Set theory and operator algebras (the final thirty minutes of the talk)

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Normal operators

Fix a separable complex Hilbert space H and consider unitary equivalence of operators on H:

$$a \sim b \Leftrightarrow (\exists u) a = ubu^*.$$

The spectral theorem provides complete invariants for unitary equivalence of normal operators. (a is *normal* if $aa^* = a^*$.) However, the invariants are quite complicated, and necessarily so.

Theorem (Kechris–Sofronidis, 1998)

The unitary equivalence of unitary operators does not admit any effectively assigned complete invariants coded by countable structures. (The same holds for self-adjoint operators.)

Equivalence modulo compact operators

a and b are *compalent* $\Leftrightarrow (\exists c) a \sim c$ and $c - b$ is compact.

Theorem (Weyl–von Neumann–Berg)

The essential spectrum, $\sigma_e(a)$, provides a complete invariant for compalence of normal operators.

Proof.

If a is normal, then there is a diagonal operator d such that $a - d$ is compact and $\sigma(d) = \sigma_e(a)$. □

Example

For $g \in S_\infty$ the equations $u_g(e_n) = e_{g(n)}$ uniquely define a unitary operator u_g. If g has arbitrarily long cycles, or an infinite cycle, then $\sigma_e(u_g) = \mathbb{S}^1$ and u_g is compalent with the bilateral shift of the basis.

The Calkin algebra

$\mathcal{B}(H)$: The algebra of all bounded operators on H.
$\mathcal{K}(H)$: Its ideal of compact operators.
$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is the *Calkin algebra*.
$\pi\colon \mathcal{B}(H) \to \mathcal{C}(H)$: The quotient map.

Fact

Operators a and b are compalent if and only if $\pi(a) \sim \pi(b)$.

Corollary

For normal operators a and b TFAE.

1. $\sigma_e(a) = \sigma_e(b)$,
2. *a and b are compalent,*
3. $\pi(a) \sim \pi(b)$ *in* $\mathcal{C}(H)$,
4. $(\exists \Phi \in \mathrm{Aut}(\mathcal{C}(H)))\, \Phi(\pi(a)) = \pi(b)$.

Let $S(e_n) = e_{n+1}$ and $\dot{S} = \pi(S)$.

Question (Brown–Douglas–Fillmore, 1977)

Is there an automorphism Φ *of* $\mathcal{C}(H)$ *such that*

1. $\Phi(\dot{S}) = \dot{S}^*$*?*
2. *there are* a *and* b *in* $\mathcal{C}(H)$ *such that* $\Phi(a) = b$ *but* $uau^* \neq b$ *for all* u*?*
3. *Is there an outer automorphism of* $\mathcal{B}(H)$*?*

The answer to (2) is negative if a and b are images of normal operators in $\mathcal{B}(H)$ (BDF).

Differences between $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and $\mathcal{C}(H)$

If Ψ_1 and Ψ_2 are automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, then so is $\Psi_1 \oplus \Psi_2$

Proposition

An automorphism of $\mathcal{C}(H)$ is inner iff its restriction to $\mathcal{C}(H_0)$ for some (any) infinite-dimensional subspace H_0 of H is implemented by a unitary.

Proof.

Fix u such that $\Phi(b) = ubu^*$ for $b \in \mathcal{C}(H_0)$.
Fix $v \in \mathcal{C}(H)$ so that $vv^* = \dot{P}_{H_0}$ and $v^*v = \dot{I}$. Then

$$\begin{aligned}\Phi(a) =& \Phi(v^*)\Phi(vav^*)\Phi(v)\\ =& \Phi(v^*)uvav^*u^*\Phi(v)\end{aligned}$$

With $w = \Phi(v^*)uv$ we have $\Phi(a) = waw^*$. □

An attempt and a misleading fact

1. Start from Rudin's nontrivial automorphism of $\mathcal{P}(\mathbb{N})$.
2. Extend it to an automorphism of $\mathcal{C}(H)$.
3. Check that it is outer.

This does not work.

Proposition

An automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *extends to an automorphism of* $\mathcal{C}(H)$ *if and only if it is trivial.*

All nontrivial elements of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ belong to the same orbit of $Aut(\mathcal{P}(\mathbb{N})/\operatorname{Fin})$.

In $\mathcal{C}(H)$ there are $2^{\aleph_0}$ many orbits.

Orbits of triples

Club many countable subalgebras of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are isomorphic to the countable atomless boolean algebra.

On finite-dimensional vector subspaces of $\mathcal{C}(H)$ one can define metric δ_{cb} - an operator space variant of the Banach–Mazur distance. This metric is an automorphism invariant.

Theorem (Junge–Pisier, 1995)

1. *δ_{cb} is separable on subspaces of any separable algebra.*
2. *δ_{cb} is nonseparable on three-dimensional subspaces of $\mathcal{C}(H)$.*

Corollary (Phillips, 2000)

No separable subalgebra of $\mathcal{C}(H)$ realizes all 3-types.
No separable C algebra is injectively universal.*

Theorem (Phillips–Weaver, 2006)

CH implies there is an outer automorphism of the Calkin algebra.

The proof uses very nontrivial methods developed for analysis of a deep extension of BDF theory, Kasparov's KK-theory, for separable C* algebras.

I will present my own elementary proof of the Phillips–Weaver theorem.

Coherent families of unitaries I

For a partition of $\mathbb{N}$ into finite intervals $\mathbb{N} = \bigcup_n J_n$

$$E_n = \overline{\mathrm{Span}}\{e_i \mid i \in J_n\}$$

is a decomposition of H into finite-dimensional orthogonal subspaces.
Then $\mathcal{D}[\vec{J}] = \mathcal{D}[\vec{E}]$ is the algebra of all operators that have each E_n as an invariant subspace.
Let

$$\vec{J}^{\mathrm{even}} = (J_{2n} \oplus J_{2n+1})_n$$
$$\vec{J}^{\mathrm{odd}} = (J_{2n+1} \oplus J_{2n+2})_n.$$

A unitary u defines an inner automorphism $\operatorname{Ad} u(a) = uau^*$.

Lemma

Assume u is a unitary and $\alpha_n \in \mathbb{C}$, $|\alpha_n| = 1$ for all n. If

$$v = \sum_n \alpha_n P_{J_n} u$$

then $\operatorname{Ad} u$ and $\operatorname{Ad} v$ agree on $\mathcal{D}[\vec{J}]$.

Pf. $a \in \mathcal{D}[\vec{J}]$ iff $a = \sum_n P_{J_n} a$. Then

$$vav^* = \sum_n \alpha_n P_n P_n a \overline{\alpha_n} P_n = \sum_n P_n a.$$

□

Write $\mathcal{U}(1) = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$.
For $\alpha \in (\mathcal{U}(1))^{\mathbb{N}}$

$$u_\alpha = \sum_n \alpha(n) P_{\{n\}}$$

is a unitary.
Define $\rho\colon \mathbb{N}^2 \times \mathcal{U}(1) \to [0, \infty)$:

$$\rho(i, j, \alpha, \beta) = |\alpha(i)\overline{\alpha(j)} - \beta(i)\overline{\beta(j)}|.$$

For $J = [m, n] \subseteq \mathbb{N}$ and α, β in $(\mathcal{U}(1))^{\mathbb{N}}$ let

$$\Delta_I(\alpha, \beta) = \sup_{m \le i < j \le n} \rho(i, j, \alpha, \beta).$$

Fact

$\Delta_I(\alpha, \beta) \approx \|P_I(\operatorname{Ad} u_\alpha - \operatorname{Ad} u_\beta)P_I\|$.

Let $u \sim_{\mathcal{D}[\vec{J}]} v$ iff $\operatorname{Ad} u(a) - \operatorname{Ad} v(a)$ is compact for all $a \in \mathcal{D}[\vec{J}]$.

Lemma

For $\alpha, \beta \in (\mathcal{U}(1))^{\mathbb{N}}$ TFAE:

1. $u_\alpha \sim_{\mathcal{D}[\vec{J}]} u_\beta$
2. $\limsup_n \Delta_{J_n}(\alpha, \beta) = 0$.

Nontrivial coherent families of unitaries

For partitions $\vec{J}$, $\vec{K}$ of $\mathbb{N}$ into finite intervals let

$$\vec{J} \ll \vec{K} \qquad \text{iff} \qquad (\forall m)(\exists n) J_m \subseteq K_n \cup K_{n+1}.$$

Fact

1. *The ordering $\ll$ is σ-directed.*
2. *It is cofinally equivalent to $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.*

Let $\mathcal{D}D[\vec{J}] = \mathcal{D}[\vec{J}^{\text{even}}] \cup \mathcal{D}[\vec{J}^{\text{odd}}]$.

A family $\mathcal{F}$ of pairs $(\vec{J}, u)$ is a *coherent family of unitaries* if

1. $\mathcal{F}_0 = \{\vec{J} \mid (\exists u_{\mathcal{F}})(\vec{J}, u) \in \mathcal{F}\}$, is $\ll$-cofinal,
2. For $\vec{J} \ll \vec{K}$ in $\mathcal{F}_0$ we have $u_{\vec{J}} \sim_{\mathcal{D}D[\vec{J}]} u_{\vec{K}}$.

Lemma

If $\mathcal{F}$ is a coherent family of unitaries then there is the unique automorphism $\Phi_{\mathcal{F}}$ of $\mathcal{C}(H)$ such that $\Phi_{\mathcal{F}}(\pi(a)) = \pi(uau^*)$ for all $(\vec{J}, u) \in \mathcal{F}$ and all $a \in \mathcal{D}[\vec{J}]$.

Pf. For any a in $\mathcal{B}(H)$ there are a fdod $\vec{J}$, $a^0 \in \mathcal{D}[\vec{J}^{\text{even}}]$ and $a^1 \in \mathcal{D}[\vec{J}^{\text{odd}}]$ such that $a - a^0 - a^1$ is compact. □

A coherent family of unitaries is *trivial* if there is u_0 such that $u_0 \sim_{\vec{J}} u$ for all $(\vec{J}, u) \in \mathcal{F}$. The automorphism $\Phi_{\mathcal{F}}$ is inner iff $\mathcal{F}$ is trivial.

Theorem (Farah, 2007)

Assume CH. Then there is a nontrivial coherent family of unitaries.

Pf. Construct $\ll$-increasing and cofinal $\mathcal{J}^\xi$, $\xi < \omega_1$, and $\alpha^\xi \in (\mathcal{U}(1))^{\mathbb{N}}$ such that for $\xi < \eta$

$$\limsup_n \Delta_{J_n^\xi}(\alpha^\xi, \alpha^\eta) = 0.$$

This defines an automorphism of $\mathcal{C}(H)$. One can build $2^{\aleph_1}$ different ones, and some of them ought to be trivial. □

Corollary (Phillips–Weaver, 2006)

CH implies there is an outer automorphism of $\mathcal{C}(H)$. □

Unlike my original proof, the above proof (due to S. Geschke) shows:

Theorem (Geschke)

$\mathfrak{d} = \aleph_1$ and $2^{\aleph_0} < 2^{\aleph_1}$ imply there are nontrivial automorphisms of $\mathcal{C}(H)$.

Todorcevic's Axiom: If $G = (V, E)$ is a graph such that

$$E = \bigcup_{n=0}^{\infty} U_n \times V_n,$$

then G is either countably chromatic or it has an uncountable clique.

TA is among the axioms that are sometimes called OCA.

Two consequences of TA

Lemma (Farah, 2007)

TA implies that every coherent family of unitaries $\mathcal{F}$ is trivial. In particular, if Φ is inner on each $\mathcal{C}[\vec{J}]$ then it is inner.

Pf. Wlog for each $\vec{E}$ we have

$$u_{\vec{J}} = u_{\alpha(\vec{J})}$$

for some $\alpha(\vec{J}) \in (\mathcal{U}(1))^{\mathbb{N}}$. For $k \in \mathbb{N}$ let G_k be the graph on $\mathcal{F}_0$ in which $\vec{J}$ and $\vec{K}$ are adjacent iff

$$\sup_{m,n} \Delta_{J_m \cap K_n}(\alpha(\vec{J}), \alpha(\vec{K})) > \frac{1}{k}$$

There are no uncountable cliques for any k.
Pick $\mathcal{F} \supseteq \mathcal{X}_1 \supseteq cX_2 \supseteq \dots$ such that each $\mathcal{X}_k$ is G_k-independent and $\ll^*$-cofinal. Then

$$\alpha = \lim_k \lim_{\vec{J} \in \mathcal{X}_k} \alpha(\vec{J})$$

works. □

For $M \subseteq \mathbb{N}$ let

$$\mathcal{D}_M[\vec{J}] = \{a \in \mathcal{D}[\vec{J}] \mid aP_{J_n} = 0 \text{ for all } n \notin M\}.$$

Lemma

Assume $|J_n| \leq |J_{n+1}|$ for all n. If Φ is implemented by a unitary on $\mathcal{C}_M[\vec{J}]$ for some infinite M then it is implemented by a unitary on $\mathcal{C}[\vec{J}]$,

Proof.

Find v such that $v^*v = I$ and $v\mathcal{D}[\vec{J}]v^* \subseteq \mathcal{D}_M[\vec{J}]$.
Then for $a \in \mathcal{D}[\vec{J}]$:

$$\begin{aligned}\Phi(a) &= \Phi(v^*)\Phi(vav^*)\Phi(v) \\ &= \Phi(v^*)uvav^*u^*\Phi(v)\end{aligned}$$

hence with $w = \Phi(v^*)uv$ we have that $\operatorname{Ad} w$ is a representation of Φ on $\mathcal{D}[\vec{J}]$. □

Representations (liftings)

Some $\Psi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ is a representation of $\Phi\colon \mathcal{C}(H) \to \mathcal{C}(H)$ if

$$\begin{array}{ccc}
\mathcal{B}(H) & \xrightarrow{\Psi} & \mathcal{B}(H) \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{C}(H) & \xrightarrow[\Phi]{} & \mathcal{C}(H).
\end{array}$$

Let Ψ be a representation of Φ

Wlog, for all a:

1. a is a projection/unitary/self-adjoint iff $\Psi(a)$ is projection/unitary/self-adjoint.
2. $\|a\| = \|\Psi(a)\|$.
3. $\Psi(a^*) = \Psi(a)^*$.

Uniformizations (selections)

A subset of a Polish space is *analytic* if it is a continuous image of a Borel set. Sets in the σ-algebra generated by analytic sets are *C-measurable*.

Theorem (Jankov, von Neumann)

If X and Y are Polish spaces and $B \subseteq X \times Y$ is analytic, then B can be uniformized by a C-measurable function.

The strong operator topology on $\mathcal{B}(H)_{\leq 1} = \{a \mid \|a\| = 1\}$ is Polish.

Theorem (Farah, 2007)

If Φ has a C-measurable representation on $\mathcal{B}(H)_{\leq 1}$, then it is inner.

Proof.

Some other time. □

Corollary

TFAE:

1. Φ *is inner,*
2. Φ *has a C-measurable representation on* $\mathcal{B}(H)_{\leq 1}$
3. $\Gamma_\Phi = \{(a, b) \mid \|a\| \leq 1, \|b\| \leq 1, \Phi(\dot{a}) = \dot{b}\}$ *is analytic.*

ε-approximations

$\Psi\colon \mathcal{B}(H)_{\leq 1} \to \mathcal{B}(H)_{\leq 1}$ is an *ε-approximation* to Φ if $\|\pi(\Psi(a)) - \Phi(\pi(a))\| \leq \varepsilon$ for all $a \in \mathcal{B}(H) \leq 1$.

$$\begin{array}{ccc} \mathcal{B}(H)_{\leq 1} & \xrightarrow{\ \Psi\ } & \mathcal{B}(H)_{\leq 1} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathcal{C}(H)_{\leq 1} & \xrightarrow[\ \Phi\ ]{} & \mathcal{C}(H)_{\leq 1}. \end{array}$$

Proposition

TFAE:

1. Φ *is inner.*
2. *For every* $\varepsilon > 0$ Φ *has a C-measurable* ε*-approximation.*
3. *For every* $\varepsilon > 0$ *the set* $\Gamma_\Phi^\varepsilon = \{(a, b) \mid \|a\| \leq 1, \|b\| \leq 1, \|\Phi(\dot{a}) - \dot{b}\| \leq \varepsilon\}$ *is analytic.*

Proof.

(3) implies (1): $\bigcap_n \Gamma_\Phi^{1/n}$ is analytic; apply Jankov, von Neumann to get a C-measurable representation. □

Lemma

Assume $|E_n| \leq |E_{n+1}|$ *for all* n*. If* $M \subseteq \mathbb{N}$ *is infinite and* Φ *has a* C*-measurable* ε*-approximation on* $\mathcal{D}_M[\vec{J}]$*, then* Φ *has a* C*-measurable* ε*-approximation on* $\mathcal{D}[\vec{J}]$*.*

Proof.

Fix v such that $v^*v = I$ and $v\mathcal{D}[\vec{J}]v^* \subseteq \mathcal{D}_M[\vec{J}]\ldots$ □

A set $\Delta \subseteq \mathcal{B}(H)_{\leq 1} \times \mathcal{B}(H)_{\leq 1}$ is ε-*narrow* if

$$(\forall a, b, c)((a, b) \in \Delta \text{ and } (a, c) \in \Delta \text{ implies } \|\dot{b} - \dot{c}\| \leq \varepsilon).$$

Fact

1. *Γ_Φ^ε is ε-narrow*
2. *If $\mathcal{X} \supseteq \Phi$ and $\mathcal{X}$ is ε-narrow, then $\mathcal{X} \subseteq \Gamma_\Phi^\varepsilon$.* □

Fix $\vec{J}$.

$$\mathcal{J}^\varepsilon = \{M \subseteq \mathbb{N} \mid \Gamma_\Phi \text{ can be covered by countably many } \varepsilon\text{-narrow analytic sets on } \mathcal{D}_M[\vec{J}]\}$$

$$= \{M \subseteq \mathbb{N} \mid \text{there are } \varepsilon\text{-narrow analytic sets } \Delta_i,\ i \in \mathbb{N}, \text{ such that}$$

$$\{(a, b) \in \Gamma_\Phi \mid a \in \mathcal{D}_M[\vec{J}]\} \subseteq \bigcup_i \Delta_i\}$$

Proposition

If $M = \dot{\bigcup}_n M_n$ is in $\mathcal{J}^{\varepsilon}$, then Γ_{Φ} has a C-measurable 4ε-approximation on $\mathcal{C}_{M_n}[\vec{J}]$ for some n.

Proof.

Diagonalization. □

Corollary

Assume that $\mathcal{J}^{\varepsilon}$ contains an infinite set for every $\varepsilon > 0$. Then Φ is inner on $\mathcal{C}[\vec{J}]$.

Lemma

For all $\varepsilon > 0$ and all $\vec{J}$, TA implies that $\mathcal{J}^{\varepsilon}_{\Phi}$ contains an infinite set.

Proof. Index $\vec{J}$ as J_s, $s \in 2^{<\mathbb{N}}$.
Let $\mathcal{X}$ be the set of all pairs (S, a) such that

1. S is contained in a maximal branch $B(S)$ in $2^{<\mathbb{N}}$
2. $a \in \mathcal{D}_S[\vec{J}]_{\leq 1}$ and $a \notin \mathcal{K}(H)\}$.

For $n \in \mathbb{N}$ define a graph G_n with vertex set $\mathcal{X}$ such that there is an edge between (S, a) and (T, b) iff all of the following hold:

(K1) $B(S) \neq B(T)$,

(K2) $(\forall i \in S \cap T)\|(a - b)P_{\{i\}}\| < 2^{-i}$.

(K3) $\|\Psi(a)\Psi(P_T) - \Psi(P_S)\Psi(b)\| > 2^{-n}$ or
$\|\Psi(P_T)\Psi(a) - \Psi(b)\Psi(P_S)\| > 2^{-n}$.

Fact

G_n has no uncountable cliques for any n.

Fact

Assume $\mathcal{X} = \bigcup_j \mathcal{X}_j$*, each* $\mathcal{X}_j$ *is* G_n*-independent and* $\mathcal{D}_j$ *is countable and dense in* $\mathcal{X}_j$*. If* $B \in 2^{\mathbb{N}}$ *is distinct from* $B(S)$ *for all* $(S, a) \in \bigcup_j \mathcal{D}_j$*, then* $\bigcup_{k \in B} J_k \in \mathcal{J}^{10/n}$*.*

□

Open problems

$M(A)$ is the *multiplier algebra* of a C^*-algebra A.
If $A = C_0(X)$ then $M(A) = C(\beta X)$.
If $A = \mathcal{K}(H)$ then $M(A) = \mathcal{B}(H)$.

Question (Elliott)

What can be said about automorphisms of the corona algebras $M(A)/A$? In particular, what if A is a UHF (uniformly hyperfinite) algebra?

Theorem (Farah, 1997)

Assume TA+MA. If $A = C_0(\xi)$ for a countable ordinal ξ then all automorphisms of $M(A)/A$ are trivial.

Each known automorphism of $\mathcal{C}(H)$ is 'pointwise inner'

$$(\forall a)(\exists u)\Phi(a) = uau^*.$$

Question

Is there an automorphism of $\mathcal{C}(H)$ that sends the unilateral shift to its adjoint?

Such an automorphism cannot be inner (Fredholm index!), hence TA implies the negative answer.