On the strengths and weaknesses of weak squares

Menachem Magidor and Chris Lambie-Hanson

1 Introduction

The term "square" refers not just to one but to an entire family of combinatorial principles. The strongest is denoted by "$\square$" or by "Global $\square$," and there are many interesting weakenings of this notion. Before introducing any particular square principle, we provide some motivating applications. In this section, the term "square" will serve as a generic term for "some particular square principle."

- Jensen introduced square principles based on work regarding the fine structure of L. In his first application, he showed that, in L, there exist κ-Suslin trees for every uncountable cardinal κ that is not weakly compact.
- Let T be a countable theory with a distinguished predicate R. A model of T is said to be of type (λ, μ) if the cardinality of the model is λ and the cardinality of the model's interpretation of R is μ. For cardinals α, β, γ, and δ, $(\alpha, \beta) \to (\gamma, \delta)$ is the assertion that for every countable theory T, if T has a model of type (α, β), then it has a model of type (γ, δ). Chang showed that under GCH, $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ holds for every regular cardinal κ. Jensen later showed that under GCH+square, $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ holds for every singular cardinal κ as well.
- Square can be used to produce examples of incompactness, i.e. structures such that every substructure of a smaller cardinality has a certain property but the entire structure does not:
  - Square allows for the construction of a family of countable sets such that every subfamily of smaller cardinality has a transversal (i.e. a $1-1$ choice function) but the entire family does not.
  - Assuming square, one can construct a first countable topological space such that every subspace of smaller cardinality is metrizable but the entire space is not.
  - We say that an abelian group G is free if, for some index set I,

    $$G \approx \sum_{i \in I} \mathbb{Z}$$

    where $\sum$ denotes the direct sum. Square can be used to construct a group G such that G is not free but every subgroup of smaller cardinality is.
  - We say that an abelian group G is free$^+$ if, for some index set I,

    $$G \subseteq \prod_{i \in I} \mathbb{Z}$$

where $\prod$ denotes the direct product. Square can be used to construct a group G such that G is not free$^+$ but every subgroup of smaller cardinality is.

This chapter will further explore these and other applications of squares as well as the consistency strengths of the failures of certain square principles. In sections 2 and 3, we introduce basic square principles and derive some immediate consequences thereof. In section 4, we present forcing arguments to separate the strengths of different square principles. Section 5 deals with scales and their interactions with squares. In section 6, we provide two examples of incompactness that can be derived from square principles. In section 7, we present a stronger version of Jensen's original construction of Suslin trees from squares. In section 8, we consider the consistency strengths of the failures of square principles. Section 9 contains results regarding weak squares at singular cardinals.

2 Jensen's Original Square Principle

Definition Let κ be a cardinal. $\square_\kappa$ is the assertion that there exists a sequence $\langle C_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ such that for all α, β limit with $\kappa < \alpha < \beta < \kappa^+$, we have the following:

1. C_α is a closed, unbounded subset of α
2. otp$(C_\alpha) < \alpha$
3. (Coherence) If α is a limit point of C_β, then $C_\beta \cap \alpha = C_\alpha$.

Such a sequence is called a $\square_\kappa$-sequence and can be thought of as a canonical way of witnessing that the ordinals between κ and κ^+ are singular.

We start with a few easy observations about $\square_\kappa$-sequences.

Proposition 2.1 *If $\square_\kappa$ holds, then there is a $\square_\kappa$-sequence $\langle D_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ such that for all α, $\mathrm{otp}(D_\alpha) \leq \kappa$. In addition, if κ is singular, then we can require that for all α, otp$(D_\alpha) < \kappa$.*

Proof Suppose that $\langle C_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ is a $\square_\kappa$-sequence. We will define $\langle D_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+\rangle$ so that $\langle D_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ works. For $\kappa < \alpha < \kappa^+$, let $C^*_\alpha = C_\alpha \setminus \kappa$. We first define D_κ to be any club subset of κ of order-type cf(κ) (if κ is regular, we can let $D_\kappa = \kappa$). If δ is a limit point of D_κ, let $D_\delta = D_\kappa \cap \delta$. For all other limit ordinals $\delta < \kappa$, let $D_\delta = \delta \setminus \sup(D_\kappa \cap \delta)$. Recursively define $D_\alpha \subseteq C^*_\alpha$ for $\kappa < \alpha < \kappa^+$ by letting $D_\alpha = \{\gamma \mid \gamma \in C^*_\alpha, \mathrm{otp}(C_\alpha \cap \gamma) \in D_{\mathrm{otp}(C^*_\alpha)}\}$. It is easy to check by induction on α that $\langle D_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ is as desired. ∎

Notice that, if $\langle D_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ is a $\square_\kappa$-sequence as given in Proposition 2.1, if we let $D^*_\alpha = \alpha$ for limit $\alpha \leq \kappa$ and $D^*_\alpha = D_\alpha \setminus \kappa$ for $\kappa < \alpha < \kappa^+$, α limit, then $\langle D^*_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+\rangle$ satisfies, for all limit $\alpha < \beta < \kappa^+$:

1. D^*_α is a club in α

2. $\mathrm{otp}(D^*_\alpha) \leq \kappa$

3. If α is a limit point of D^*_β, then $D^*_\beta \cap \alpha = D^*_\alpha$.

Therefore, $\Box_\kappa$ is equivalent to the existence of such a sequence $\langle D^*_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+ \rangle$, and we will sometimes refer to such a sequence as a $\Box_\kappa$-sequence.

Soon after introducing this square principle, Jensen showed that, in L, $\Box_\kappa$ holds for every infinite cardinal κ. In fact, it is the case that in certain other canonical inner models (all Mitchell-Steel core models, for example), $\Box_\kappa$ holds for every infinite cardinal κ. The proof that $\Box_\kappa$ holds in L can be found in [4] and [7]. For more recent work concerning other inner models, see [10].

3 Weak Squares

A natural question to ask is whether one can weaken the square principle and still get interesting combinatorial results. One such weakening of square is given by the following notion, introduced by Schimmerling.

Definition $\Box_{\kappa,\lambda}$ is the assertion that there exists a sequence $\langle \mathcal{C}_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+ \rangle$ such that for all α, $|\mathcal{C}_\alpha| \leq \lambda$ and for every $C \in \mathcal{C}_\alpha$,

1. C is a club in α

2. $\mathrm{otp}(C) \leq \kappa$

3. If β is a limit point of C, then $C \cap \beta \in \mathcal{C}_\beta$

$\Box_{\kappa,<\lambda}$ is defined similarly, except, for each α, we require $|\mathcal{C}_\alpha| < \lambda$.

Note that $\Box_{\kappa,\lambda}$ weakens as λ grows. $\Box_{\kappa,1}$ is simply $\Box_\kappa$. $\Box_{\kappa,\kappa}$ is often called *weak square* and written as $\Box^*_\kappa$. $\Box_{\kappa,\kappa^+}$ is often called *silly square*. It is a theorem of ZFC that $\Box_{\kappa,\kappa^+}$ holds for every infinite cardinal κ: for every limit α such that $\kappa < \alpha < \kappa^+$, let C_α be a club in α. For limit β such that $\kappa < \beta < \kappa^+$, let $\mathcal{C}_\beta = \{C_\alpha \cap \beta \mid \alpha \text{ limit}, \beta < \alpha < \kappa^+\}$. It is easy to verify that $\langle \mathcal{C}_\beta \mid \beta \text{ limit}, \kappa < \beta < \kappa^+ \rangle$ is a $\Box_{\kappa,\kappa^+}$-sequence.

Definition Let κ be an infinite cardinal. A κ^+-Aronszajn tree T is *special* if there is a function $f : T \to \kappa$ such that, for all $x, y \in T$, if $x <_T y$, then $f(x) \neq f(y)$.

Theorem 3.1 *There is a special κ^+-Aronszajn tree if and only if $\Box^*_\kappa$ holds.*

Proof We will prove only the forward direction. The proof of the reverse direction can be found in [1].

Let T be a special κ^+-Aronszajn tree, as witnessed by $f : T \to \kappa$. Let U_α denote the nodes of T in level α. By thinning out the tree if necessary, we can assume without loss of generality that the nodes in a branch below a limit level β uniquely determine the node of the branch at level β. For α limit, $\kappa < \alpha < \kappa^+$, we define $\mathcal{C}_\alpha$ as follows.

Let $x \in U_\alpha$. We will construct, for some $\gamma \le \kappa$, $\langle x_\beta \mid \beta < \gamma \rangle$, an increasing sequence in the tree such that, for every $\beta < \gamma$, $x_\beta <_T x$. Let x_0 be the root of the tree. If x_α has been chosen and $x_\alpha <_T x$, let $\delta_\alpha = \min(\{f(y) \mid x_\alpha <_T y <_T x\}$. Let $x_{\alpha+1}$ be the unique y such that $x_\alpha < y < x$ and $f(y) = \delta_\alpha$. If α is a limit ordinal and x_β has been chosen for every $\beta < \alpha$, let x_α be the least upper bound of $\{x_\beta \mid \beta < \alpha\}$. Continue this construction as long as $x_\alpha < x$ and $\sup(\{\delta_\beta \mid \beta < \alpha\}) < \kappa$. In fact, we claim that if $x_\alpha <_T x$, then $\sup(\{\delta_\beta \mid \beta < \alpha\} < \kappa$. Suppose for sake of contradiction that there is α such that $x_\alpha <_T x$ but $\sup(\{\delta_\beta \mid \beta < \alpha\} = \kappa$. Then $f(x_\alpha) < \delta_\beta$ for some $\beta < \alpha$, contradicting the choice of δ_β. Therefore, we can continue the construction until we reach $\gamma \le \kappa$ such that $x_\gamma = x$.

Now let $C_x = \{\text{level}(x_\alpha) \mid \alpha < \gamma\}$. It is easy to verify that C_x is a club in α and that $\text{otp}(C_x) = \gamma \le \kappa$. Let $\mathcal{C}_\alpha = \{C_x \mid x \in U_\alpha\}$. Since T is Aronszajn, $|\mathcal{C}_\alpha| \le \kappa$. It remains to check the coherence condition. Let β be a limit point of C_x. Then β is the level of some x_β, where $\langle x_\alpha \mid \alpha < \gamma \rangle$ is the sequence leading up to x used to define C_x. Let $\langle y_\alpha \mid \alpha < \gamma' \rangle$ be the sequence leading up to x_β used to define C_{x_β}. Notice that when defining $\langle y_\alpha \mid \alpha < \gamma' \rangle$, we went through the same steps as we went through when defining $\langle x_\alpha \mid \alpha < \gamma \rangle$, so it is easy to check by induction that, for all $\alpha < \beta$, $x_\alpha = y_\alpha$, so $C_{x_\beta} = C_x \cap \beta$, so $C_x \cap \beta \in \mathcal{C}_\beta$. ∎

Definition A κ^+-tree T is *normal if it satisfies the following properties:*

1. *T has a unique least element.*
2. *For every $x \in T$, x has κ-many immediate successors in T.*
3. *For every $\alpha < \beta < \kappa^+$ and every x in level α of T, there is a y in level β of T such that $x <_T y$.*
4. *For every limit ordinal $\beta < \kappa^+$, if x and y are in level β of T and $\{z \mid z <_T x\} = \{z \mid z <_T y\}$, then $x = y$.*

We now show that if κ is a regular cardinal, then $\square^*_\kappa$ automatically holds under sufficient cardinal arithmetic assumptions.

Theorem 3.2 *Suppose that $\kappa^{<\kappa} = \kappa$. Then there is a normal special κ^+-Aronszajn tree.*

Proof Let Q be the set ${}^{<\omega}\kappa$ equipped with the lexicographic ordering. That is, if $s, t \in Q$, then $s <_l t$ iff

1. There is $n \in \text{dom}(s) \cap \text{dom}(t)$ such that $s(n) < t(n)$ and $s \restriction n = t \restriction n$ or
2. $\text{dom}(s) < \text{dom}(t)$ and $t \restriction \text{dom}(s) = s$.

We will construct a special κ^+-Aronszajn tree T. For $\alpha < \kappa^+$, the α-th level of the tree will be denoted U_α. For all $\alpha < \kappa^+$, U_α will consist of increasing sequences from Q of length $\alpha + 1$. The tree will be ordered so that for all $x, y \in T$, $x \le_T y$ iff $x \subseteq y$. T cannot have a branch of length κ^+, as such a branch would correspond to an increasing sequence from Q of length κ^+. This is a contradiction, since $|Q| = \kappa$. Thus, T will be an Aronszajn tree provided that $U_\alpha \ne \varnothing$ and $|U_\alpha| \le \kappa$ for all $\alpha < \kappa^+$. It will also follow that T is special: Fix a bijection F between Q and κ. If $x \in U_\alpha$ for some $\alpha < \kappa^+$, let $f(x) = F(x(\alpha))$. Then f witnesses that T is special.

We will construct U_α by recursion on $\alpha < \kappa^+$ so that each U_α satisfies the following conditions:

1. $|U_\alpha| \leq \kappa$.

2. For every $\beta < \alpha$ and $x \in U_\beta$, if $|x(\beta)| = n+1$, there is $y \in U_\alpha$ such that $x \subset y$ and $y(\alpha) \leq_l (x(\beta) \restriction n) \frown\langle x(\beta)(n)+1\rangle$.

Let $U_0 = \{\langle\langle 0\rangle\rangle\}$. If $\alpha = \beta+1$, let $U_\alpha = \{x\frown s \mid x \in U_\beta,\ x(\beta) <_l s\}$. It is clear that U_α satisfies conditions 1 and 2.

Suppose α is a limit ordinal of cofinality $< \kappa$. Let T_α denote the tree below level α. We say b is a branch through T_α if b is an increasing α-sequence from Q such that, for all $\beta < \alpha$, $b \restriction (\beta+1) \in T_\alpha$ and such that there exists $\sup(\mathrm{ran}(b)) \in Q$. Let $U_\alpha = \{b \frown\langle s\rangle \mid$ b is a branch through T_α and $\sup(\mathrm{ran}(b)) = s\}$. U_α satisfies condition 1 because $\kappa^{<\kappa} = \kappa$, so there are at most κ many branches through T_α. We claim that U_α also satisfies condition 2.

To show this, fix $\beta < \alpha$ and $x \in U_\beta$ with $|x(\beta)| = n+1$. Fix an increasing, continuous seqence of ordinals $\langle \alpha_\gamma \mid \gamma < \mathrm{cf}(\alpha)\rangle$ cofinal in α such that $\alpha_0 = \beta$. For $\gamma < \mathrm{cf}(\alpha)$, let $s_\gamma = x(\beta) \frown\langle\gamma\rangle$. Note that $\langle s_\gamma \mid \gamma < \mathrm{cf}(\alpha)\rangle$ is strictly increasing and $s = \sup(\{s_\gamma \mid \gamma < \mathrm{cf}(\alpha)\}) = x(\beta) \frown\langle \mathrm{cf}(\alpha)\rangle$. Now we will define a sequence $\langle x_\gamma \mid \gamma < \mathrm{cf}(\alpha)\rangle$ such that:

1. For all $\gamma < \mathrm{cf}(\alpha)$, $x_\gamma \in U_{\alpha_\gamma}$ or $x_\gamma \in U_{\alpha_\gamma+1}$.

2. For all $\gamma < \mathrm{cf}(\alpha)$, $x_\gamma(\alpha_\gamma) = s_\gamma$ or $x_\gamma(\alpha_\gamma+1) = s_\gamma$.

3. For all $\delta < \gamma < \mathrm{cf}(\alpha)$, $x_\delta \subset x_\gamma$.

We go by recursion on $\gamma < \mathrm{cf}(\alpha)$. Let $x_0 = x \frown\langle s_0\rangle$. If $\gamma = \gamma'+1$, then let $\bar{x}_\gamma \in U_{\alpha_\gamma}$ be such that $x_{\gamma'} \subset \bar{x}_\gamma$ and $\bar{x}_\gamma(\alpha_\gamma) \leq_l s_\gamma$. Such an $\bar{x}_\gamma$ exists because U_{α_γ} satisfies condition 2. If $\bar{x}_\gamma(\alpha_\gamma) = s_\gamma$, let $x_\gamma = \bar{x}_\gamma$. Otherwise, let $x_\gamma = \bar{x}_\gamma \frown\langle s_\gamma\rangle$. If γ is a limit ordinal, then $\bigcup_{\delta<\gamma} x_\delta$ is a branch through T_γ, and $\sup(\mathrm{ran}(\bigcup_{\delta<\gamma} x_\delta)) = s_\gamma$. By the way we constructed U_γ, $(\bigcup_{\delta<\gamma} x_\delta)\frown\langle s_\gamma\rangle \in U_\gamma$. Let $x_\gamma = (\bigcup_{\delta<\gamma} x_\delta) \frown\langle s_\gamma\rangle$.

Now $b = \bigcup_{\gamma<\mathrm{cf}(\alpha)} x_\gamma$ is a branch through T_α and $\sup(\mathrm{ran}(b)) = s$. Let $y = b\frown\langle s\rangle$. It is easy to see that y is as desired, so U_α satisfies condition 2.

Finally, suppose α is a limit ordinal of cofinality κ. Note that we can not extend all branches through T_α, as there are possibly more than κ many of them. We claim that for each $\beta < \alpha$ and $x \in U_\beta$, if $|x(\beta)| = n+1$, there is a branch b through T_α such that $x \subset b$ and $\sup(\mathrm{ran}(b)) = (x(\beta) \restriction n) \frown\langle x(\beta)(n)+1\rangle$. To show this, fix an increasing, continuous sequence of ordinals $\langle \alpha_\gamma \mid \gamma < \kappa\rangle$ cofinal in α such that $\alpha_0 = \beta$. For $\gamma < \kappa$, let $s_\gamma = x(\beta) \frown\langle\gamma\rangle$. $\langle s_\gamma \mid \gamma < \kappa\rangle$ is increasing and $s = \sup(\{s_\gamma \mid \gamma < \kappa\}) = (x(\beta) \restriction n) \frown\langle x(\beta)(n)+1\rangle$. Exactly as above, define a sequence $\langle x_\gamma \mid \gamma < \kappa\rangle$ such that:

1. For all $\gamma < \kappa$, $x_\gamma \in U_{\alpha_\gamma}$ or $x_\gamma \in U_{\alpha_\gamma+1}$.

2. For all $\gamma < \kappa$, $x_\gamma(\alpha_\gamma) = s_\gamma$ or $x_\gamma(\alpha_\gamma+1) = s_\gamma$.

3. For all $\delta < \gamma < \kappa$, $x_\delta \subset x_\gamma$.

Then $b = \bigcup_{\gamma<\kappa} x_\gamma$ is a branch through T_α such that $x \subset b$ and $\sup(\operatorname{ran}(b)) = s$. Now, for each $x \in T_\alpha$, choose such a branch, b_x. Let $U_\alpha = \{b_x{}^\frown\langle s\rangle \mid x \in T_\alpha,\ \sup(\operatorname{ran}(b_x)) = s\}$. By construction, U_α is easily seen to satisfy conditions 1 and 2. This completes the construction of T. It is easy to see that T is in fact a normal tree, thus concluding the proof of the theorem. ∎

We would like to understand the extent to which these weak squares are sufficient to obtain some of the implications of the original square principle. We are interested in particular in some combinatorial principles that serve as intermediaries between the square principles and their applications in algebra, topology, and other fields. A basic example of such a combinatorial principle is given by stationary reflection.

Definition Let μ be an uncountable, regular cardinal, and let $S \subseteq \mu$ be stationary. We say that S *reflects* at α if $\alpha < \mu$, cf$(\alpha) > \omega$, and $S \cap \alpha$ is stationary in α. S *does not reflect* if there is no $\alpha < \mu$ such that S reflects at α.

Proposition 3.3 *Suppose that $\square_\kappa$ holds. Then for every stationary $S \subseteq \kappa^+$, there is a stationary $S^* \subseteq S$ such that S^* does not reflect.*

Proof Let $\langle C_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ be a $\square_\kappa$-sequence. Let $S \subseteq \kappa^+$ be stationary. By thinning out S if necessary, we may assume that S consists entirely of limit ordinals and that $S \subseteq \kappa^+ \setminus \kappa$. Define a function $f : S \to \kappa$ by letting $f(\alpha) = \operatorname{otp}(C_\alpha)$ for all $\alpha \in S$. Then f is a regressive function, so, by Fodor's Lemma, there is a stationary $S^* \subseteq S$ and a $\mu \leq \kappa$ such that for all $\alpha \in S^*$, $\operatorname{otp}(C_\alpha) = \mu$. Now suppose for sake of contradiction that there is $\beta < \kappa^+$ such that $\operatorname{cf}(\beta) > \omega$ and $S^* \cap \beta$ is stationary. Let C'_β be the set of limit points of C_β. Then, since C'_β is a club in β, $C'_\beta \cap S^*$ is unbounded in β. Let $\gamma_1 < \gamma_2 \in C'_\beta \cap S^*$. $C_\beta \cap \gamma_1 = C_{\gamma_1}$ and $C_\beta \cap \gamma_2 = C_{\gamma_2}$, so $C_{\gamma_1} \subsetneq C_{\gamma_2}$. But this is a contradiction, since $\operatorname{otp}(C_{\gamma_1}) = \operatorname{otp}(C_{\gamma_2})$. ∎

Notice that we have actually shown something more: for every limit α such that $\kappa < \alpha < \kappa^+$, $C'_\alpha \cap S^*$ consists of at most one point. Note also that if, for every limit α such that $\kappa < \alpha < \kappa^+$, we define $D_\alpha = C'_\alpha \setminus \gamma$ if $\gamma \in C'_\alpha \cap S^*$ and $D_\alpha = C'_\alpha$ otherwise, then $\langle D_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ is a $\square_\kappa$-sequence. We thus obtain the following corollary, which plays an important role in Jensen's proof that, in L, κ^+ Suslin trees exist for every infinite cardinal κ:

Corollary 3.4 *Suppose that $\square_\kappa$ holds. Then for every stationary $S \subseteq \kappa^+$, there is a non-reflecting stationary $S^* \subseteq S$ and a $\square_\kappa$-sequence $\langle D_\alpha \mid \alpha \text{ limit}, \kappa < \alpha < \kappa^+\rangle$ such that, for every α, $D_\alpha \cap S^* = \emptyset$.*

4 Separating Squares

In this section, we show that, for an uncountable cardinal κ and cardinals μ, ν such that $1 \leq \mu < \nu \leq \kappa$, $\square_{\kappa,\mu}$ and $\square_{\kappa,\nu}$ are in fact distinct principles. We first introduce two forcing posets.

The first, denoted $\mathbb{S}(\kappa,\lambda)$, adds a $\square_{\kappa,\lambda}$-sequence while preserving all cardinals up to and including κ^+, where κ is an uncountable cardinal and $1 \le \lambda \le \kappa$. Conditions of $\mathbb{S}(\kappa,\lambda)$ are functions s such that:

1. $\mathrm{dom}(s) = \{\beta \le \alpha \mid \beta$ is a limit ordinal$\}$ for some limit ordinal $\alpha < \kappa^+$.
2. For all $\beta \in \mathrm{dom}(s)$, $1 \le |s(\beta)| \le \lambda$.
3. For all $\beta \in \mathrm{dom}(s)$, $s(\beta)$ is a set of clubs in β of order type $\le \kappa$. If $\mathrm{cf}(\beta) < \kappa$, then $s(\beta)$ is a set of clubs β of order type $< \kappa$.
4. For all $\beta \in \mathrm{dom}(s)$, if $C \in s(\beta)$ and γ is a limit point of C, then $C \cap \gamma \in s(\gamma)$.

For all $s,t \in \mathbb{S}(\kappa,\lambda)$, $t \le s$ iff t end-extends s (i.e. $s \subseteq t$).

Fact 4.1 *$\mathbb{S}(\kappa,\lambda)$ is κ^+-distributive.*

We next introduce a forcing poset that kills a square sequence.

Definition Let $\overrightarrow{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa^+\rangle$ be a $\square_{\kappa,\lambda}$-sequence in V. Let W be an outer model of V. Then $C \in W$ *threads* $\overrightarrow{\mathcal{C}}$ iff C is a club in κ^+ and for every limit point α of C, $C \cap \alpha \in \mathcal{C}_\alpha$. It is clear from order-type considerations that if there is $C \in W$ such that C threads $\overrightarrow{\mathcal{C}}$, then $\overrightarrow{\mathcal{C}}$ is not a $\square_{\kappa,\lambda}$-sequence in W.

Given a $\square_{\kappa,\lambda}$-sequence $\overrightarrow{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa^+\rangle$, let γ be a regular cardinal such that $\gamma \le \kappa$. We will define a threading poset $\mathbb{T}_\gamma(\overrightarrow{\mathcal{C}})$. Conditions of the poset are sets c such that:

1. c is a closed, bounded subset of κ^+.
2. c has order type $< \gamma$.
3. For all limit points β of c, $c \cap \beta \in \mathcal{C}_\beta$.

For all $c,d \in \mathbb{T}_\gamma(\overrightarrow{\mathcal{C}})$, $d \le c$ iff d end-extends c (i.e. $d \cap (\max(c)+1) = d$).

If $\overrightarrow{\mathcal{C}}$ is introduced by forcing with $\mathbb{S}(\kappa,\lambda)$, then $\mathbb{T}_\gamma(\overrightarrow{\mathcal{C}})$ behaves quite nicely.

Lemma 4.2 *Suppose κ is an uncountable cardinal, λ is a cardinal such that $1 \le \lambda \le \kappa$, and γ is a regular cardinal $\le \kappa$. Let $\mathbb{S} = \mathbb{S}(\kappa,\lambda)$, and let $\mathbb{T} = \mathbb{T}_\gamma(\overrightarrow{\mathcal{C}})_{V^{\mathbb{S}}}$, where $\overrightarrow{\mathcal{C}}$ is the $\square_{\kappa,\lambda}$-sequence added by forcing with $\mathbb{S}$. Then*

1. *$\mathbb{S} * \mathbb{T}$ has a dense γ-closed subset.*
2. *$\mathbb{T}$ adds a set of order type γ which threads $\overrightarrow{\mathcal{C}}$, and $(\kappa^+)^V$ has cofinality γ in $V^{\mathbb{S}*\mathbb{T}}$.*

Namely, the dense γ-closed subset of $\mathbb{S} * \mathbb{T}$ is the set of conditions $(s,\dot{c})$ such that, for some $c \in V$, $s \Vdash \dot{c} = \check{c}$ and $\max(\mathrm{dom}(s)) = \max(c)$. The proof of the above Lemma can be found in [2]. We will also need the following Lemma:

Lemma 4.3 *Let ρ, κ, and λ be cardinals such that ρ is regular and $\rho < \kappa < \lambda$. Suppose that, in $V^{\mathrm{Coll}(\rho,<\kappa)}$, $\mathbb{P}$ is a ρ-closed poset and $|\mathbb{P}| < \lambda$. Let i be the canonical complete embedding of $\mathrm{Coll}(\rho, < \kappa)$ into $\mathrm{Coll}(\rho, < \lambda)$ (namely, i is the identity map). Then i can be extended to a complete embedding j of $\mathrm{Coll}(\rho, < \kappa) * \mathbb{P}$ into $\mathrm{Coll}(\rho, < \lambda)$ so that the quotient forcing, $\mathrm{Coll}(\rho, < \lambda)/j[\mathrm{Coll}(\rho, < \kappa) * \mathbb{P}]$ is ρ-closed in $V^{j[\mathrm{Coll}(\rho,<\kappa)*\mathbb{P}]}$..*

Theorem 4.4 *Let ρ be a regular, uncountable cardinal and let $\mu > \rho$ be Mahlo. Then $\square_{\rho,<\rho}$ fails in $V^{\mathrm{Coll}(\rho,<\mu)}$.*

Proof Let G be $\mathrm{Coll}(\rho, < \mu)$-generic over V and suppose for sake of contradiction that $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \mu \rangle$ is a $\square_{\rho,<\rho}$-sequence in $V[G]$. For $\alpha < \mu$, let $G \restriction \alpha$ denote the pointwise image of G under the canonical projection from $\mathrm{Coll}(\rho, < \mu)$ onto $\mathrm{Coll}(\rho, < \alpha)$. By a standard nice names argument, the set $\{\alpha < \mu \mid \text{for all } \beta < \alpha,\ \mathcal{C}_\beta \in V[G \restriction \alpha]\}$ is club in μ. Thus, since μ is Mahlo, there is an inaccessible $\kappa < \mu$ such that for every $\beta < \kappa$, $\mathcal{C}_\beta \in V[G \restriction \kappa]$. Since $G \restriction \kappa$ is $\mathrm{Coll}(\rho, < \kappa)$-generic over V, $\kappa = \rho^+$ in $V[G \restriction \kappa]$. It can easily be verified that $\langle \mathcal{C}_\beta \mid \beta < \kappa \rangle$ is a $\square_{\rho,<\rho}$-sequence in $V[G \restriction \kappa]$. Note that the quotient forcing $\mathrm{Coll}(\rho, < \mu)/\mathrm{Coll}(\rho, < \kappa)$ is ρ-closed. Note also that the sequence $\langle \mathcal{C}_\beta \mid \beta < \kappa \rangle$ is threaded in $V[G]$, namely by any element of $\mathcal{C}_\kappa$. The following Lemma therefore suffices to prove the theorem:

Lemma 4.5 *Suppose λ is a regular, uncountable cardinal, $\vec{\mathcal{D}} = \langle \mathcal{D}_\alpha \mid \alpha < \lambda^+ \rangle$ is a $\square_{\lambda,<\lambda}$-sequence, and $\mathbb{P}$ is a λ-closed forcing poset. Then $\mathbb{P}$ does not add a thread through $\vec{\mathcal{D}}$.*

Proof Assume for sake of contradiction that $\dot{D}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\dot{D}$ is club in λ^+ and for all limit points $\alpha \in \dot{D}$, $\dot{D} \cap \alpha \in \mathcal{D}_\alpha$". First suppose that λ is not strongly inaccessible. Let γ be the least cardinal such that $2^\gamma \geq \lambda$. We will construct $\langle p_s \mid s \in {}^{\leq\gamma}2 \rangle$ and $\langle \alpha_\beta \mid \beta \leq \gamma \rangle$ such that:

1. For all $s, t \in {}^{\leq\gamma}2$ such that $s \subseteq t$, we have $p_s, p_t \in \mathbb{P}$ and $p_t \leq p_s$.
2. $\langle \alpha_\beta \mid \beta \leq \gamma \rangle$ is a strictly increasing, continuous sequence of ordinals less than λ^+.
3. For all $s \in {}^{<\gamma}2$, there is $\alpha < \alpha_{|s|+1}$ such that $p_{s^\frown\langle 0 \rangle}$ and $p_{s^\frown\langle 1 \rangle}$ decide the statement "$\alpha \in \dot{D}$" in opposite ways.
4. For all limit ordinals $\beta \leq \gamma$ and all $s \in {}^{\beta}2$, $p_s \Vdash$ "α_β is a limit point of $\dot{D}$", and there is $D_s \in \mathcal{D}_{\alpha_\beta}$ such that $p_s \Vdash$ "$\dot{D} \cap \alpha_\beta = D_s$".

Assume for a moment that we have successfully constructed these sequences. For all $s \in {}^{\gamma}2$, there is $D_s \in \mathcal{D}_{\alpha_\gamma}$ such that $p_s \Vdash$ "α_γ is a limit point of $\dot{D}$ and $\dot{D} \cap \alpha_\beta = D_s$". But if $s, t \in {}^{\gamma}2$, $s \neq t$, then there is $\alpha < \alpha_\gamma$ such that p_s and p_t decide the statement "$\alpha \in \dot{D}$" in opposite ways, so $D_s \neq D_t$. But, since $2^\gamma \geq \lambda$, this contradicts the fact that $|\mathcal{D}_{\alpha_\gamma}| < \lambda$.

We now turn to the construction of $\langle p_s \mid s \in {}^{\leq\gamma}2 \rangle$ and $\langle \alpha_\beta \mid \beta \leq \gamma \rangle$. Let $p_{\langle\rangle} = 1_{\mathbb{P}}$ and $\alpha_0 = 0$. Fix $\beta < \gamma$ and suppose that $\langle p_s \mid s \in {}^{\beta}2 \rangle$ and α_β are given. Fix $s \in {}^{\beta}2$. Since $\Vdash_{\mathbb{P}}$ "$\dot{D}$ is club in λ^+", we can find $p'_s \leq p_s$ and $\alpha > \alpha_\beta$ such that $p'_s \Vdash$ "$\alpha \in \dot{D}$". Since $\Vdash_{\mathbb{P}}$ "$\dot{D} \notin V$", we can find $\alpha_s > \alpha$ and $p_0, p_1 \leq p'$ such that p_0 and p_1 decide the statement

"$\alpha_s \in \dot{D}$" in opposite ways. Let $p_{s^\frown\langle 0\rangle} = p_0$ and $p_{s^\frown\langle 1\rangle} = p_1$. Do this for all $s \in {}^{\beta}2$, and let $\alpha_\beta = \sup\{\alpha_s \mid s \in {}^{\beta}2\}$. $2^\beta < \lambda$, so $\alpha_\beta < \lambda^+$.

Suppose $\beta \leq \gamma$ is a limit ordinal and that $\langle p_s \mid s \in {}^{<\beta}2\rangle$ and $\langle \alpha_\delta \mid \delta < \beta\rangle$ have been constructed. Let $\alpha_\beta = \sup\{\alpha_\delta \mid \delta < \beta\rangle\}$. Fix $s \in {}^{\beta}2$. As $\mathbb{P}$ is λ-closed, we can find a $p \in \mathbb{P}$ such that, for every $\delta < \beta$, $p \leq p_{s\restriction\delta}$. Note that for every $\delta < \beta$, there is $\alpha > \alpha_\delta$ such that $p_{s\restriction\delta+1} \Vdash$ "$\alpha \in \dot{D}$". Thus, $p \Vdash$ "α_β is a limit point of $\dot{D}$", so $p \Vdash$ "$\dot{D} \cap \alpha_\beta \in \mathcal{D}_{\alpha_\beta}$". Find $p' \leq p$ and $D_s \in \mathcal{D}_{\alpha_\beta}$ such that $p' \Vdash$ "$\dot{D} \cap \alpha_\beta = D_s$". Let $p_s = p'$. It is easy to see that this is as desired.

Now suppose that λ is strongly inaccessible. We modify the previous argument slightly. First, use Fodor's Lemma to fix a $\gamma < \lambda$ and a stationary $S \subseteq \lambda^+$ such that, for every $\alpha \in S$, $|\mathcal{D}_\alpha| \leq \gamma$. Construct sequences $\langle p_s \mid s \in {}^{\leq\gamma}2\rangle$ and $\langle \alpha_\beta \mid \beta \leq \gamma\rangle$ exactly as in the previous case. For each $s \in {}^{\gamma}2$, let $E_s = \{\alpha > \alpha_\gamma \mid$ there is $q \leq p_s$ such that $q \Vdash$ "α is a limit point of $\dot{D}$"$\}$. Since $\Vdash_{\mathbb{P}}$ "$\dot{D}$ is club in λ^+", each E_s contains a club, so $E = \bigcap_{s\in{}^{\gamma}2} E_s$ contains a club in λ^+. Fix $\alpha \in E \cap S$. For each $s \in {}^{\gamma}2$, find $D'_s \in \mathcal{D}_\alpha$ and $q_s \leq p_s$ such that $q_s \Vdash$ "$\dot{D} \cap \alpha = D'_s$. If $s, t \in {}^{\gamma}2$, $s \neq t$, then, as in the previous case, $D'_s \neq D'_t$, but this contradicts the fact that, since $\alpha \in S$ $|\mathcal{D}_\alpha| \leq \gamma$. This finishes the prove of the lemma and hence of the theorem. ∎ ∎

Note that if GCH holds in V, then $(\rho$ is regular and $\rho^{<\rho} = \rho)^{V^{\mathrm{Coll}(\rho,<\mu)}}$. Thus, by theorems 3.1 and 3.2, $\square^*_\rho$ holds in $V^{\mathrm{Coll}(\rho,<\mu)}$, so we have the following consistency result:

Corollary 4.6 *Suppose μ is a Mahlo cardinal, $\rho < \mu$ is a regular, uncountable cardinal, and GCH holds in V. Then there is a generic extension in which*

1. *All cardinals less than or equal to ρ are preserved and $\mu = \rho^+$.*
2. *$\square^*_\rho$ holds.*
3. *$\square_{\rho,<\rho}$ fails.*

Remark Mitchell [8] showed that if $\rho > \omega_1$ is regular and there is a Mahlo cardinal $\mu > \rho$, then there is a forcing extension in which all cardinals $\leq \rho$ are preserved and there are no special ρ^+-Aronszajn trees (and hence $\square^*_\rho$ fails).

We will now prove another specific instance of the consistency of the separation of different square principles. This theorem is due to Jensen, who proved the result using a Mahlo cardinal rather than a measurable [6].

Theorem 4.7 *Suppose κ is a measurable cardinal and $\rho < \kappa$ is a regular, uncountable cardinal. Then there is a generic extension in which*

1. *All cardinals less than or equal to ρ are preserved and $\kappa = \rho^+$.*
2. *$\square_{\rho,2}$ holds.*
3. *$\square_\rho$ fails.*

Proof Let $\mathbb{P} = \mathrm{Coll}(\rho, < \kappa)$. Let $\mathbb{S} = \mathbb{S}(\rho, 2)_{V^{\mathbb{P}}}$ and let $\mathbb{T} = \mathbb{T}_\rho(\overrightarrow{\mathcal{C}})_{V^{\mathbb{P}*\mathbb{S}}}$, where $\overrightarrow{\mathcal{C}}$ is the $\square_{\rho,2}$-sequence added by $\mathbb{S}$. $V^{\mathbb{P}*\mathbb{S}}$ will be the model in which the desired conclusion will hold.

Fix an elementary embedding $j : V \to M$ witnessing that κ is measurable. $j \restriction \mathbb{P}$ is the identity map and thus gives the natural complete embedding of $\mathbb{P}$ into $j(\mathbb{P}) = \mathrm{Coll}(\rho, < j(\kappa))$. In $V^{\mathbb{P}}$, $|\mathbb{S} * \mathbb{T}| < j(\kappa)$ and, by Lemma 4.2, $\mathbb{S} * \mathbb{T}$ has a dense ρ-closed subset. Thus, by Lemma 4.3, we can extend $j \restriction \mathbb{P}$ to a complete embedding of $\mathbb{P} * \mathbb{S} * \mathbb{T}$ into $j(\mathbb{P})$ so that the quotient forcing $j(\mathbb{P})/\mathbb{P} * \mathbb{S} * \mathbb{T}$ is ρ-closed in $V^{\mathbb{P}*\mathbb{S}*\mathbb{T}}$.

Now let G be $\mathbb{P}$-generic over V, let H be $\mathbb{S}$-generic over $V[G]$, let I be $\mathbb{T}$-generic over $V[G * H]$, and let J be $j(\mathbb{P})/G * H * I$-generic over $V[G * H * I]$. Then, by letting $j(\tau_G) = j(\tau)_{G*H*I}$ for all $\mathbb{P}$-names τ, we can extend j to $j : V[G] \to M[G * H * I * J]$. We now show how to further extend j so that its domain is $V[G * H]$.

Let $\overrightarrow{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \text{ limit}, \alpha < \kappa \rangle = \bigcup_{s \in H} s$ (so $\overrightarrow{\mathcal{C}}$ is the $\square_{\rho,2}$-sequence added by H). Let C be the club in κ added by I. Note that for all $s \in H$, $j(s) = s$, and $j''C = C$. $\overrightarrow{\mathcal{C}}$ is not a condition in $j(\mathbb{S})$, since it has no top element. However, it is easy to see that $S = \overrightarrow{\mathcal{C}} \cup \{(\kappa, \{C\})\}$ is a condition and that $S \le s = j(s)$ for every $s \in H$.

Now let K be $j(\mathbb{S})$-generic over $V[G * H * I * J]$ such that $S \in K$. j"$H \subseteq K$, so we can further extend j to $j : V[G * H] \to M[G * H * I * J * K]$.

Suppose for sake of contradiction that $\overrightarrow{D} = \langle D_\alpha \mid \alpha \text{ limit}, \alpha < \kappa \rangle$ is a $\square_\rho$-sequence in $V[G * H]$.

Claim 4.8 *In $V[G * H * I * J]$, there is a club $F \subseteq \kappa$ such that for every limit point α of F, $F \cap \alpha = D_\alpha$.*

Let $j(\overrightarrow{D}) = \overrightarrow{E} = \langle E_\alpha \mid \alpha \text{ limit}, \alpha < j(\kappa) \rangle$. Let $F = E_\kappa$. $F \in M[G * H * I * J * K]$, but since $j(\mathbb{S})$ is $j(\kappa)$-distributive, we have $F \in M[G * H * I * J]$. For all $\alpha < \kappa$, $D_\alpha = E_\alpha$, so $F \cap \alpha = D_\alpha$ for every limit point α of F. Thus, F is as desired.

Note that, since $j(\mathbb{P})/G * H * I$ is ρ-closed, by Lemma 4.5 we may assume that $F \in V[G * H * I]$.

Claim 4.9 *$F \in V[G * H]$.*

Suppose not. Then there is an $\mathbb{S} * \mathbb{T}$-name $\dot{F} \in V[G]$ such that $\dot{F}^{H*I} = F$ and $\Vdash^{V[G]}_{\mathbb{S}*\mathbb{T}}$ "$\dot{F} \notin V[G][G_{\mathbb{S}}]$". We claim that for all $(s, t) \in \mathbb{S} * \mathbb{T}$, there are $s' \le s$, t_0, t_1, and α such that $(s', t_0), (s', t_1) \le (s, t)$ and the conditions (s', t_0) and (s', t_1) decide the statement "$\alpha \in \dot{F}$" in opposite ways. For, if not, we can define in $V[G]$ an $\mathbb{S}$-name $\dot{F}'$ such that for all $s' \le s$ and all $\alpha < \rho^+$, $s' \Vdash^{V[G]}$ "$\alpha \in \dot{F}'$" if and only if there is t' such that $(s', t') \le (s', t)$ and $(s', t') \Vdash^{V[G]}$ "$\alpha \in \dot{F}$". Then $(s, t) \Vdash^{V[G]}$ "$\dot{F}' = \dot{F}$", contradicting the assumption that $F \notin V[G * H]$.

Fix a condition (s, t) such that $(s, t) \Vdash^{V[G]}$ "For every limit point α of $\dot{F}, \dot{F} \cap \alpha = D_\alpha$". Fix $s' \le s$, $t_0, t_1 \le t$, and $\alpha < \rho^+$ such that $(s', t_0), (s', t_1) \le (s, t)$ and (s', t_0) and (s', t_1) decide the statement "$\alpha \in \dot{F}$" in opposite ways. Now recursively construct s^i_j, t^i_j, and α^i_j for $i \in \omega$ and $j \in \{0, 1\}$ such that:

1. $s^0_0 \le s'$ and, for all $i \in \omega$, $s^{i+1}_0 \le s^i_1 \le s^i_0$.

2. $\alpha < \alpha_0^0$ and, for all $i \in \omega$, $\alpha_0^i < \alpha_1^i < \alpha_0^{i+1}$.

3. For each $j \in \{0,1\}$, $(s_j^0) \leq (s', t_j)$ and, for all $i \in \omega$, $(s_j^{i+1}, t_j^{i+1}) \leq (s_j^i, t_j^i)$.

4. For each $i \in \omega$ and $j \in \{0,1\}$, $(s_j^i, t_j^i) \Vdash^{V[G]}$ "$\alpha_j^i \in \dot{F}$".

The construction is straightforward. Now let $\alpha^* = \sup\{\alpha_j^i \mid i \in \omega, j \in \{0,1\}\}$. For $j \in \{0,1\}$, let $t_j^* = \bigcup_i t_j^i \cup \{\alpha^*\}$, and let $s^* = \bigcup_{i,j} s_j^i \cup \{(\alpha^*, \{t_j^* \cap \alpha^* \mid j \in \{0,1\}\})\}$ (note that each $t_j^* \cap \alpha^* \in V[G]$, since $\mathbb{S}$ is ρ^+-distributive in $V[G]$). Now $s^* \in \mathbb{S}$ and $(s^*, t_j^*) \in \mathbb{S} * \mathbb{T}$ for $j \in \{0,1\}$. Find $\bar{s} \leq s^*$ such that $\bar{s}$ decides the value of D_{α^*}. For each $j \in \{0,1\}$, $(\bar{s}, t_j^*) \Vdash^{V[G]}$ "α^* is a limit point of $\dot{F}$, so $\dot{F} \cap \alpha^* = D_{\alpha^*}$". But $\alpha < \alpha^*$, and $(\bar{s}, t_0^*)$ and $(\bar{s}, t_1^*)$ decide the statement "$\alpha \in \dot{F}$" in opposite ways. Contradiction.

Thus, $F \in V[G * H]$. But F threads $\overrightarrow{D}$, which was supposed to be a $\square_\rho$-sequence in $V[G * H]$. This is a contradiction, so $\square_\rho$ fails in $V[G * H]$, thus proving the theorem. ∎

Slight modifications of this proof will yield separation results for any $\square_{\rho,\mu}$ and $\square_{\rho,<\mu}$ where ρ is regular and $1 < \mu < \rho$. Cummings, Foreman, and Magidor, in [2], provided a further modification to obtain a similar result at singular cardinals. Their result is specifically about $\aleph_\omega$, but similar methods work at other singular cardinals:

Theorem 4.10 *Suppose κ is a supercompact cardinal and $2^{\kappa^{+\omega}} = \kappa^{+\omega+1}$. Let μ and ν be cardinals such that $1 \leq \mu < \nu < \aleph_\omega$. Then there is a generic extension in which:*

1. *All cardinals less than or equal to ν are preserved.*

2. *$\aleph_\omega = \kappa_V^{+\omega}$.*

3. *$\square_{\aleph_\omega,\nu}$ holds.*

4. *$\square_{\aleph_\omega,\mu}$ fails.*

5 Scales

We now introduce another intermediary combinatorial principle which has useful applications and follows from weakenings of square.

Let λ be a singular cardinal. Let $\overrightarrow{\mu} = \langle \mu_i \mid i < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence of regular cardinals cofinal in λ. For f and g in $\prod_{i<\mathrm{cf}(\kappa)} \mu_i$, we say that $f <^* g$ if $\{j < \mathrm{cf}(\lambda) \mid f(j) \geq g(j)\}$ is bounded in $\mathrm{cf}(\lambda)$. Similarly, $f \leq^* g$ if $\{j < \mathrm{cf}(\lambda) \mid f(j) > g(j)\}$ is bounded in $\mathrm{cf}(\lambda)$.

Definition If λ and $\overrightarrow{\mu}$ are as above, a *$(\lambda^+, \overrightarrow{\mu})$-scale* is a sequence $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

1. For every $\alpha < \lambda^+$, $f_\alpha \in \prod_{i<cf(\kappa)} \mu_i$

2. For every $\alpha < \beta < \lambda^+$, $f_\alpha <^* f_\beta$

3. For every $g \in \prod_{i<cf(\kappa)} \mu_i$, there is $\alpha < \lambda^+$ such that $g <^* f_\alpha$

Shelah, as part of PCF theory, proved the following [12]:

Theorem 5.1 *If λ is a singular cardinal, then there is a sequence $\vec{\mu}$ such that there is a $(\lambda^+, \vec{\mu})$-scale.*

Definition Let D be a set of ordinals and let $\langle f_\delta \mid \delta \in D\rangle$ be a sequence of functions in ${}^{\mathrm{cf}(\lambda)}\mathrm{OR}$ such that, for all $\delta, \delta' \in D$, if $\delta < \delta'$, then $f_\delta <^* f_{\delta'}$. The sequence is said to be *strongly increasing* if, for each $\delta \in D$, there is an $i_\delta \in \mathrm{cf}(\lambda)$ such that, for all $\delta, \delta' \in D$, if $\delta < \delta'$ and $j \geq i_\delta, i_{\delta'}$, then $f_\delta(j) < f_{\delta'}(j)$.

The following are useful strengthenings of the notion of a scale:

Definition

1. A λ^+-scale $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ is *good* if, for every limit ordinal $\alpha < \lambda^+$, there is $D_\alpha \subseteq \alpha$ such that D_α is cofinal in α and $\langle f_\beta \mid \beta \in D_\alpha\rangle$ is strongly increasing.
2. A λ^+-scale $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ is *better* if in the definition of a good scale one can assume in addition that each D_α is club in α.
3. A λ^+-scale $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ is *very good* if in the definition of a good scale one can assume in addition that each D_α is club in α and that there is a $j \in \mathrm{cf}(\lambda)$ such that, if $i \geq j$, $\beta, \gamma \in D_\alpha$, and $\beta < \gamma$, then $f_\beta(i) < f_\gamma(i)$.

There is a relationship between square principles and the existence of good scales. For example, the following theorem, a proof of which can be found in [2], provides a sufficient condition for the existence of very good scales.

Theorem 5.2 *If λ is singular, $\kappa < \lambda$, and $\square_{\lambda,\kappa}$ holds, then there is a very good λ^+-scale.*

We give the proof here of an analogous theorem, also from [2], relating weak square and the existence of better scales.

Theorem 5.3 *If λ is singular and $\square^*_\lambda$ holds, then there is a better λ^+-scale.*

Proof Let $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ be a $(\lambda^+, \vec{\mu})$-scale for some $\vec{\mu} = \langle \mu_i \mid i < \mathrm{cf}(\lambda)\rangle$. We will improve this scale to a better $(\lambda^+, \vec{\mu})$-scale, $\langle g_\alpha \mid \alpha < \lambda^+\rangle$. Fix a $\square^*_\lambda$-sequence, $\langle \mathcal{C}_\alpha \mid \alpha \text{ limit}, \alpha < \lambda^+\rangle$ such that for all α and all $C \in \mathcal{C}_\alpha$, $\mathrm{otp}(C) < \lambda$. We will define $\langle g_\alpha \mid \alpha < \lambda^+\rangle$ by induction.

If g_α has been defined, choose $g_{\alpha+1}$ such that $f_{\alpha+1} \leq^* g_{\alpha+1}$ and $g_\alpha <^* g_{\alpha+1}$.

Suppose α is a limit ordinal and g_β has been defined for all $\beta < \alpha$. For each $C \in \mathcal{C}_\alpha$, define $h_C \in \prod \mu_i$ so that

$$h_C(i) = \begin{cases} 0 & : \mu_i \leq \mathrm{otp}(C) \\ \sup_{\beta \in C}(g_\beta(i)) & : \mathrm{otp}(C) < \mu_i \end{cases}$$

Since $|\mathcal{C}_\alpha| \leq \lambda$, we can choose g_α such that $f_\alpha \leq^* g_\alpha$ and $h_C <^* g_\alpha$ for every $C \in \mathcal{C}_\alpha$.

It is immediate from the construction that $\langle g_\alpha \mid \alpha < \lambda^+\rangle$ is a $(\lambda^+, \vec{\mu})$-scale. We claim that it is in fact a better scale. To show this, let $\alpha < \lambda^+$ be a limit ordinal. If $\mathrm{cf}(\alpha) = \omega$, then any D which has order type ω, is cofinal in α, and consists of successor ordinals witnesses that $\langle g_\alpha \mid \alpha < \lambda^+\rangle$ is a better scale. So, suppose that $\mathrm{cf}(\alpha) > \omega$. Pick $C \in \mathcal{C}_\alpha$. Let D be the club subset of α consisting of the limit points of C. For $\beta \in D$, $C \cap \beta \in \mathcal{C}_\beta$. Thus, in defining g_β, we considered the function $h_{C\cap\beta}$, so $h_{C\cap\beta} \leq^* g_\beta$. Pick $i_\beta < \mathrm{cf}(\lambda)$ such that for all $i_\beta < j < \mathrm{cf}(\lambda)$, $\mathrm{otp}(C) < \mu_j$ and $h_{C\cap\beta}(j) \leq g_\beta(j)$. Now let $\beta, \beta' \in D$ with $\beta < \beta'$. If $j \geq i_\beta, i_{\beta'}$, then $g_\beta(j) < h_{C\cap\beta'}(j) \leq g_{\beta'}(j)$. Thus, D witnesses that $\langle g_\alpha \mid \alpha < \lambda^+\rangle$ is a better scale. ∎

Scales can be useful as tools for constructing interesting objects. An example is given by the following [2]:

Theorem 5.4 *If λ is a singular cardinal and there exists a better λ^+-scale, then there is a sequence $\langle A_\alpha \mid \alpha < \lambda^+\rangle$ such that:*

1. *For each $\alpha < \lambda^+$, $|A_\alpha| = \mathrm{cf}(\lambda)$.*
2. *For each $\alpha < \lambda^+$, A_α is a cofinal subset of λ.*
3. *For each $\beta < \lambda^+$, there is a function $g_\beta : \beta \to \lambda$ such that $\{A_\alpha \setminus g_\beta(\alpha) \mid \alpha < \beta\}$ consists of mutually disjoint sets.*

Remark Note that there can be no function $g : \lambda^+ \to \lambda$ such that $\{A_\alpha \setminus g(\alpha) \mid \alpha < \lambda^+\}$ consists of disjoint sets. This theorem therefore gives an example of incompactness.

Proof Let $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ be a better $(\lambda^+, \vec{\mu})$-scale. For each $\alpha < \lambda^+$, let A_α be a subset of λ which codes f_α in a canonical way. By induction on β, we will show that for every $\beta < \lambda^+$, there is a function $g_\beta : \beta \to \lambda$ such that $\{A_\alpha \setminus g_\beta(\alpha) \mid \alpha < \beta\}$ consists of pairwise disjoint sets.

First, suppose that $\beta = \beta' + 1$. Let $g_\beta(\beta') = 0$. If $\alpha < \beta'$, let $k_\alpha \in \mathrm{cf}(\lambda)$ be large enough so that $\mu_{k_\alpha} > g_{\beta'}(\alpha)$ and, if $j \geq k_\alpha$, then $f_\alpha(j) < f_{\beta'}(j)$. Then, let $g_\beta(\alpha) = \mu_{k_\alpha}$. It is clear that this g_β is as required.

Now suppose that β is a limit ordinal. Since $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ is a better scale, there is D, a club in β, such that, for each $\gamma \in D$, there is an $i_\gamma < \mathrm{cf}(\lambda)$ such that, for every $\gamma < \gamma'$ in D, if $j \geq i_\gamma, i_{\gamma'}$, then $f_\gamma(j) < f_{\gamma'}(j)$. Let $\alpha < \beta$. Then there is a unique $\gamma \in D$ such that $\gamma \leq \alpha < \bar{\gamma}$, where $\bar{\gamma}$ denotes the smallest ordinal of D larger than γ. Define $k_\alpha \in \mathrm{cf}(\lambda)$ such that

- $k_\alpha > i_\gamma, i_{\bar{\gamma}}$
- If $j \geq k_\alpha$, then $f_\gamma(j) < f_\alpha(j) < f_{\bar{\gamma}}(j)$, and
- $\mu_{k_\alpha} > g_{\bar{\gamma}}(\alpha)$

Then, let $g_\beta(\alpha) = \mu_{k_\alpha}$. We claim that this g_β works. To show this, take $\alpha < \alpha' < \beta$. If α and α' belong to the same interval of D (i.e., if there is $\gamma \in D$ such that $\gamma < \alpha < \alpha' < \bar{\gamma}$), then $g_\beta(\alpha) > g_{\bar{\gamma}}(\alpha)$ and $g_\beta(\alpha') > g_{\bar{\gamma}}(\alpha')$, so $((A_\alpha \setminus g_\beta(\alpha)) \cap (A_{\alpha'} \setminus g_\beta(\alpha'))) \subseteq ((A_\alpha \setminus g_{\bar{\gamma}}(\alpha)) \cap (A_{\alpha'} \setminus g_{\bar{\gamma}}(\alpha'))) = \varnothing$.

Suppose that α and α' do not belong to the same interval. Let $\gamma, \gamma' \in D$ be such that $\gamma < \alpha < \bar{\gamma}$ and $\gamma' < \alpha' < \bar{\gamma'}$. Note that $\bar{\gamma} \leq \gamma'$. Now, if $\mu_j > g_\beta(\alpha), g_\beta(\alpha')$, then $f_\alpha(j) < f_{\bar{\gamma}}(j) \leq f_{\gamma'}(j) < f_{\alpha'}(j)$. Thus, g_β is as required. ∎

6 Examples of Incompactness

We will now use the result of Theorem 5.4 to construct two concrete examples of incompactness, one of a topological nature and the other algebraic.

Theorem 6.1 *Let λ be a singular cardinal with $\mathrm{cf}(\lambda) = \omega$. If $\square^*_\lambda$ holds, then there is a first countable topological space X such that X is not metrizable, but every subspace $Y \subset X$ with $|Y| < \lambda^+$ is metrizable.*

Proof Since $\square^*_\lambda$ holds and $\mathrm{cf}(\lambda) = \omega$, there is a sequence $\langle A_\beta \mid \lambda < \beta < \lambda^+ \rangle$ such that, for every β,

1. A_β is a cofinal subset of λ
2. A_β is countable
3. There is a function $g_\beta : \beta \to \lambda$ such that $\{A_\alpha \setminus g_\beta(\alpha) \mid \lambda < \alpha < \beta\}$ consists of pairwise disjoint sets.

We define a topological space $X = \lambda \cup (\lambda, \lambda^+)$. λ is endowed with the discrete topology. In general, a subset U of X is open if for all α such that $\lambda < \alpha < \lambda^+$, if $\alpha \in U$, then $A_\alpha \setminus U$ is finite. Note that X is first countable: if $\alpha < \lambda$, then $\{\alpha\}$ is a neighborhood base for α. If $\alpha \in (\lambda, \lambda^+)$, then the cofinite subsets of A_α form a neighborhood base.

We show that every subspace $Y \subset X$ such that $|Y| < \lambda^+$ is metrizable. First note that every such subspace Y is contained in $\lambda \cup (\lambda, \beta)$ for some $\beta < \lambda^+$. It thus suffices to prove that $\lambda \cup (\lambda, \beta)$ is metrizable for every $\beta < \lambda^+$. Fix such a β. Pick $g_\beta : \beta \to \lambda$ such that $\{A_\alpha \setminus g_\beta(\alpha) \mid \lambda < \alpha < \beta\}$ consists of mutually disjoint sets. For each $\lambda < \alpha < \beta$, enumerate A_α as $\{\eta^\alpha_n \mid n < \omega\}$. Set $d(\alpha, \eta^\alpha_n) = 1/n$ if $\eta^\alpha_n \in A_\alpha \setminus g_\beta(\alpha)$ and $d(\alpha, \gamma) = 1$ in all other cases. It is routine to check that d is a metric and induces the subspace topology on $\lambda \cup (\lambda, \beta)$.

Finally, we show that X is not metrizable. Suppose for sake of contradiction that d is a metric compatible with X. Note that, if $\lambda < \alpha < \lambda^+$, then $\{\alpha\} \cup A_\alpha$ is an open set. Thus, there is an $n_\alpha < \omega$ such that if $d(\alpha, x) < 1/n_\alpha$, then $x \in A_\alpha$. Also, as $\alpha = \lim_{k\to\infty} \eta^\alpha_k$, there is an $\eta_\alpha \in A_\alpha$ such that $d(\alpha, \eta_\alpha) < 1/(2n_\alpha)$. Find $\alpha < \alpha'$ such that $n_\alpha = n_{\alpha'} = n$ and $\eta_\alpha = \eta_{\alpha'} = \eta$. Then $d(\alpha, \eta) < 1/(2n)$ and $d(\alpha', \eta) < 1/(2n)$, so $d(\alpha, \alpha') < 1/n$. But this means that $\alpha' \in A_\alpha$, which is a contradiction, since $\alpha' \notin \lambda$. ∎

Theorem 6.2 *Let κ be a singular cardinal with $\mathrm{cf}(\kappa) = \omega$. If $\square^*_\kappa$ holds, then there is an abelian group G of cardinality κ^+ such that every subgroup of G of cardinality $< \kappa^+$ is free but G is not free itself.*

Proof As before, fix a sequence $\langle A_\beta \mid \kappa < \beta < \kappa^+ \rangle$ such that, for all β,

1. A_β is a cofinal subset of κ
2. A_β is countable
3. There is a function $g_\beta : \beta \to \kappa$ such that $\{A_\alpha \setminus g_\beta(\alpha) \mid \kappa < \alpha < \beta\}$ consists of pairwise disjoint sets.

Enumerate each A_β as $\langle \eta^n_\beta \mid n < \omega \rangle$. Let G be the abelian group generated by elements $\{X_\eta \mid \eta < \kappa\} \cup \{Z^n_\beta \mid n < \omega, \kappa < \beta < \kappa^+\}$ subject to the relations $2Z^{n+1}_\beta - Z^n_\beta = X_{\eta^n_\beta}$ for every $n < \omega$ and $\kappa < \beta < \kappa^+$. G can be thought of us the quotient of the free abelian group, F, generated by $\{X_\eta \mid \eta < \kappa\} \cup \{Z^n_\beta \mid n < \omega, \kappa < \beta < \kappa^+\}$ with respect to these relations (so G consists of cosets of F). To simplify notation, we will use X_η and Z^n_β to refer to the cosets of F in G containing X_η and Z^n_β, respectively.

Claim 6.3 *If H is a subgroup of G and $|H| < \kappa^+$, then H is free.*

Because a subgroup of a free group is necessarily free, it suffices to prove that if H is generated by $\{X_\eta \mid \eta < \kappa\} \cup \{Z^n_\alpha \mid n < \omega, \kappa < \alpha < \beta\}$ for some $\beta < \kappa^+$, then H is free. For each $\alpha < \beta$, let $k_\alpha = g_\beta(\alpha)$, and let $A^*_\alpha = \{\eta^i_\alpha \mid i \geq k_\alpha\}$ (so $\langle A^*_\alpha \mid \kappa < \alpha < \beta \rangle$ is a sequence of pairwise disjoint sets). We claim that H is generated freely by $S = \{X_\eta \mid \eta \notin \bigcup_{\alpha<\beta} A^*_\alpha\} \cup \{Z^i_\alpha \mid \kappa < \alpha < \beta,\ i \geq k_\alpha\}$.

Let H' be the group generated by S. We will show that $H'{=}H$. First, fix $\eta < \kappa$. If $\eta \notin \bigcup_{\alpha<\beta} A^*_\beta$, then X_η is a generator of H'. So, suppose that $\eta \in A^*_\alpha$ for some $\alpha < \beta$. Then $\eta = \eta^i_\alpha$ for some $i \geq k_\alpha$. But then Z^{i+1}_α and Z^i_α are in S, so, since $2Z^{i+1}_\alpha - Z^i_\alpha = X_{\eta^i_\alpha}$, we have that $X_\eta \in H'$. Thus, $X_\eta \in H'$ for every $\eta < \kappa$. Now fix α such that $\kappa < \alpha < \beta$. $Z^{k_\alpha - 1}_\alpha \in H'$, since $2Z^{k_\alpha}_\alpha - Z^{k_\alpha-1}_\alpha = X_{\eta^{k_\alpha-1}_\alpha}$ and both $Z^{k_\alpha}_\alpha$ and $X_{\eta^{k_\alpha-1}_\alpha}$ are in H'. Continuing inductively in this way, one shows that $Z^i_\alpha \in H'$ for every $\kappa < \alpha < \beta$ and $i < \omega$. Thus, $H \subseteq H'$, so in fact $H = H'$.

We now check that S generates H freely. To do this, suppose we have a relation $\sum r_i Z^{\ell_i}_{\beta_i} + \sum s_j X_{\eta_j} = 0$ which holds in H (and hence in G), where all $Z^{\ell_i}_{\beta_i}$ and X_{η_j} are from S. Then, by the construction of G, it must be the case that this relation is a linear combination of our basic relations of the form $2Z^{n+1}_\alpha - Z^n_\alpha - X_{\eta^n_\alpha} = 0$ for $n < \omega$ and $\kappa < \alpha < \kappa^+$. Say that $\sum r_i Z^{\ell_i}_{\beta_i} + \sum s_j X_{\eta_j} = \sum t_k R_k$, where the R_k are of the form $2Z^{n+1}_\alpha - Z^n_\alpha - X_{\eta^n_\alpha}$. Let LHS denote $\sum r_i Z^{\ell_i}_{\beta_i} + \sum s_j X_{\eta_j}$ and RHS denote $\sum t_k R_k$.

Subclaim 6.4 *If $\kappa < \alpha < \kappa^+$ and $i < \omega$ are such that Z^i_α is not in S, then $2Z^{i+1}_\alpha - Z^i_\alpha - X_{\eta^i_\alpha}$ cannot appear in the RHS.*

First note that if $Z^i_\alpha \notin S$, then $Z^j_\alpha \notin S$ for all $j < i$. Now suppose for sake of contradiction that $Z^i_\alpha \notin S$ but $2Z^{i+1}_\alpha - Z^i_\alpha - X_{\eta^i_\alpha}$ does appear in the RHS. Then, since Z^i_α does not appear in the LHS, it must be canceled by another term in the RHS. But the only term that can do this is $2Z^i_\alpha - Z^{i-1}_\alpha - X_{\eta^{i-1}_\alpha}$, so this term must appear in the RHS. But then, continuing inductively, we find that $2Z^1_\alpha - Z^0_\alpha - X_{\eta^0_\alpha}$ must appear in the RHS. $Z^0_\alpha \notin S$, so it doesn't appear in the LHS. However, there is nothing that can cancel it in the RHS. This is a contradiction and proves the subclaim.

We now claim that the LHS is not of the form $\sum s_j X_{\eta_j}$ (where at least one s_j is nonzero). To show this, suppose for sake of contradiction that it is of this form. Suppose η is such that X_η appears in the LHS. Then X_η must appear in the RHS. Then there is $\kappa < \alpha < \kappa^+$ and $i < \omega$ such that $\eta = \eta^i_\alpha$ and $2Z^{i+1}_\alpha - Z^i_\alpha - X_{\eta^i_\alpha}$ appears in the RHS. But Z^i_α does not appear in the LHS, so something must cancel it in the RHS. By the same argument as in the subclaim, we arrive at a contradiction.

Now suppose that some r_i in the LHS is non-zero. Fix α such that $Z_\alpha^{\ell_i}$ appears in the LHS for some ℓ_i. Let ℓ be smallest such that Z_α^ℓ appears in the LHS. Note that, by the subclaim, $2Z_\alpha^\ell - Z_\alpha^{\ell-1} - X_{\eta_\alpha^{\ell-1}}$ cannot appear in the RHS. Thus, $2Z_\alpha^{\ell+1} - Z_\alpha^\ell - X_{\eta_\alpha^\ell}$ appears in the RHS. $\eta_\alpha^\ell \in A_\alpha^*$, so $X_{\eta_\alpha^\ell} \notin S$, so it does not appear in the LHS. It must therefore be canceled in the LHS. This implies that there is $\gamma \neq \alpha$ and $j < \omega$ such that $\eta_\alpha^\ell = \eta_\gamma^j$ and $2Z_\gamma^{j+1} - Z_\gamma^j - X_{\eta_\gamma^j}$ appears on the RHS. But, since $\gamma \neq \alpha$, either $\gamma \geq \beta$ or $\gamma < \beta$ and $\eta_\gamma^j \notin A_\gamma^*$ (so $j < k_\gamma$). In either case, $Z_\gamma^j \notin S$, contradicting the sublaim. Thus, the relation is trivial, so S generates H freely.

Claim 6.5 *G is not free.*

Suppose for sake of contradiction that G is free. Fix a set of T of elements of G such that T generates G freely. By the regularity of κ^+, we can find a $\beta < \kappa^+$ such that, if H is the subgroup generated by $\{X_\eta \mid \eta < \kappa\} \cup \{Z_\alpha^n \mid n < \omega,\ \alpha < \beta\}$, then H is generated freely by $T \cap H$. It follows that the quotient group G/H is free.

Now, in G/H, we have that $2Z_\beta^{n+1} - Z_\beta^n = 0$ for all $n < \omega$. Thus, for all $n < \omega$, $Z_\beta^0 = 2^n Z_\beta^n$. In particular, Z_β^0 is infinitely divisible. Since G/H is free, this means that, in G/H, $Z_\beta^0 = 0$. This implies that $Z_\beta^0 = \sum k_i Z_{\alpha_i}^{n_i} + \sum \ell_j X_{\eta_j}$, where each $\alpha_i < \beta$. Thus, the relation $Z_\beta^0 - \sum k_i Z_{\alpha_i}^{n_i} - \sum \ell_j X_{\eta_j}$ must hold in G, so this relation must be a linear combination of basic relations of the form $2Z_\alpha^{n+1} - Z_\alpha^n - X_{\eta_\alpha^n} = 0$. But this is impossible, since, to account for the Z_β^0 term, any such linear combination must contain some Z_β^n, where $n > 0$. Thus, G is not free, and, in light of the fact that every subgroup of G of cardinality $< \kappa^+$ is free, we get also that $|G| = \kappa^+$. ∎

7 Suslin trees

Definition If κ is an infinite cardinal, then a *Suslin tree* on κ is a tree T such that the nodes of T are ordinals less than κ and every branch and every antichain of T has cardinality $< \kappa$.

One of the first applications of the square principle was the following theorem of Jensen [7]:

Theorem 7.1 *If V=L, then, for all infinite cardinals κ, there is a Suslin tree on κ^+.*

The proof of this theorem actually shows that, if there are $\vec{C}$ and S such that $\vec{C} = \langle C_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+\rangle$ is a $\square_\kappa$-sequence, $S \subseteq \kappa^+$ is stationary such that, for all α limit, $\alpha < \kappa$, $C'_\alpha \cap S = \emptyset$ (where C'_α denotes the limit points of C_α), and $\diamondsuit(S)$ holds, then there is a Suslin tree on κ^+.

We are interested in determining the minimal assumptions required to guarantee the existence of a Suslin tree. The situation is rather complex for successors of singular cardinals. For example, if κ is a singular cardinal, it is unknown whether one can obtain a model in which there are no Suslin trees on κ^+ without killing all κ^+-Aronszajn trees.

The following result of Shelah [11] provides a slightly better result than Jensen's original theorem:

Theorem 7.2 *If κ is an infinite cardinal, $2^\kappa = \kappa^+$, and $S \subseteq \kappa^+$ is stationary such that, for all $\alpha \in S$, $\mathrm{cf}(\alpha) \neq \mathrm{cf}(\kappa)$, then $\diamondsuit(S)$ holds.*

Corollary 7.3 *If κ is an infinite cardinal, $\square_\kappa$ holds, and $2^\kappa = \kappa^+$, then there is a Suslin tree on κ^+.*

We prove here a strengthening of this result, showing that one can obtain a Suslin tree on κ^+ from weaker assumptions.

Theorem 7.4 *If κ is an infinite cardinal, $\square_{\kappa,<\mathrm{cf}(\kappa)}$ holds, and $2^\kappa = \kappa^+$, then there is a Suslin tree on κ^+.*

Proof We begin with the following claim:

Claim 7.5 *Suppose $\langle \mathcal{C}_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+\rangle$ is a $\square_{\kappa,<\mathrm{cf}(\kappa)}$-sequence. Then, for every stationary $S \subseteq \kappa^+$, there is a stationary $S^* \subseteq S$ and a $\square_{\kappa,<\mathrm{cf}(\kappa)}$-sequence $\langle \mathcal{C}^*_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+\rangle$ such that for all α, if $C \in \mathcal{C}^*_\alpha$, then $C' \cap S^* = \emptyset$, where C' denotes the limit points of C.*

We will prove this claim in parallel for singular and regular κ. If κ is singular, let $\langle \kappa_i \mid i < \mathrm{cf}(\kappa)\rangle$ be a sequence of regular cardinals cofinal in κ such that, for all i, $\mathrm{cf}(\kappa) < \kappa_i$. If κ is regular, let $\kappa_i = \kappa$ for all $i < \kappa$. We will now define, by induction on $\alpha < \kappa^+$, a sequence $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ (not necessarily a scale) such that, for all $\alpha, \alpha' < \kappa^+$, we have $f_\alpha \in \prod \kappa_i$ and, if $\alpha < \alpha'$, then $f_\alpha <^* f'_\alpha$.

If f_α has been defined, we simply let $f_{\alpha+1}$ be such that $f_\alpha <^* f_{\alpha+1}$. Suppose that α is a limit ordinal and $\langle f_\beta \mid \beta < \alpha\rangle$ has been defined. If $\mathrm{cf}(\alpha) < \kappa$, let $s(\alpha) = \sup\{\mathrm{otp}(C) \mid C \in \mathcal{C}_\alpha,\ \mathrm{otp}(C) < \kappa\}$. Note that, since $|\mathcal{C}_\alpha| < \mathrm{cf}(\kappa)$, we have $s(\alpha) < \kappa$. Now, for each $C \in \mathcal{C}_\alpha$, define $h_C \in \prod \kappa_i$ by

$$h_C(i) = \begin{cases} 0 & : \kappa_i \leq \mathrm{otp}(C) \\ \sup_{\beta \in C}\{f_\beta(i) + 1\} & : \mathrm{otp}(C) < \kappa_i \end{cases}$$

If κ is singular, then, for all $i < \mathrm{cf}(\kappa)$, let $f_\alpha(i) = \sup_{C \in \mathcal{C}_\alpha}\{h_C(i)\}$. If κ is regular and $\mathrm{cf}(\alpha) < \kappa$, then let $f_\alpha(i) = \sup_{C \in \mathcal{C}_\alpha}\{h_C(i)\}$. If κ is regular and $\mathrm{cf}(\alpha) = \kappa$, then simply let f_α be any $<^*$ bound for $\langle f_\beta \mid \beta < \alpha\rangle$.

Let $S \subseteq \kappa^+$ be stationary. Assume that, for all $\alpha \in S$, $\mathrm{cf}(\alpha) < \kappa$. By Fodor's Lemma, we can find a stationary $\bar{S} \subseteq S$ and a $\mu < \kappa$ such that $s(\alpha) = \mu$ for all $\alpha \in \bar{S}$. Fix i such that $\mu < \kappa_i$. Apply Fodor's Lemma again to obtain a stationary $S^* \subseteq \bar{S}$ and an $\eta < \kappa_i$ such that $f_\alpha(i) = \eta$ for every $\alpha \in S^*$.

Let $\beta < \kappa^+$ be such that $\mathrm{cf}(\beta) > \omega$. We claim that for every $C \in \mathcal{C}_\beta$, $C' \cap S^*$ contains at most one point. Suppose for sake of contradiction that $\alpha < \alpha'$ are such that $\alpha, \alpha' \in C' \cap S^*$. Then $C \cap \alpha' \in \mathcal{C}_{\alpha'}$, so we considered $C \cap \alpha'$ when we defined $f_{\alpha'}$. Since $\mathrm{otp}(C \cap \alpha') \leq \mu < \kappa_i$, $h_{C\cap\alpha'} = \sup_{\gamma \in C \cap \alpha'}\{f_\gamma(i) + 1\}$. Then $f_{\alpha'}(i) = \sup_{D \in \mathcal{C}_{\alpha'}}\{h_D(i)\} \geq h_{C\cap\alpha'}(i) > f_\alpha(i)$. But this contradicts the fact that $\alpha, \alpha' \in S^*$. Thus, $C' \cap S^*$ contains at most one point, so, as before, we can adjust the $\square_{\kappa,<\mathrm{cf}(\kappa)}$-sequence so that it avoids S^*. This finishes the claim.

We will now sketch the construction of a κ^+-Suslin tree. The construction is very much like Jensen's original construction, which can be found in [7]. The reader is directed there for more details.

By the claim, we can assume that $\langle \mathcal{C}_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+ \rangle$ is a $\square_{\kappa,<\mathrm{cf}(\kappa)}$ -sequence, $S \subseteq \kappa^+$ is stationary such that $\mathrm{cf}(\alpha) \neq \mathrm{cf}(\kappa)$ for all $\alpha \in S$ and, for all limit $\alpha < \kappa^+$ and $C \in \mathcal{C}_\alpha$, we have $C' \cap S = \varnothing$. By the above theorem of Shelah, $\diamondsuit(S)$ holds, i.e., there is a sequence $\langle B_\alpha \mid \alpha \in S \rangle$ such that, for all $X \subseteq \kappa^+$, $\{\alpha \mid \alpha \in S,\ X \cap \alpha = B_\alpha\}$ is stationary in κ^+.

We will define a Suslin tree on κ^+ by recursion on the levels of the tree. At the successor stage, we will simply split above each node, so that every node on level α of the tree has two immediate successors in level $\alpha + 1$. If α is a limit ordinal, we define level α of the tree as follows. Let T_α be the tree up to level α. For every $x \in T_\alpha$ and every $C \in \mathcal{C}_\alpha$, we will define a branch in T_α, $b_{x,C}$, that will be continued. Let $\mathrm{lev}(x)$ denote the level of x in T_α.

Suppose first that $\alpha \notin S$. Let $x_0 = x$. Let x_1 be the least ordinal in T_α above x_0 in level β_0, where β_0 is the least $\beta \in C$ such that $\beta > \mathrm{lev}(x_0)$. If x_γ has been defined, let $x_{\gamma+1}$ be the least ordinal above x_γ in level β_γ of the tree, where β_γ is the least $\beta \in C$ such that $\beta > \mathrm{lev}(x_\gamma)$. If γ is a limit ordinal, let x_γ be the least ordinal in level $\sup_{\eta<\gamma}\{\beta_\eta\}$ of the tree such that x_γ is above x_η for every $\eta < \gamma$. Continue in this manner until reaching a stage δ such that $\{\mathrm{lev}(x_\gamma) \mid \gamma < \delta\}$ is cofinal in α. By the same argument used in Jensen's original proof, the coherence of the square sequence ensures that the construction will not break down before this point. Let $b_{x,C}$ be the downward closure of $\{x_\gamma \mid \gamma < \delta\}$, and place one node above $b_{x,C}$ in level α of the tree.

If $\alpha \in S$, then, if possible, let x_0 be the least ordinal above x in T_α such that $x_0 \in B_\alpha$ and then continue defining $b_{x,C}$ as above.

It is routine to check by induction on $\alpha < \kappa^+$ that $|T_\alpha| \leq \kappa$. The rest of the argument that T is a Suslin tree is exactly as in Jensen's original proof. ■

8 The failure of square

In this section, we investigate the consistency strength of the failure of various square principles. We start with the following proposition of Burke and Kanamori (see [9]).

Proposition 8.1 *Suppose κ is a strongly compact cardinal, μ is a regular cardinal, and $\kappa \leq \mu$. Then, for all stationary $S \subseteq \mu$ such that $\mathrm{cf}(\alpha) < \kappa$ for all $\alpha \in S$, S reflects to some $\beta < \mu$.*

Corollary 8.2 *If κ is a strongly compact cardinal, then $\square_{\lambda,<\mathrm{cf}(\lambda)}$ fails for every $\lambda \geq \kappa$.*

The following result of Shelah provides a stronger result for singular cardinals above a strongly compact.

Theorem 8.3 *Suppose κ is a strongly compact cardinal, λ is a singular cardinal, and $\mathrm{cf}(\lambda) < \kappa$. Then there is no good λ^+-scale.*

Corollary 8.4 *If κ is strongly compact, λ is a singular cardinal, and $\mathrm{cf}(\lambda) < \kappa$, then $\Box^*_\lambda$ fails.*

However, a result of Cummings, Foreman, and Magidor [2] limits the extent to which the preceding results can be strengthened:

Theorem 8.5 *Suppose the existence of a supercompact cardinal is consistent. Then it is consistent that there is a supercompact cardinal κ such that $\Box_{\lambda,\mathrm{cf}(\lambda)}$ holds for all singular cardinals λ such that $\mathrm{cf}(\lambda) \geq \kappa$.*

We showed in Section 4 how to force to obtain the failure of square at a regular cardinal. Forcing to obtain the failure of square at a singular cardinal is more difficult. The following large cardinal notion will be of help in achieving this goal.

Definition A cardinal κ is *subcompact* if, for all $A \subseteq H_{\kappa^+}$, there is a $\mu < \kappa$, a $B \subseteq H_{\mu^+}$, and a $\pi : \langle H_{\mu^+}, \in, B\rangle \to \langle H_{\kappa^+}, \in, A\rangle$ such that π is an elementary embedding with critical point μ.

Proposition 8.6 *If κ is a subcompact cardinal, then $\Box_{\kappa,<\kappa}$ fails.*

Proof Suppose for sake of contradiction that $\langle \mathcal{C}_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+\rangle$ is a $\Box_{\kappa,<\kappa}$-sequence. We can code this sequence in a canonical way as a subset A of H_{κ^+}. By subcompactness, there is a $\mu < \kappa$, a $B \subseteq H_{\mu^+}$, and a $\pi : \langle H_{\mu^+}, \in, B\rangle \to \langle H_{\kappa^+}, \in, A\rangle$ such that π is elementary with critical point μ. By absoluteness of our coding, B codes a $\Box_{\mu,<\mu}$-sequence, $\langle \mathcal{C}^*_\beta \mid \beta \text{ limit}, \beta < \mu^+\rangle$. Let $D = \{\pi(\rho) \mid \rho < \mu^+\}$, and let $\eta = \sup(D)$. Let $C \in \mathcal{C}_\eta$, and let $E = C \cap D$. Note that E is a $<\mu$-closed, unbounded subset of η. Now, for every limit $\alpha < \mu^+$, since $|\mathcal{C}^*_\alpha| < \mu$, $\pi[\mathcal{C}^*_\alpha] = \mathcal{C}_{\pi(\alpha)}$. Thus, if $\pi(\alpha) \in E$, then $C \cap \pi(\alpha)$ is in the range of π. Therefore, if $F = \bigcup\{\pi^{-1}(C \cap \pi(\alpha)) \mid \pi(\alpha) \in E\}$, then F is an unbounded subset of μ^+ such that $F \cap \alpha \in \mathcal{C}^*_\alpha$ for every α that is a limit point of E. But this contradicts the fact that $\langle \mathcal{C}^*_\alpha \mid \alpha \text{ limit}, \alpha < \mu^+\rangle$ is a $\Box_{\mu,<\mu}$-sequence. Thus $\Box_{\kappa,<\kappa}$ fails. ∎

Another notion that will be of use to us is that of Prikry forcing. Let κ be a measurable cardinal, and fix a normal measure U on κ. The Prikry forcing poset $\mathbb{P}_\kappa$ consists of conditions of the form $\langle \vec{\beta}, A\rangle$, where $\vec{\beta}$ is a finite, increasing sequence from κ and $A \in U$. We say that $\langle \vec{\beta}^*, A^*\rangle \leq \langle \vec{\beta}, A\rangle$ if and only if $A^* \subseteq A$, $\vec{\beta}^*$ is an end extension of $\vec{\beta}$, and $\vec{\beta}^* \setminus \vec{\beta} \subseteq A$. In $V^{\mathbb{P}_\kappa}$, κ is a singular cardinal of countable cofinality. An important feature of this forcing is that it has the Prikry property: Given a statement Φ in the forcing language and a condition $\langle \vec{\beta}, A\rangle$, there is an $A^* \subseteq A$ such that $\langle \vec{\beta}, A^*\rangle$ decides the truth value of Φ.

We now present a result, due to Zeman, on the consistency of the failure of square at singular cardinals of countable cofinality.

Theorem 8.7 *Suppose κ is a subcompact measurable cardinal, and let $\mathbb{P}_\kappa$ be Prikry forcing for κ with respect to a normal measure U. Then $\Box_\kappa$ fails in $V^{\mathbb{P}_\kappa}$.*

Proof Suppose for sake of contradiction that $\square_\kappa$ holds in $V^{\mathbb{P}_\kappa}$. Let $\langle \dot{C}_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+\rangle$ be a sequence of $\mathbb{P}_\kappa$-names forced to be a $\square_\kappa$ sequence. $\mathbb{P}_\kappa$ and $\langle \dot{C}_\alpha \mid \alpha \text{ limit}, \alpha < \kappa^+\rangle$ can be coded by a single set $A \subseteq H_{\kappa^+}$. As κ is subcompact, there are $\mu < \kappa$, $\bar{A} \subseteq H_{\mu^+}$, and $\pi : \langle H_{\mu^+}, \in, \bar{A}\rangle \to \langle H_{\kappa^+}, \in, A\rangle$ such that π is elementary and $\mu = \text{crit}(\pi)$. By decoding $\bar{A}$, we obtain a forcing poset $\mathbb{P}_\mu$ and a sequence of $\mathbb{P}_\mu$-names, $\langle \dot{\bar{C}}_\alpha \mid \alpha \text{ limit}, \alpha < \mu^+\rangle$. By the elementarity of π, we may assume that every member of $\mathbb{P}_\mu$ is of the form $\langle \overrightarrow{\beta}, B\rangle$, where $\overrightarrow{\beta} \in {}^{<\omega}\mu$ and $B \subseteq \mu$ is such that $\pi(B) \in U$.

For $\alpha < \mu^+$ of countable cofinality, fix a condition $\langle \overrightarrow{\beta}_\alpha, B_\alpha\rangle \in \mathbb{P}_\mu$ and an $\eta_\alpha < \mu$ such that $\langle \overrightarrow{\beta}_\alpha, B_\alpha\rangle \Vdash \text{otp}(\dot{\bar{C}}_\alpha) = \check{\eta}_\alpha$. By Fodor's Lemma, we get a fixed $\overrightarrow{\beta}$ and η such that $S = \{\alpha \mid \text{cf}(\alpha) = \omega,\ \exists B_\alpha(\langle \overrightarrow{\beta}, B_\alpha\rangle \Vdash \text{otp}(\dot{\bar{C}}_\alpha) = \check{\eta})\}$ is stationary in μ^+. Note that, for any $\alpha, \alpha' \in S$, $\langle \overrightarrow{\beta}, B_\alpha\rangle$ and $\langle \overrightarrow{\beta}, B_{\alpha'}\rangle$ are compatible. Let $\rho = \sup \pi\text{"}\mu^+$. $\pi\text{"}\mu^+$ is ω-closed and cofinal in ρ, so, as S is stationary in μ^+, $\pi\text{"}S$ is a stationary subset of ρ.

Let $D = \{\gamma \mid \gamma < \rho,\ \text{cf}(\gamma) = \omega,\ \exists B \in U(\langle \overrightarrow{\beta}, B\rangle \Vdash \text{"}\gamma \text{ is a limit point of } \dot{C}_\rho\text{"})\}$. We claim first that D is ω-closed. To show this, let $\langle \gamma_i \mid i < \omega\rangle$ be an increasing sequence from D. For each $i < \omega$, there is $B_i \in U$ such that $\langle \overrightarrow{\beta}, B_i\rangle \Vdash \check{\gamma}_i \in \dot{C}_\rho$. Then $\langle \overrightarrow{\beta}, \bigcap_{i<\omega} B_i\rangle \Vdash \sup(\check{\gamma}_i) \in \dot{C}_\rho$.

We next claim that D is unbounded in ρ. Suppose for sake of contradiction that D is bounded. Let F be a club in ρ such that $\text{otp}(F) = \mu^+ < \kappa$ and, for every $\delta \in F$, $\sup(D) < \delta$. Then for every $\delta \in F$, there is a $B_\delta \in U$ such that $\langle \overrightarrow{\beta}, B_\delta\rangle \Vdash \text{"}\check{\delta} \text{ is not a limit point of } \dot{C}_\rho\text{"}$. Then $\langle \overrightarrow{\beta}, \bigcap_{\delta \in F} B_\delta\rangle \Vdash \check{F} \cap \dot{C}_\delta = \varnothing$. But $\dot{C}_\rho$ is forced to be a club in ρ, and $\text{cf}(\rho)^{V_{\mathbb{P}_\kappa}} = \mu^+$, so F is a club subset of ρ in $V^{\mathbb{P}_\kappa}$. This is a contradiction.

Thus, D is an unbounded, ω-closed subset of ρ. Since $\text{cf}(\alpha) = \omega$ for all $\alpha \in \pi\text{"}S$, we know that $\pi\text{"}S \cap D$ is unbounded in ρ. Let $\gamma_1, \gamma_2 < \mu^+$ be such that $\pi(\gamma_1), \pi(\gamma_2) \in \pi\text{"}S \cap D$. We know that there are B_1^* and B_2^* such that $\langle \overrightarrow{\beta}, B_1^*\rangle \Vdash_{\mathbb{P}_\mu} \text{otp}(\dot{\bar{C}}_{\gamma_1}) = \check{\eta}$ and $\langle \overrightarrow{\beta}, B_2^*\rangle \Vdash_{\mathbb{P}_\mu} \text{otp}(\dot{\bar{C}}_{\gamma_2}) = \check{\eta}$. Thus, appealing to the elementarity of π, there are $B_1, B_2 \in U$ such that $\langle \overrightarrow{\beta}, B_1\rangle \Vdash \text{otp}(\dot{C}_{\pi(\gamma_1)}) = \check{\eta}$ and $\langle \overrightarrow{\beta}, B_2\rangle \Vdash \text{otp}(\dot{C}_{\pi(\gamma_2)}) = \check{\eta}$. Also, there are $B_3, B_4 \in U$ such that $\langle \overrightarrow{\beta}, B_3\rangle \Vdash \pi(\check{\gamma}_1)$ is a limit point of $\dot{C}_\rho$ and $\langle \overrightarrow{\beta}, B_4\rangle \Vdash \pi(\check{\gamma}_2)$ is a limit point of $\dot{C}_\rho$. But then $\langle \overrightarrow{\beta}, B_1 \cap B_2 \cap B_3 \cap B_4\rangle \Vdash \dot{C}_{\pi(\gamma_1)} = \dot{C}_\rho \cap \pi(\gamma_1)$, $\dot{C}_{\pi(\gamma_2)} = \dot{C}_\rho \cap \pi(\gamma_2)$, and $\text{otp}(\dot{C}_{\pi(\gamma_1)}) = \text{otp}(\dot{C}_{\pi(\gamma_2)})$. This is a contradiction. Thus, $\square_\kappa$ fails in $V^{\mathbb{P}_\kappa}$. ∎

There is a limit to how far we can extend this result, though, as evidenced by the following theorem of Cummings and Schimmerling [3].

Theorem 8.8 *Suppose that κ is a measurable cardinal and $\mathbb{P}_\kappa$ is Prikry forcing for κ. Then $\square_{\kappa,\omega}$ holds in $V^{\mathbb{P}_\kappa}$.*

9 Weak squares at singular cardinals

We end with a result showing that it is difficult to avoid weak squares at singular cardinals. The theorems in this section are due both to Gitik and to Dzamonja and Shelah. We start with a definition.

Definition Let $S \subseteq \kappa^+$ be a set of ordinals. We say that $\langle C_\alpha \mid \alpha \text{ limit, } \alpha \in S\rangle$ is a *partial square sequence* if, for all limit $\alpha \in S$:

1. C_α is a club in α.
2. If β is a limit point of C_α, then $\beta \in S$ and $C_\beta = C_\alpha \cap \beta$.
3. $\operatorname{otp}(C_\alpha) \leq \kappa$.

If such a sequence exists, then we say that S carries a partial square sequence.

The following fact is due to Shelah and can be found in [12].

Proposition 9.1 *Suppose κ is a regular cardinal and $\kappa > \aleph_1$. Then there is a sequence of sets $\langle S_i \mid i < \kappa\rangle$ such that*

1. *$\bigcup_{i<\kappa} S_i = \{\alpha < \kappa^+ \mid \operatorname{cf}(\alpha) < \kappa\}$.*
2. *For each $i < \kappa$, S_i carries a partial square sequence, $\langle C^i_\alpha \mid \alpha \text{ limit, } \alpha \in S_i\rangle$.*

Moreover, if κ is weakly inaccessible, then for every $i < \kappa$, there is $\mu_i < \kappa$ such that for all limit $\alpha \in S_i$, $\operatorname{otp}(C^i_\alpha) < \mu_i$.

Theorem 9.2 *Suppose W is an outer model of V, κ is an inaccessible cardinal in V and a singular cardinal in W, and $(\kappa^+)^V = (\kappa^+)^W$. In V, let $\langle D_\alpha \mid \alpha < \kappa^+\rangle$ be a sequence of clubs in κ. Then, in W, there is a sequence $\langle \delta_i \mid i < \operatorname{cf}(\kappa)\rangle$ cofinal in κ such that, for each $\alpha < \kappa^+$, $\{\delta_i \mid i < \operatorname{cf}(\kappa)\rangle \setminus D_\alpha$ is bounded in κ. Moreover, if $\mu < \kappa$, then we may assume that for every $i < \operatorname{cf}(\kappa)$, $\operatorname{cf}(\delta_i) \geq \mu$.*

Remark We will omit the proof of the "Moreover" clause at the end of the theorem and refer the interested reader to [5]. We give the proof of the remainder of the theorem here.

Proof Let $\operatorname{cf}(\kappa)^W = \mu$. In V, let $\langle S_i \mid i < \kappa\rangle$ be as given by Proposition 9.1. For each $i < \kappa$, let $\langle C^i_\alpha \mid \alpha \text{ limit, } \alpha \in S_i\rangle$ be a partial square sequence and let $\mu_i < \kappa$ be such that for all limit $\alpha \in S_i$, $\operatorname{otp}(C^i_\alpha) < \mu_i$. Since $(\kappa^+)^V = (\kappa^+)^W$ and the relevant notions are absolute, the following holds in W:

1. $\bigcup_{i<\kappa} S_i \supseteq \{\alpha < \kappa^+ \mid \operatorname{cf}(\alpha) \neq \mu\}$.
2. For each $i < \kappa$, $\langle C^i_\alpha \mid \alpha \text{ limit, } \alpha \in S_i\rangle$ is a partial square sequence.
3. For each $i < \kappa$, for all limit $\alpha \in S_i$, $\operatorname{otp}(C^i_\alpha) < \mu_i$.

We now work in W.

Claim 9.3 *There is an $i^* < \kappa$ such that, if $C \in W$ is a club in κ^+, then for stationarily many $\alpha \in S_{i^*}$, $C^{i^*}_\alpha \cap C$ is a club in α and $\operatorname{cf}(\alpha) = \mu^+$.*

$S^{\kappa^+}_{\mu^+} = \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \mu^+\}$ is stationary in κ^+ and $S^{\kappa^+}_{\mu^+} \subseteq \bigcup_{i<\kappa} S_i$, so we can find an $i^* < \kappa$ such that $S_{i^*} \cap S^{\kappa^+}_{\mu^+}$ is stationary. But then, if C is a club in κ^+ and $\alpha \in S_{i^*} \cap S^{\kappa^+}_{\mu^+} \cap C'$, then $C \cap C^{i^*}_\alpha$ is a club in α and $\mathrm{cf}(\alpha) = \mu^+$.

Claim 9.4 *In V, there is a sequence $\langle D^*_\alpha \mid \alpha < \kappa^+\rangle$ such that:*

1. *For each $\alpha < \kappa^+$, D^*_α is a club in κ and $D^*_\alpha \subseteq D_\alpha$.*
2. *If $\alpha < \beta < \kappa^+$, then $|D^*_\beta \setminus D^*_\alpha| < \kappa$.*
3. *If $\beta \in C^{i^*}_\alpha$, then $D^*_\alpha \subseteq D^*_\beta$.*

Work in V. We will prove this claim by recursion on $\alpha < \kappa^+$. Let $D^*_0 = D_0$. Suppose $\langle D^*_\beta \mid \beta < \alpha\rangle$ has been defined. Let $D^\alpha = \Delta_{\beta<\alpha} D^*_\beta$. If $\alpha \in S_{i^*}$, let $D^*_\alpha = D_\alpha \cap D^\alpha \cap \bigcap_{\beta \in C^{i^*}_\alpha} D^*_\beta$. If $\alpha \notin S_{i^*}$, let $D^*_\alpha = D_\alpha \cap D^\alpha$. Note that D^*_α is a club in κ and that the sequence $\langle D^*_\alpha \mid \alpha < \kappa^+\rangle$ is as required.

Move back to W. Let $\langle \rho_\gamma \mid \gamma < \mu\rangle$ be a strictly increasing sequence cofinal in κ. Assume moreover that the sequence is anti-continuous, i.e., for every limit $\gamma < \mu$, $\sup_{\gamma'<\gamma} \rho_{\gamma'} < \rho_\gamma$. Let F_γ be the interval $(\sup_{\gamma'<\gamma} \rho_{\gamma'}, \rho_\gamma)$. For every $\alpha < \kappa^+$, if $D^*_\alpha \cap F_\gamma \neq \emptyset$, let $\rho^\alpha_\gamma = \sup(D^*_\alpha \cap F_\gamma)$. If $D^*_\alpha \cap F_\gamma = \emptyset$, then ρ^α_γ is not defined. Let $d_\alpha = \{\gamma \mid D^*_\alpha \cap F_\gamma \neq \emptyset\}$. Let $E_\alpha = \{\rho^\alpha_\gamma \mid \gamma \in d_\alpha\}$. Note that if $\alpha < \alpha'$, then $D^*_{\alpha'} \setminus D^*_\alpha$ is bounded in κ, so $d_{\alpha'} \setminus d_\alpha$ is bounded in μ.

Suppose there is an $\alpha < \kappa^+$ such that for every $\alpha < \alpha' < \kappa^+$, $|E_{\alpha'} \Delta E_\alpha| < \mu$. Then it is easy to verify that, if $\langle \delta_i \mid i < \mu\rangle$ is an enumeration of E_α, then $\langle \delta_i \mid i < \mu\rangle$ is as required, and we are done.

If $\alpha < \alpha'$, we say that there is a *major change* between α and α' if $|E_\alpha \Delta E_{\alpha'}| = \mu$. Note that if there is a major change between α and α' and $\alpha' < \alpha''$, then there is a major change between α and α''.

Suppose now that for every $\alpha < \kappa^+$, there is an $f(\alpha) > \alpha$ such that there is a major change between α and $f(\alpha)$. Let $C = \{\alpha < \kappa^+ \mid \alpha \text{ is closed under } f\}$. C is a club in κ^+, so there is an $\alpha^* \in S_{i^*}$ such that $C^{i^*}_{\alpha^*} \cap C$ is a club in α^* and $\mathrm{cf}(\alpha^*) = \mu^+$. Let $\langle \alpha_\xi \mid \xi < \mu^+\rangle$ be an increasing enumeration of a cofinal subsequence of the limit points of $C^{i^*}_{\alpha^*}$. Note that for $\xi < \xi' < \mu^+$, $D^*_{\alpha_{\xi'}} \subseteq D^*_{\alpha_\xi}$, so $d_{\alpha_{\xi'}} \subseteq d_{\alpha_\xi}$. Thus, since $\langle d_{\alpha_\xi} \mid \xi < \mu^+\rangle$ is a decreasing sequence of length μ^+ of subsets of μ, there is a fixed $d \subset \mu$ such that $d_{\alpha_\xi} = d$ for sufficiently large ξ.

If $\gamma \in d$ and $\xi < \xi' < \mu^+$, then, since $D^*_{\alpha_{\xi'}} \subseteq D^*_{\alpha_\xi}$, $\rho^{\alpha_{\xi'}}_\gamma \leq \rho^{\alpha_\xi}_\gamma$. Thus, for every $\gamma \in d$, there is a $\xi_\gamma < \mu^+$ and a $\bar{\rho}_\gamma$ such that for all $\xi > \xi_\gamma$, $\rho^{\alpha_\xi} = \bar{\rho}_\gamma$. Let $\xi^* = \sup\{\xi_\gamma \mid \gamma \in d\}$. Then, for every $\xi > \xi^*$ and every $\gamma \in d$, we have $\rho^{\alpha_\xi}_\gamma = \bar{\rho}_\gamma$, so there is a fixed E such that, for all $\xi > \xi^*$, we have $E_{\alpha_\xi} = E$.

Now let $\xi^* < \xi < \xi'$ be such that $\xi, \xi' \in C$. Then $f(\alpha_\xi) < \alpha_{\xi'}$, so there is a major change between α_ξ and $\alpha_{\xi'}$. But $E_{\alpha_\xi} = E = E_{\alpha_{\xi'}}$. This is a contradiction, and we are finished. ∎

Theorem 9.5 *Suppose W is an outer model of V, κ is an inaccessible cardinal in V and a singular cardinal of countable cofinality in W, and $(\kappa^+)^V = (\kappa^+)^W$. Then $\square_{\kappa,\omega}$ holds in W.*

Proof We will define a sequence $\langle \mathcal{C}_\alpha \mid \alpha$ limit, $\kappa < \alpha < \kappa^+\rangle$ in W witnessing $\square_{\kappa,\omega}$. We define $\mathcal{C}_\alpha$ if and only if $\mathrm{cf}(\alpha)^V \neq \kappa$. If $\mathrm{cf}(\alpha)^V = \kappa$, then $\mathrm{cf}(\alpha)^W = \omega$, and the definition of a suitable $\mathcal{C}_\alpha$ is trivial.

Let χ be a sufficiently large regular cardinal, and let $<_\chi$ be a well-ordering of H_χ. Work in V. For α limit, $\kappa < \alpha < \kappa^+$, let $\langle M^\alpha_\gamma \mid \gamma < \kappa\rangle$ be a continuous $\subseteq$-increasing sequence of elementary submodels of H_χ such that:

1. $\alpha, \kappa \in M^\alpha_0$.
2. For every $\gamma < \kappa$, $|M^\alpha_\gamma| < \kappa$.
3. For every $\gamma < \kappa$, $M^\alpha_\gamma \cap \kappa$ is an ordinal.

Note that $\alpha \subseteq \bigcup_{\gamma<\kappa} M^\alpha_\gamma$, since $\kappa \subseteq \bigcup_{\gamma<\kappa} M^\alpha_\gamma$ and there is a function in H_χ mapping κ onto α.

For each limit ordinal α with $\kappa < \alpha < \kappa^+$, let $D_\alpha = \{M^\alpha_\gamma \cap \kappa \mid \gamma \leq \kappa\}$. D_α is a club in κ, so, by Theorem 9.2, there is in W a sequence $\langle \delta_n \mid n < \omega\rangle$ cofinal in κ such that, for every limit ordinal α with $\kappa < \alpha < \kappa^+$, $\{\delta_n \mid n < \omega\} \setminus D_\alpha$ is finite and, for every $n < \omega$, $\mathrm{cf}(\delta_n) > \omega$.

Claim 9.6 *If $\mathrm{cf}(\gamma) > \omega$ and $M^\alpha_\gamma \cap \alpha$ is cofinal in α, then $M^\alpha_\gamma \cap \alpha$ is ω-closed.*

Suppose $\langle \beta_n \mid n < \omega\rangle$ is an increasing sequence from $M^\alpha_\gamma \cap \alpha$ with $\beta_\omega = \sup_{n<\omega}(\beta_n) < \alpha$. Suppose for sake of contradiction that $\beta_\omega \notin M^\alpha_\gamma$. Let $\bar\beta_\omega$ be the minimal element of $M^\alpha_\gamma \cap \alpha$ above β_ω. It is easy to see that $\mathrm{cf}(\bar\beta_\omega) = \kappa$: If $\mathrm{cf}(\bar\beta_\omega) < \kappa$, then $\mathrm{cf}(\bar\beta_\omega) + 1 \subseteq M^\alpha_\gamma$, so M^α_γ is cofinal in $\bar\beta_\omega$. But this is a contradiction, since there are no points in M^α_γ between β_ω and $\bar\beta_\omega$. Thus, there is $E \in M^\alpha_\gamma$ such that E has order type κ and is cofinal in $\bar\beta_\omega$. Then, for each $n < \omega$, there is $\bar\beta_n \in M^\alpha_\gamma \cap E$ such that $\bar\beta_n \geq \beta_n$. But $\langle \bar\beta_n \mid n < \omega\rangle$ can not be cofinal in $M^\alpha_\gamma \cap E$, because $M^\alpha_\gamma \cap E$ has order type $M^\alpha_\gamma \cap \kappa$, and $\mathrm{cf}(M^\alpha_\gamma \cap \kappa) = \mathrm{cf}(\gamma) > \omega$. Thus, there is $\beta \in M^\alpha_\gamma \cap E$ such that $\beta > \beta_\omega$. But this contradicts our choice of $\bar\beta_\omega$, thus proving the claim.

If α is a limit ordinal, $\kappa < \alpha < \kappa^+$, and $\mathrm{cf}(\alpha) \neq \kappa$, then there is $\gamma < \kappa$ such that for all $\gamma' \geq \gamma$, $M^\alpha_{\gamma'}$ is cofinal in α. Now let $\mathcal{C}_\alpha = \{\overline{M^\beta_\gamma \cap \alpha} \mid \beta \geq \alpha,\ \gamma < \kappa,\ M^\beta_\gamma \cap \alpha$ is cofinal in α, $M^\beta_\gamma \cap \beta$ is cofinal in β, and $M^\beta_\gamma \cap \kappa = \delta_n$ for some $n < \omega\}$, where $\overline{M^\beta_\gamma \cap \alpha}$ denotes the closure of $M^\beta_\gamma \cap \alpha$. By construction, $\mathcal{C}_\alpha$ consists of clubs in α of order type $< \kappa$, and, by the choice of $\langle \delta_n \mid n < \omega\rangle$, each $\mathcal{C}_\alpha$ is nonempty. Also, if $\delta < \alpha$ is a limit point of $\overline{M^\beta_\gamma \cap \alpha}$, then it is immediate from our construction that $\overline{M^\beta_\gamma \cap \delta} \in \mathcal{C}_\delta$, so the coherence property holds.

It remains to show that $|\mathcal{C}_\alpha| \leq \omega$. Suppose we are given β, β', γ, and γ' such that, for some $n < \omega$, $M^\beta_\gamma \cap \kappa = \delta_n = M^{\beta'}_{\gamma'} \cap \kappa$ and both M^β_γ and $M^{\beta'}_{\gamma'}$ are cofinal in α. Notice that, since $\mathrm{cf}(\delta_n) > \omega$, it must be that $\mathrm{cf}(\gamma) > \omega$, so, by our claim, $M^\beta_\gamma \cap \beta$ is ω-closed. We claim that $M^\beta_\gamma \cap \alpha = M^{\beta'}_{\gamma'} \cap \alpha$. Note that this claim implies $|\mathcal{C}_\alpha| \leq \omega$, thus finishing the proof of the theorem.

First, suppose $\mathrm{cf}(\alpha) = \omega$. Then $\alpha \in M^\beta_\gamma, M^{\beta'}_{\gamma'}$. But then, since M^β_γ and $M^{\beta'}_{\gamma'}$ are elementary submodels of H_χ having the same intersection with κ, they also have the same functions from κ to α, so $M^\beta_\gamma \cap \alpha = M^{\beta'}_{\gamma'} \cap \alpha$.

Finally, suppose $\mathrm{cf}(\alpha) > \omega$. If $\delta \in M^{\beta}_{\gamma} \cap M^{\beta'}_{\gamma'} \cap \alpha$ then, by the argument of the previous paragraph, $M^{\beta}_{\gamma} \cap \delta = M^{\beta'}_{\gamma'} \cap \delta$. However, since $M^{\beta}_{\gamma} \cap \alpha$ and $M^{\beta'}_{\gamma'} \cap \alpha$ are ω-closed and cofinal in α, $M^{\beta}_{\gamma} \cap M^{\beta'}_{\gamma'} \cap \alpha$ is also ω-closed and cofinal in α, so $M^{\beta}_{\gamma} \cap \alpha = M^{\beta'}_{\gamma'} \cap \alpha$. ∎

References

[1] James Cummings. Notes on singular cardinal combinatorics. *Notre Dame J. Formal Logic*, 46(3):251-282, 2005.

[2] James Cummings, Matthew Foreman, and Menachem Magidor. Squares, scales, and stationary reflection. *J. Math. Log.*, 1(1): 35-98, 2001.

[3] James Cummings and Ernest Schimmerling. Indexed squares. *Israel J. Math.*, 131:61-99, 2002.

[4] Keith J. Devlin. *Constructibility*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1984.

[5] Mirna Džamonja and Saharon Shelah. On squares, outside guessing of clubs and $I_{<f}[\lambda]$. *Fund. Math.*, 148(2):165-198, 1995.

[6] Ronald B. Jensen. Some remarks on $\square$ below $0^{\P}$. Circulated notes.

[7] Ronald B. Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4(3):229-308, 1972.

[8] William Mitchell. Aronszajn trees and the independence of the transfer property. *Ann. Math. Logic*, 5(1):21-46, 1972.

[9] Ernest Schimmerling. Combinatorial principles in the core model for one Woodin cardinal. *Ann. Pure Appl. Logic*, 74(2):153-201, 1995.

[10] Ernest Schimmerling and Martin Zeman. Square in core models. *Bull. Symbolic Logic*, 7(3):305-314, 2001.

[11] Saharon Shelah. On the successors of singular cardinals. In *Logic Colloquium 78*, volume 97 of *Stud. Logic Foundations Math*, pages 357-380. North-Holland, Amsterdam-New York, 1979.

[12] Saharon Shelah. *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, New York, 1994.