_n(\mathbb{Q})$ on

$$X = \{ [A] \mid A \in R(\mathbb{Q}^n) \}$$

of $\approx_n$-classes. (Of course, $X$ is **not** a standard Borel space.)
Stabilizers of $\approx_n$-classes

**Definition**

For each $A \in R(\mathbb{Q}^n)$, the ring of quasi-endomorphisms is

$$QE(A) = \{ \varphi \in \text{Mat}_n(\mathbb{Q}) \mid (\exists m \geq 1) m \varphi \in \text{End}(A) \}.$$ 

Clearly $QE(A)$ is a $\mathbb{Q}$-subalgebra of $\text{Mat}_n(\mathbb{Q})$; and so there are only countably many possibilities for $QE(A)$.

**Definition**

$Q\text{Aut}(A)$ is the group of units of the $\mathbb{Q}$-algebra $QE(A)$.

**Lemma**

If $A \in R(\mathbb{Q}^n)$, then $Q\text{Aut}(A)$ is the setwise stabilizer of $[A]$ in $GL_n(\mathbb{Q})$. 
Defining the cocycle

For each $x \in (2)^G$, let $A_x = f(x) \in R(\mathbb{Q}^n)$.

After deleting a nullset and slightly adjusting $f$, we can suppose that the setwise stabilizer of each $[A_x]$ is a fixed subgroup $L \leq GL_n(\mathbb{Q})$.

Note that the quotient group $H = N_{GL_n(\mathbb{Q})}(L)/L$ acts freely on the corresponding set $Y = \{[A] | Q\text{Aut}(A) = L\}$ of $\approx_n$-classes.

Hence we can define a corresponding cocycle

$$\alpha : G \times (2)^G \to H$$

by setting

$$\alpha(g, x) = \text{the unique } h \in H \text{ such that } h \cdot [A_x] = [A_{g \cdot x}].$$
A suitably chosen $S$

- Let $S$ be a countable simple nonamenable group which does not embed into any of the countably many possibilities for $H$.
- By Popa, after deleting a nullset and slightly adjusting $f$, we can suppose that
  \[ \alpha : G = SL_3(\mathbb{Z}) \times S \to H \]
  is a homomorphism.
- Since $S \leq \ker \alpha$, it follows that $f : (2)^G \to R(\mathbb{Q}^n)$ is a Borel homomorphism from the $S$-action on $(2)^G$ to the hyperfinite quasi-equality $\approx_n$-relation.
- Since $S$ is nonamenable, the $S$-action on $(2)^G$ is $E_0$-ergodic and hence $\mu$-almost all $x$ are mapped to a single $\approx_n$-class, which is a contradiction.