Countable Borel Equivalence Relations III

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A quick recap

The Fundamental Question

Let G be a countable group and let X be a standard Borel G-space. To what extent does the data (X, E_G^X) "remember" the group G and its action on X?

More accurately, to what extent does the data $(C, E_G^X \restriction C)$ "remember" the group G and its action on X, where C is an arbitary Borel complete section?

Further Hypotheses

We shall usually also assume that:

- G acts freely on X; i.e. $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$.
- There exists a G-invariant probability measure μ on X.

Ergodicity

Definition

Let G be a countable group and let X be a standard Borel G-space. Then the G-invariant probability measure μ is said to be ergodic iff $\mu(A) = 0, 1$ for every G-invariant Borel subset $A \subseteq X$.

Theorem

If μ is a G-invariant probability measure on the standard Borel G-space X, then the following statements are equivalent.

- *The action of G on (X, μ) is ergodic.*
- *If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant function.*

Strong mixing

Definition

The action of G on the standard probability space (X, μ) is strongly mixing iff for any Borel subsets $A, B \subseteq X$, we have that

$$\mu(g(A) \cap B) \to \mu(A) \cdot \mu(B) \quad \text{as } g \to \infty.$$

In other words, if $\langle g_n \mid n \in \mathbb{N} \rangle$ is any sequence of distinct elements of G, then

$$\lim_{n \to \infty} \mu(g_n(A) \cap B) = \mu(A) \cdot \mu(B).$$

Observation

If $H \leqslant G$ is an infinite subgroup of G, then the action of H on (X, μ) is also strongly mixing.

Strong mixing continued

Observation

If the action of G on (X, μ) is strongly mixing, then G acts ergodically on (X, μ).

Proof.

If $A \subseteq X$ is a G-invariant Borel subset, then

$$\mu(A)^2 = \lim_{g \to \infty} \mu(g(A) \cap A) = \lim_{g \to \infty} \mu(A) = \mu(A).$$

Hence $\mu(A) = 0, 1$. $\square$

Remark

With more effort, it can be shown that for each $n \geq 2$, the diagonal action of G on (X^n, μ^n) is also ergodic.

Bernoulli actions are strongly mixing

Theorem

The action of G on $(\,(2)^G, \mu\,)$ is strongly mixing.

- Consider the case when there exist finite subsets $S, T \subset G$ and subsets $\mathcal{F} \subseteq 2^S$, $\mathcal{G} \subseteq 2^T$ such that $A = \{f \in (2)^G \mid f \restriction S \in \mathcal{F}\}$ and $B = \{f \in (2)^G \mid f \restriction T \in \mathcal{G}\}$.
- If $\langle g_n \mid n \in \mathbb{N} \rangle$ is a sequence of distinct elements of G, then

$$g_n(S) \cap T = \emptyset$$

  for all but finitely many n.
- This means that $g_n(A)$, B are independent events and so

$$\mu(g_n(A) \cap B) = \mu(g_n(A)) \cdot \mu(B) = \mu(A) \cdot \mu(B).$$

Borel cocycles

- Let G be a countable group and let X be a standard Borel G-space with invariant ergodic probability measure μ.
- Suppose that the countable group H has a free Borel action on Y and that

$$f : X \to Y$$

is a Borel homomorphism between the corresponding orbit equivalence relations.
- Then we can define a Borel cocycle

$$\alpha : G \times X \to H$$

by setting

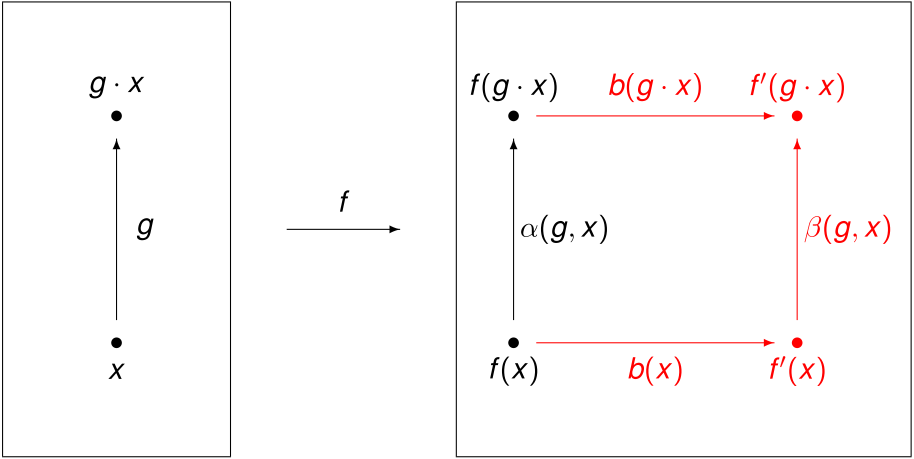
$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

The cocycle identity

- Note that
$$f(x) \xrightarrow{\alpha(g,x)} f(g \cdot x) \xrightarrow{\alpha(h,g \cdot x)} f(hg \cdot x)$$
and hence we have the identity:
$$\alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x) \quad \mu\text{-a.e } x$$

- In particular, f is a permutation group homomorphism iff
$$\alpha(g, x) = \alpha(g)$$
is a group homomorphism.

Cocycle equivalence



(X, μ)

$$\beta(g, x) = b(g \cdot x)\alpha(g, x)b(x)^{-1} \quad \mu\text{-a.e } x$$

Popa's Cocycle Superrigidity Theorem

Theorem (Popa)

Let Γ be a countably infinite Kazhdan group and let G be a countable group such that $\Gamma \trianglelefteq G$. If H is any countable group, then every Borel cocycle

$$\alpha : G \times (2)^G \to H$$

is equivalent to a group homomorphism of G into H.

Remarks

- For example, we let $\Gamma = SL_n(\mathbb{Z})$ for any $n \geq 3$.
- For example, we can let $G = \Gamma \times S$, where S is any countable group.

An easy consequence of Popa Superrigidity

Definition

E_G denotes the orbit equivalence relation of the Bernoulli action of the countable group on $(\,(2)^G, \mu\,)$.

Theorem

- *Let $G = SL_3(\mathbb{Z}) \times S$, where S is any countable group.*
- *Let H be any countable group and let Y be a free standard Borel H-space.*

If there exists a μ-nontrivial Borel homomorphism from E_G to E_H^Y, then there exists a virtual embedding $\pi : G \to H$.

Proof of Theorem

- Suppose that $f : (2)^G \to Y$ is a μ-nontrivial Borel homomorphism from E_G to E_H^Y.
- Then we can define a Borel cocycle $\alpha : G \times (2)^G \to H$ by

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

- By Popa, after deleting a nullset and slightly adjusting f, we can suppose that $\alpha : G \to H$ is a group homomorphism.
- Suppose that $K = \ker \alpha$ is infinite.
- First note that if $k \in K$, then $f(k \cdot x) = \alpha(k) \cdot f(x) = f(x)$ and so $f : (2)^G \to X$ is K-invariant.
- Next note that since the action of G is strongly mixing, it follows that K acts ergodically on $(\,(2)^G, \mu\,)$.
- But then the K-invariant function $f : (2)^G \to X$ is μ-a.e. constant and so f is μ-trivial, which is a contradiction!

Torsion-free abelian groups of finite rank

Definition

An additive subgroup $G \leqslant \mathbb{Q}^n$ has rank n iff G contains n linearly independent elements.

Definition

Let $\cong_n$ denote the isomorphism relation on the standard Borel space $R(\mathbb{Q}^n)$ of torsion-free abelian groups of rank n.

Recall that if $A, B \in R(\mathbb{Q}^n)$, then

$$A \cong B \quad \text{iff} \quad \exists g \in GL_n(\mathbb{Q}) \quad g(A) = B.$$

In other words, $\cong_n$ is the orbit equivalence relation for the action of $GL_n(\mathbb{Q})$ on the space $R(\mathbb{Q}^n)$.

Some History

- In 1937, Baer gave a satisfactory classification of the rank 1 groups. (In fact, the isomorphism relation is hyperfinite.)
- In 1938, Kurosh and Malcev independently gave an unsatisfactory classification of the higher rank groups.

Problem (Fuchs 1973)

Characterize the torsion-free abelian groups of rank 2 by invariants.

Conjecture (Hjorth-Kechris 1996)

The isomorphism relation for the torsion-free abelian groups of rank 2 is countable universal.

- In 1998, Hjorth proved that the classification problem for the rank 2 groups was strictly harder than that for the rank 1 groups.

An application of Superrigidity

Theorem (Thomas 2000)

The complexity of the classification problems for the torsion-free abelian groups of rank n increases strictly with the rank n.

Corollary

For each $n \geq 1$, the isomorphism relation for the torsion-free abelian groups of rank n is not countable universal.

A slightly embarrassing question

Is the isomorphism relation on the space of torsion-free abelian groups of finite rank countable universal?

Theorem (Thomas 2006)

The isomorphism relation on the space of torsion-free abelian groups of finite rank is not countable universal.

E_0-ergodicity

Definition

- *Let E, F be countable Borel equivalence relations on X, Y and let μ be an E-invariant probability measure on X.*
- *Then E is F-ergodic iff every Borel homomorphism $f : X \to Y$ from E to F is μ-trivial.*

Remark

- Thus $\mathrm{id}_{\mathbb{R}}$-ergodicity coincides with the usual ergodicity.
- Clearly if E is F-ergodic and $F' \leq_B F$, then E is also F'-ergodic.

E_0-ergodicity

Theorem (Jones-Schmidt)

E is E_0-ergodic iff E has no “nontrivial almost invariant subsets”.

Definition

Let $E = E_G^X$ be a countable Borel equivalence relation and let μ be an E-invariant probability measure on X. Then E has nontrivial almost invariant subsets iff there exists a sequence of Borel subsets

$$\langle A_n \subseteq X \mid n \in \mathbb{N} \rangle$$

satisfying the following conditions:

- $\mu(g \cdot A_n \triangle A_n) \to 0$ *for all* $g \in G$.
- *There exists* $\delta > 0$ *such that* $\delta < \mu(A_n) < 1 - \delta$ *for all* $n \in \mathbb{N}$.

E_0-ergodicity

Theorem (Jones-Schmidt)

Let G be a countable group and let $H \leqslant G$ be a nonamenable subgroup. Then the shift action of H on $(\, (2)^G, \mu\,)$ is E_0-ergodic.

Remark

The proof makes use of the associated unitary representation of H on the Hilbert space $L^2(\, (2)^G, \mu\,)$.

Remark

For later use, note that if E is E_0-ergodic and F is hyperfinite, then E is F-ergodic.

The non-universality proof begins

- Let S be a suitably chosen countable group and let $G = SL_3(\mathbb{Z}) \times S$.
- Let $E = E_G$ be the orbit equivalence relation of the action of G on $(\,(2)^G, \mu\,)$.
- Suppose that
$$f : (2)^G \to \bigsqcup_{n \geq 1} R(\mathbb{Q}^n)$$
is a Borel reduction from E to the isomorphism relation.
- After deleting a nullset, we can suppose that
$$f : (2)^G \to R(\mathbb{Q}^n)$$
for some fixed $n \geq 1$.

The non-universality proof begins

- At this point, we would like to define a corresponding Borel cocycle

$$\alpha : G \times (2)^G \to GL_n(\mathbb{Q}).$$

- Then we could choose S to be a group which doesn't embed into $GL_n(\mathbb{Q})$ for all n.
- Unfortunately $GL_n(\mathbb{Q})$ doesn't act freely on $R(\mathbb{Q}^n)$.
- In fact, if $B \in R(\mathbb{Q}^n)$, then the stabilizer of B in $GL_n(\mathbb{Q})$ is precisely its automorphism group $\mathrm{Aut}(B)$.
- What to do? Change the category!

The quasi-equality relation

Definition

If $A, B \in R(\mathbb{Q}^n)$, then A and B are said to be quasi-equal, written $A \approx_n B$, iff $A \cap B$ has finite index in both A and B.

Theorem (Thomas)

The quasi-equality relation $\approx_n$ is hyperfinite.

For each $A \in R(\mathbb{Q}^n)$, let $[A]$ be the $\approx_n$-class containing A. We shall consider the induced action of $GL_n(\mathbb{Q})$ on

$$X = \{[A] \mid A \in R(\mathbb{Q}^n)\}$$

of $\approx_n$-classes. (Of course, X is not a standard Borel space.)

Stabilizers of $\approx_n$-classes

Definition

For each $A \in R(\mathbb{Q}^n)$, the ring of quasi-endomorphisms is

$$\mathrm{QE}(A) = \{\varphi \in \mathrm{Mat}_n(\mathbb{Q}) \mid (\exists m \geq 1)\, m\varphi \in \mathrm{End}(A)\}.$$

Clearly $\mathrm{QE}(A)$ is a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$; and so there are only countably many possibilities for $\mathrm{QE}(A)$.

Definition

$\mathrm{QAut}(A)$ *is the group of units of the $\mathbb{Q}$-algebra* $\mathrm{QE}(A)$.

Lemma

If $A \in R(\mathbb{Q}^n)$, then $\mathrm{QAut}(A)$ is the setwise stabilizer of $[A]$ in $GL_n(\mathbb{Q})$.

Defining the cocycle

- For each $x \in (2)^G$, let $A_x = f(x) \in R(\mathbb{Q}^n)$.
- After deleting a nullset and slightly adjusting f, we can suppose that the setwise stabilizer of each $[A_x]$ is a fixed subgroup $L \leqslant GL_n(\mathbb{Q})$.
- Note that the quotient group $H = N_{GL_n(\mathbb{Q})}(L)/L$ acts freely on the corresponding set $Y = \{[A] \mid \mathrm{QAut}(A) = L\}$ of $\approx_n$-classes.
- Hence we can define a corresponding cocycle

$$\alpha : G \times (2)^G \to H$$

by setting

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot [A_x] = [A_{g \cdot x}].$$

A suitably chosen S

- Let S be a countable simple nonamenable group which does not embed into any of the countably many possibilities for H.
- By Popa, after deleting a nullset and slightly adjusting f, we can suppose that
$$\alpha : G = SL_3(\mathbb{Z}) \times S \to H$$
is a homomorphism.
- Since $S \leqslant \ker \alpha$, it follows that $f : (2)^G \to R(\mathbb{Q}^n)$ is a Borel homomorphism from the S-action on $(2)^G$ to the hyperfinite quasi-equality $\approx_n$-relation.
- Since S is nonamenable, the S-action on $(2)^G$ is E_0-ergodic and hence μ-almost all x are mapped to a single $\approx_n$-class, which is a contradiction.