Appalachian Set Theory
Workshop on Coherent Sequences
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1 Introduction

2 Preliminaries

Let's begin by defining the basic structure for our work. A *C-sequence* is a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ so that the following holds for any $\alpha < \omega_1$,

1. $C_{\alpha+1} = \{\alpha\}$, and
2. if α is a non-zero limit ordinal, then
   (a) $\sup C_\alpha = \alpha$
   (b) o.t.$(C_\alpha) = \omega$, where o.t. stands for order type, and
   (c) C_α does not contain any succesor ordinal.

For the rest of the notes C_α will always denote the αth term of our C-sequence.

The porpouse of this section is to stablish some of the basic structures and properties linked to a C-sequence. We start with the upper and full lower trace.

2.1 Definition. Let $\alpha < \beta < \omega_1$.

1. The *upper trace of the walk from β to α* is defined as $\mathrm{Tr}(\alpha, \beta) = \langle \beta_i : i \leq n \rangle$, where
   (a) $\beta_0 = \beta$,
   (b) $\beta_n = \alpha$, and
   (c) $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$, for $i < n$.

2. The *full lower trace of the walk from β to α* is defined recursively as

$$F(\alpha,\beta) := F(\alpha,\min(C_\beta \setminus \alpha)) \cup \bigcup\{F(\xi,\alpha) : \xi \in C_\beta \cap \alpha\},$$

and $F(\alpha,\alpha) := \{\alpha\}$.

Note that $\mathrm{Tr}(\alpha,\beta)$ is a decreasing sequence.
The following statements can be proven by induction on γ.

2.2 Proposition. If $\alpha \le \beta \le \gamma$, then

1. $F(\alpha,\gamma) \subseteq F(\alpha,\beta) \cup F(\beta,\gamma)$.
2. $F(\alpha,\beta) \subseteq F(\alpha,\gamma) \cup F(\beta,\gamma)$.

Assume that c is a function with domain $[\omega_1]^2$. We will use the symbol $c(\alpha,\beta)$ to denote $c(\{\alpha,\beta\})$ when $\alpha < \beta$. It will be a common practice to use recursion to define the value of $c(\alpha,\beta)$, and sometimes the expression $c(\alpha,\alpha)$ will be involved!

If s and t are sequences, $s^\frown t$ is the concatenation of s followed by t.

2.3 Definition. The *full code* of the walk is the function $\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$ given by

$$\rho_0(\alpha,\beta) := \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha,\min(C_\beta \setminus \alpha)),$$

and $\rho_0(\alpha,\alpha) := \emptyset$.

Note that if we know the full code, then we can get the upper trace by

$$\mathrm{Tr}(\alpha,\beta) = \{\xi : \rho_0(\xi,\beta) \sqsubseteq \rho_0(\alpha,\beta)\},$$

where $s \sqsubseteq t$ means that s is an initial segment of the sequence t.

2.4 Lemma. Let $\xi < \alpha < \beta$. If $\overline{\alpha_\xi} := \min(F(\alpha,\beta) \setminus \xi)$, then

1. $\rho_0(\xi,\alpha) = \rho_0(\overline{\alpha_\xi},\alpha)^\frown \rho_0(\xi,\overline{\alpha_\xi})$.
2. $\rho_0(\xi,\beta) = \rho_0(\overline{\alpha_\xi},\beta)^\frown \rho_0(\xi,\overline{\alpha_\xi})$.

The *right lexicographical order*, $<_r$, on $\omega^{<\omega}$ is the linear ordering defined by letting

$$s <_r t \text{ iff } t \sqsubset s \text{ or } s(m) < t(m), \text{ where } m := \min\{i : s(i) \ne t(i)\}.$$

2.5 Lemma. If $\xi_0 < \xi_1 < \beta$, then $\rho_0(\xi_0,\beta) <_r \rho_0(\xi_1,\beta)$. In other words, the function $\rho(\,\cdot\,,\beta) \restriction \alpha$ is strictly increasing when $\alpha \le \beta$.

Given $\alpha \le \beta < \omega_1$, let $\rho_{0\beta} \restriction \alpha := \{\rho_0(\xi,\beta) : \xi < \alpha\}$. Now define

$$T(\rho_0) := \{\rho_{0\beta} \restriction \alpha : \alpha \le \beta < \omega_1\}$$

and order it by end-extension, i.e. for all $a, b \in T(\rho_0)$ let $a <_e b$ iff

$$a \subset b \text{ and } (\forall x \in a)(\forall y \in b \setminus a)(x <_r y).$$

Then we have the following result.

2.6 Proposition. $T(\rho_0)$ is an Aronszajn tree.

2.7 Definition. Let $\mathrm{Tr}(\alpha,\beta) = \langle \beta_i : i \le n \rangle$. The *lower trace of the walk from β to α* is the increasing sequence $L(\alpha,\beta) = \langle \lambda_i : i < n \rangle$, where

$$\lambda_i = \max\{\max(C_\xi \cap \alpha) : \rho_0(\xi,\beta) \sqsubseteq \rho_0(\beta_i,\beta)\}$$

for any $i < n$.

Observe that $L(\alpha,\beta) \subseteq F(\alpha,\beta)$.

Let a and b be two sets (or sequences) of ordinals. The symbol $a < b$ means that any ordinal from a is smaller than any element of b. We will write $a < \alpha$ instead of $a < \{\alpha\}$.

2.8 Proposition. If $L(\alpha,\beta) < \xi < \alpha < \beta$, then $\rho_0(\xi,\beta) = \rho_0(\alpha,\beta)^\frown\rho_0(\xi,\alpha)$.

3 Second Session

3.1 Definition. The *maximal weight* of the walk is the function $\rho_1 : [\omega_1]^2 \to \omega$ given by

$$\rho_1(\alpha,\beta) := \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\},$$

and $\rho_1(\alpha,\alpha) = 0$.

In other words, $\rho_1(\alpha,\beta)$ is the largest integer appearing in the sequence $\rho_0(\alpha,\beta)$.

3.2 Proposition. For all $\alpha < \beta < \omega_1$ and $n < \omega$, the following sets are finite

1. $\{\xi < \alpha : \rho_1(\xi,\alpha) \neq \rho_1(\xi,\beta)\}$.
2. $\{\xi < \alpha : \rho_1(\xi,\alpha) \le n\}$.

If s and t are sequences with the same domain, then $s =^* t$ means that $\{\xi : s(\xi) \neq t(\xi)\}$ is finite.

3.3 Proposition. The set $T(\rho_1) := \{t \in \omega^{<\omega_1} : t =^* \rho_1(\,\cdot\,, \mathrm{dom}(t))\}$ ordered by the relative ordering from $\omega^{<\omega_1}$ is an $\mathbb{R}$-embeddable (i.e. there exists an increasing map from $T(\rho_1)$ to $\mathbb{R}$) Aronszajn tree.

Let T be an arbitrary tree. Given $x, y \in T$, define

$$\Delta(x,y) := \mathrm{o.t.}(\{t \in T : t \le x \wedge t \le y\}).$$

One should view Δ as some sort of distance function on T by interpreting inequalities like $\Delta(x,y) > \Delta(x,z)$ as saying that x is closer to y than z. A map $g \subseteq T \times T$ is *Lipschtiz*, if g is level preserving and $\Delta(g(x), g(y)) \ge \Delta(x,y)$ for all $x, y \in \mathrm{dom}(g)$. Finally, T is *Lipschtiz* if every function $f \subseteq T \times T$ with uncountable domain which is level preserving is Lipschitz on an uncountable subset of $\mathrm{dom}(f)$ or f^{-1} is Lipschitz on an uncountable subset of its domain.

3.4 Proposition. $T(\rho_0)$ and $T(\rho_1)$ are Lipschitz trees.

3.5 Question. Let T be a tree. Does

$$\mathcal{U}(T) := \{A \subseteq \omega_1 : (\exists X \subseteq T)(|X| > \omega \wedge \{\Delta(x, y) : x, y \in X\} \subseteq A)\}$$

extend the club filter? The conjecture is **no**.

Remember that a set $X \subseteq \mathbb{R}$ has *strong measure zero* if for every sequence of positive real numbers $\langle \varepsilon_n : n < \omega \rangle$ there is a sequence $\langle I_n : n < \omega \rangle$ of open intervals so that $X \subseteq \bigcup_n I_n$ and the diameter of I_n is at most ε_n. This metric property has several closely related topological notions. For example *Rothberger's property C''*: for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of a space X one can choose $U_n \in \mathcal{U}_n$ for each n so that $X = \bigcup_n U_n$.

3.6 Proposition. $T(\rho_1)$ has property C'' when considered with the topology generated by the sets $\{s \in T(\rho_1) : t \subseteq s\}$, $t \in T(\rho_1)$.

4 Third Session

4.1 Definition. The *number of steps* of the minimal walk from β to α is defined by

$$\rho_2(\alpha, \beta) := \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1,$$

and $\rho_2(\alpha, \alpha) := 0$. In other words, $\rho_2(\alpha, \beta)$ is the size of $\mathrm{Tr}(\alpha, \beta)$.

Observe that $\rho_2 : [\omega_1]^2 \to \omega$. This is an interesting map which is particularly useful on higher cardinalities.

4.2 Definition. The *last step function* $\rho_3 : [\omega_1]^2 \to 2$ is defined by letting $\rho_3(\alpha, \beta) = 1$ iff the last element of the sequence $\rho_0(\alpha, \beta)$ is $\rho_1(\alpha, \beta)$. In other words, we let $\rho_3(\alpha, \beta) = 1$ only in case the last step of the walk from β to α comes with the maximal weight.

The key idea to prove the next result is Lemma 2.4.

4.3 Proposition. ρ_3 satisfies the following

1. For all $\alpha < \beta < \omega_1$,

   $$\rho_3(\,\cdot\,, \beta) =^* \rho_3(\,\cdot\,, \beta) \restriction \alpha,$$

   in other words, the set $\{\xi < \alpha : \rho_3(\xi, \alpha) \neq \rho_3(\xi, \beta)\}$ is finite.

2. There is no function $h : \omega_1 \to 2$ so that $h \restriction \alpha =^* \rho_3(\,\cdot\,, \alpha)$ for every $\alpha < \omega_1$.

Now observe that $T(\rho_3) := \{\rho_3(\,\cdot\,, \beta) \restriction \alpha : \alpha \leq \beta < \omega_1\}$ is a tree with the relative ordering from $2^{<\omega_1}$.

4.4 Proposition. The following holds.

1. $T(\rho_3)$ is an Aronszajn tree.
2. $T(\rho_3)$ is not necessarily $\mathbb{R}$-embeddable.
3. Every uncountable $X \subseteq T(\rho_3)$ contains an uncountable $Y \subseteq X$ such that the cartesian square $(Y \times Y, <_\ell)$ is the union of countable many antichains, where $<_\ell$ is the lexicographical ordering.

We say that a function $c : [\omega_1]^2 \to \omega$ is *unbounded* if for any uncountable subset $A \subseteq \omega_1$, the set $c''[A]^2$ is unbounded in ω. The number of steps function satisfies this condition and even more:

4.5 Lemma. For every uncountable familiy A consisting of pairwise disjoint finite subsets of ω_1 and for all $n < \omega$ there exists an uncountable $B \subseteq A$ so that for any pair $a, b \in B$ with $a < b$ we have

$$(\forall \alpha \in a)(\forall \beta \in b)(\rho_2(\alpha, \beta) > n).$$

5 Fourth Session

The work we have done for ω_1 can be generalized for any cardinal number θ as follows.

5.1 Definition. $\langle C_\alpha : \alpha < \theta \rangle$ is a *C-sequence* in θ if for any $\alpha < \theta$ we have

1. C_α is a club in α whenever α is a limit ordinal,
2. $C_{\alpha+1} = \{\alpha\}$, and
3. for all $\beta \in C_\alpha$, if o.t.$(C_\alpha \cap \beta)$ is a succesor, then β is a succesor too.

From now on, $\langle C_\alpha : \alpha < \theta \rangle$ will be always a C-sequenece on θ.

Observe that our definition of $\mathrm{Tr}(\alpha, \beta)$ makes perfect sense when $\alpha < \beta < \theta$. In the case of ρ_0 a slight modification is needed:

$$\rho_0(\alpha, \beta) := \langle \text{o.t.}(C_\beta \cap \alpha) \rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

and $\rho_0(\alpha, \alpha) := \emptyset$. Note that $\rho_0 : [\theta]^2 \to \theta^{<\omega}$.

A quick review of the notions involved in the definition of the tree $T(\rho_0)$ shows that all of them make sense when one changes ω_1 by θ (and $\theta^{<\omega}$ by $\omega^{<\omega}$ in the case of the right lexicographical ordering).

The *maximal weight* and the *number of steps* functions are defined by

1. $\rho_1(\alpha, \beta) := \max\{\text{o.t.}(C_\beta \cap \alpha), \rho_1(\alpha, \min(C_\beta \setminus \alpha)\}$ with boundary value $\rho_1(\alpha, \alpha) := 0$, and
2. $\rho_2(\alpha, \beta) := \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1$ with boundary value $\rho_2(\alpha, \alpha) := 0$,

respectively

Without any doubt the C-sequence $C_\alpha = \alpha$ is the most trivial choice. The following notion of triviality seems to be only marginally different.

5.2 Definition. $\langle C_\alpha : \alpha < \theta \rangle$ is *trivial* if there exists a club $C \subseteq \theta$ so that

$$(\forall \alpha < \theta)(\exists \beta \geq \alpha)(C \cap \alpha \subseteq C_\beta).$$

5.3 Theorem. The following are equivalent for any regular uncountable θ.

1. $\langle C_\alpha : \alpha < \theta \rangle$ is not trivial.
2. For every family A of θ pairwise disjoint finite subsets of θ and every integer n, there exists a subfamily B of A of size θ such that $\rho_2(\alpha, \beta) > n$ for all $\alpha \in a$, $\beta \in b$, and $a < b$ in B.

5.4 Question. Can you characterize weak compactness of θ by the following property: For all $f : [\theta]^2 \to 3$ there exists an unbounded $X \subseteq \theta$ so that $f''[X]^2 \neq 3$?

Let x and y be arbitrary subsets of θ. Define

$$\operatorname{osc}(x, y) := |(x \setminus (\sup(x \cap y) + 1)) / \sim|,$$

where $\sim$ is the equivalence relation on $x \setminus (\sup(x \cap y) + 1)$ defined by letting $\alpha \sim \beta$ iff the closed interval determined by α and β contains no point from y. So, if x and y are disjoint, $\operatorname{osc}(x, y)$ is simply the number of convex pieces in which the set x is split by the set y. The oscillation map has proven to be a useful device in various schemes for coding information.

5.5 Definition. A family $\mathfrak{X} \subseteq \mathcal{P}(\theta)$ is *unbounded* if for every club $C \subseteq \theta$ there exists $x \in \mathfrak{X}$ and an increasing sequence $\langle \delta_n : n < \omega \rangle \subseteq C$ such that $\sup(x \cap \delta_n) < \delta_n$ and $[\delta_n, \delta_{n+1}) \cap x \neq \emptyset$, for all $n < \omega$.

This notion of unboundedness has proven to be the key behind a number of results asserting the complex behaviour of the oscillation map on $\mathfrak{X}^2$.

5.6 Lemma. If $\mathfrak{X}$ is an unbounded family consisting of closed and unbounded subsets of θ, then for every positive integer n there exist $x, y \in \mathfrak{X}$ such that $\operatorname{osc}(x, y) = n$.

Let $\Gamma \subseteq \theta$ be arbitrary. We say that $\langle C_\alpha : \alpha \in \Gamma \rangle$ is a *stationary subsequence* if the set

$$\bigcup_{\alpha \in \Gamma} \{\xi < \theta : \sup(C_\alpha \cap \xi) = \xi\}$$

is stationary in θ.

5.7 Lemma. Any stationary subsequence of a nontrivial C-sequence is an unbounded family.

6 Fifth Session

It is known that nontrivial C-sequences exist only on successor cardinals. In fact it is possible to show that nontrivial C-sequences exist for some inaccessible cardinals quite high in the Mahlo-hierarchy. To show how close this is to the notion of weak compactness, we will give the following characterization.

6.1 Theorem. The following are equivalent for an inaccessible θ.

1. θ is weakly compact.
2. For any C-sequence $\langle C_\alpha : \alpha < \theta \rangle$ there is a club C such that

$$(\forall \alpha)(\exists \beta \geq \alpha)(C_\beta \cap \alpha = C \cap \alpha)$$

It turns out that in the previous result we cannot replace (2) by *every C-sequence on θ is trivial*. One can show this using a model of Kunen [5].

One of the most basic questions frequently asked about set-theoretical trees is the question whether they contain a *cofinal branch*, a branch that intersects every level of the tree. The fundamental importance of this question has already been realized in the work of Kurepa [6] and then later in the work of Erdős and Tarski in their respective attempts to develop the theory of partition calculus [2] and large cardinals. A tree T of height equal to some regular cardinal θ may not have a cofinal branch for a very special reason as the following definition indicates.

6.2 Definition. Let $T = \langle T, <_T \rangle$ be a tree.

1. A function $f : T \to T$ is *regressive* if $f(t) <_T t$ for every $t \in T$ that is not a minimal node of T.
2. If T has height θ, then T is *special* if there is a regressive map $f : T \to T$ such that $f^{-1}[t]$ can be covered by less than θ antichains in T.

When $\theta = \omega_1$, this definition reduces to the old concept of special tree: a tree that can be decomposed into countably many antichains. Moreover, we have that if $\theta = \kappa^+$ and T has height θ, then T is special if and only if T is the union of at mos κ antichains.

Recall the well-known characterization of weakly compact cardinals due to Tarski and his collaborators: a strongly inaccesssible cardinal θ is weakly compact iff there are no Aronszajn trees of height θ. Using C-sequences we can prove the following.

6.3 Theorem. The following are equivalent for a strongly inacccesssible θ:

1. θ is Mahlo.
2. There are no special Aronszajn trees of height θ.

For each set $X \subseteq \theta$, denote by X' the set of all limit points of X.

6.4 Definition. A C-sequence $\langle C_\alpha : \alpha < \theta\rangle$ is a $\square$-*sequence* if it is *coherent*, i.e. we have $C_\alpha = C_\beta \cap \alpha$ whenever $\alpha \in C'_\beta$.

Observe that our notion of triviality and the condition mentioned in Theorem 6.1 coincide in the realm of $\square$-sequences:

6.5 Lemma. A $\square$-sequence $\langle C_\alpha : \alpha < \theta\rangle$ is trivial if and only if there is a club $C \subseteq \theta$ so that $C_\alpha = C \cap \alpha$ for all $\alpha \in C'$.

It is known that ω_1 admits a nontrivial $\square$-sequence.

Define the function $\Lambda : [\theta]^2 \to \theta$ by

$$\Lambda(\alpha, \beta) := \max(\{0\} \cup (C_\beta \cap (\alpha + 1))').$$

With the aid of Λ we are ready to define $F : [\theta]^2 \to [\theta]^{<\omega}$, the *full lower trace* function:

$$F(\alpha, \beta) := F(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup\{F(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\},$$

with $F(\alpha, \alpha) := \{\alpha\}$ for all $\alpha < \theta$.

Under the assumptions that $\langle C_\alpha : \alpha < \theta\rangle$ is a nontrivial $\square$-sequence and θ is a regular uncountable cardinal, Proposition 2.2 and Lemma 2.4 hold, and therefore we get the following result.

6.6 Corollary. For any nontrivial $\square$-sequence on a regular uncountable θ we have

$$\sup_{\xi<\alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \infty$$

for all $\alpha < \beta < \theta$.

Proof. Apply Lemma 2.4 to obtain

$$\sup_{\xi<\alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| \leq \sup_{\xi\in F(\alpha,\beta)} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|.$$

$\square$

It is a known fact that there is no function $h : \theta \to \omega$ so that

$$\sup_{\xi<\alpha} |\rho_2(\xi, \alpha) - h(\xi)| < \infty$$

for all $\alpha < \theta$.

Let $\mathfrak{I}$ be an ideal of subsets of some set S. Recall that $\mathfrak{I}$ is a P*-ideal* if for every sequence $\langle A_n : n < \omega\rangle \subseteq \mathfrak{I}$ there is $B \in \mathfrak{I}$ so that $A_n \setminus B$ is finite for all $n < \omega$. A set $X \subseteq S$ is *orthogonal* to $\mathfrak{I}$ (in symbols, $X \perp \mathfrak{I}$) if $X \cap A$ is finite for all $A \in \mathfrak{I}$.

The following statement is known as the P*-ideal Dichotomy*: For every P-ideal $\mathfrak{I}$ of countable subsets of some set S either

1. There is an uncountable $X \subseteq S$ such that $[X]^\omega \subseteq \mathfrak{I}$, or

2. S can be decomposed into countably many sets orthogonal to $\mathfrak{I}$.

The P-ideal dichotomy is a consequence of the Proper Forcing Axiom and, moreover, it does not contradict the Continuum Hypothesis [8].

6.7 Theorem. If θ is regular and uncountable, then the P-ideal Dichotomy implies that a nontrivial $\square$-sequence can exist only on $\theta = \omega_1$.

Proof. The family

$$\mathfrak{I} := \left\{ A \in [\theta]^\omega : (\forall \alpha < \theta)(\forall \Delta \in [A \cap \alpha]^\omega) \left(\sup_{\xi \in \Delta} \rho_2(\xi, \alpha) = \infty \right) \right\}$$

is a P-ideal of countable subsets of θ. Thus we have two possibilities:

1. There is an uncountable $X \subseteq \theta$ so that $[X]^\omega \subseteq \mathfrak{I}$, or
2. There is a decomposition $\theta = \bigcup_n O_n$ so that $O_n \perp \mathfrak{I}$ for all $n < \omega$.

If (1) holds, Corollary 6.6 implies that $X \cap \alpha$ is countable for each $\alpha < \theta$ and thus θ must have cofinality ω_1. Therefore $\theta = \omega_1$.

Now assume (2) is true. Fix $k < \omega$ so that O_k is unbounded in θ. Since $O_k \perp \mathfrak{I}$ we have that $\rho_2(\,\cdot\,, \alpha)$ is bounded in $O_k \cap \alpha$ for each $\alpha < \theta$. Hence there exist an unbounded set $\Gamma \subseteq \theta$ and an integer m such that $\rho_2(\alpha, \beta) \leq m$ for any $\beta \in \Gamma$ and $\alpha \in O_k \cap \beta$. Theorem 5.3 implies that the $\square$-sequence we started with must be trivial. $\square$

The *tightness* of a point x in a topological space X, $t(x, X)$, is equal to κ if κ is the minimal cardinal such that for any set $A \subseteq X \setminus \{x\}$, if $x \in \overline{A}$ (the clousure of A), then there exists $B \in [A]^{\leq \kappa}$ so that $x \in \overline{B}$. The tightness of X is $\sup\{t(x, X) : x \in X\}$.

The *sequential fan* with θ edges is the space obtained on $(\theta \times \omega) \cup \{\infty\}$ by declaring ∞ as the only nonisolated point, while a typical neighborhood for ∞ has the form

$$U_f := \{(\alpha, n) : \alpha < \theta,\ n \geq f(\alpha)\} \cup \{\infty\},$$

where $f : \theta \to \omega$ is arbitrary. We will denote this space by S_θ.

6.8 Theorem. Let θ be regular and uncountable. If there is a nontrivial $\square$-sequence on θ, then the tightness of (∞, ∞) in the topological product $S_\theta^2 := S_\theta \times S_\theta$ is equal to θ.

Proof. Assume that $\langle C_\alpha : \alpha < \theta \rangle$ is a nontrivial $\square$-sequence on θ and define $d : [\theta]^2 \to \omega$ by

$$d(\alpha, \beta) := \sup_{\xi \leq \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|.$$

For each $\gamma \leq \theta$ let $W_\gamma := \{((\alpha, d(\alpha, \beta)), (\beta, d(\alpha, \beta))) : \alpha < \beta < \gamma\}$. W_θ is a subset of $S_\theta^2 \setminus \{(\infty, \infty)\}$ of size θ and $(\infty, \infty) \in \overline{W_\theta}$. On the other hand,

$(\infty, \infty) \notin \overline{W_\xi}$ for all $\xi < \theta$. Finally, if $B \in [W_\theta]^{<\theta}$, then there exists $\gamma < \theta$ so that $B \subseteq W_\gamma$ and hence B does not accumulate to (∞, ∞). □

Since ω_1 supports a nontrivial □-sequence, the previous result leads to the following result of Gruenhage and Tanaka [3].

6.9 Corollary. $S^2_{\omega_1}$ is not countably tight.

We have seen that the case $\theta = \omega_1$ is quite special when one considers the problem of existence of various nontrivial □-sequences on θ. It should be noted that a similar result about the problem of the tightness of S^2_θ is not available. In particular, it is not known whether the P-ideal dichotomy or a similar consistent hypothesis of set theory implies that the tightness of the square of, say, S_{ω_2} is smaller than ω_2. It is interesting that considerably more is known about the dual question, the question of initial compactness of the Tychonoff cube ω^θ. For example, if one defines $B_{\alpha\beta} := \{f \in \omega^\theta : f(\alpha),\ f(\beta) \leq d(\alpha, \beta)\}$ $(\alpha < \beta < \theta)$ one gets an open cover of ω^θ without a subcover of size $< \theta$. However, for small θ such as $\theta = \omega_2$ one is able to find such a cover of ω^θ without any additional set-theoretic assumption and in particular without the assumption that θ carries a nontrivial □-sequence.

6.10 Question. What is the tightness of $S^2_{\omega_2}$?

7 Sixth Session

From now on let's fix a □-sequence $\langle C_\alpha : \alpha < \theta \rangle$ on a regular uncountable cardinal θ. We are going to discuss some properties of the distance function $\rho : [\theta]^2 \to \theta$ defined by

$$\rho(\alpha, \beta) := \max\{\text{o.t.}(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\},$$

where we stipulate $\rho(\alpha, \alpha) = 0$ for all $\alpha < \theta$. ρ has the following subadditive properties.

7.1 Lemma. If $\alpha < \beta < \theta$ then

1. $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$
2. $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$

Let $D : [\theta]^2 \to [\theta]^{<\theta}$ be defined by

$$D(\alpha, \beta) := \{\xi < \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}.$$

Note that $D(\alpha, \beta) = \{\xi < \alpha : \rho(\xi, \beta) \leq \rho(\alpha, \beta)\}$ so we could take the formula

$$D\{\alpha, \beta\} = \{\xi < \min\{\alpha, \beta\} : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}$$

as our definition of $D\{\alpha, \beta\}$ when there is no implicit assumption about the ordering between α and β as there is whenever we write $D(\alpha, \beta)$.

Recall that a cardinal κ is λ-inaccessible if $\nu^\tau < \kappa$ for all $\nu < \kappa$ and $\tau < \lambda$.

7.2 Lemma. If κ is λ-inaccessible for some $\lambda < \kappa$ and $\theta = \kappa^+$, then for every family $A \subseteq [\theta]^{<\lambda}$ with $|A| = \kappa$ there exists $B \in [A]^\kappa$ such that for all $a, b \in B$ and all $\alpha \in a \setminus b$, $\beta \in b \setminus a$, and $\gamma \in a \cap b$:

1. $\alpha, \beta > \gamma$ implies $D\{\alpha, \gamma\} \cup D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$,
2. $\beta > \gamma$ implies $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$,
3. $\alpha > \gamma$ implies $D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$, and
4. $\gamma > \alpha, \beta$ implies $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$ or $D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$.

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has *property* Δ if for every uncountable set A of finite subsets of ω_2 there existst a and b in A such that for all $\alpha \in a \setminus b$, $\beta \in b \setminus a$, and $\gamma \in a \cap b$:

1. $\alpha, \beta > \gamma$ implies $\gamma \in f\{\alpha, \beta\}$.
2. $\beta > \gamma$ implies $f\{\alpha, \gamma\} \subseteq f\{\alpha, \beta\}$.
3. $\alpha > \gamma$ implies $f\{\beta, \gamma\} \subseteq f\{\alpha, \beta\}$.

This definition is due to Baumgartner and Shelah [1] who have used it in their well-known forcing construction. It should be noted that they were also able to force a function with the property Δ using a σ-closed ω_2-cc poset.

As shown above the function D has property Δ. However Lemma 7.2 shows that D has many other properties that are of independent interest and that are likely to be needed in other forcing constructions of this sort. The papers of Koszmider [4] and Rabus [7] are good examples of further work in this area.

There are some generalizations of the ρ-functions we have analyzed. Recall that an ordinal α divides an ordinal γ if there there is β such that $\gamma = \alpha \cdot \beta$, i.e. γ can be written as the union of an increasing β-sequence of intervals of order type α. Let $\kappa \leq \theta$ be a fixed infinite regular cardinal. Let $\Lambda_\kappa : [\theta]^2 \to \theta$ be defined by

$$\Lambda_\kappa(\alpha, \beta) := \max\{\xi \in C_\beta \cap (\alpha + 1) : \kappa \text{ divides } \text{o.t.}(C_\beta \cap \xi)\}.$$

Observe that the function Λ we introduced before is Λ_ω, i.e. $\Lambda = \Lambda_\omega$.

Our object of study is the function $\rho^\kappa : [\theta]^2 \to \theta$ defined recursively by

$$\rho^\kappa(\alpha, \beta) := \sup\{\text{o.t.}(C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha), \rho^\kappa(\alpha, \min(C_\beta \setminus \alpha)), \\ \rho^\kappa(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)\},$$

and $\rho^\kappa(\alpha, \alpha) = 0$ for all $\alpha < \theta$.

7.3 Lemma. If $\alpha < \beta < \theta$ then

1. $\rho^\kappa(\alpha, \gamma) \leq \max\{\rho^\kappa(\alpha, \beta), \rho^\kappa(\beta, \gamma)\}$
2. $\rho^\kappa(\alpha, \beta) \leq \max\{\rho^\kappa(\alpha, \gamma), \rho^\kappa(\beta, \gamma)\}$

For $\alpha < \beta < \theta$ and $\nu < \kappa$ set

$$\alpha <^{\kappa}_{\nu} \beta \text{ if and only if } \rho^{\kappa}(\alpha, \beta) \leq \nu.$$

The following result is a corollary of Lemma 7.3.

7.4 Proposition.

1. $(\theta, <^{\kappa}_{\nu})$ is a tree for all $\nu < \kappa$.
2. If $\nu < \mu < \kappa$ then $<^{\kappa}_{\nu} \subseteq <^{\kappa}_{\mu}$.
3. $\in \upharpoonright \theta = \bigcup_{\nu < \kappa} <^{\kappa}_{\nu}$.

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