Operator algebras and set theory

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$$
\begin{aligned}
H\text{:} \quad & \text{a complex, infinite-dimensional Hilbert space} \\
(e_n)\text{:} \quad & \text{an orthonormal basis of } H \\
(\xi|\eta)\text{:} \quad & \text{the inner product on } H \\
\|\xi\| = \quad & \sqrt{(\xi|\xi)} \\
a\colon H \to H\text{:} \quad & \text{a linear operator} \\
\|a\| = \quad & \sup\{\|a\xi\| \mid \xi \in H, \|\xi\| = 1\} \\
a \text{ is } \textit{bounded} \quad & \text{if } \|a\| < \infty. \\
(\mathcal{B}(H), +, \cdot, {}^*, \|\cdot\|)\text{:} \quad & \text{the algebra of all bounded operators on } H.
\end{aligned}
$$

The *adjoint*, a^*, is defined implicitly by

$$(a^*\xi|\eta) = (\xi|a\eta)$$

for all ξ, η in H.

Lemma

For all a, b we have

1. $(a^*)^* = a$,
2. $\|a\| = \|a^*\|$,
3. $\|ab\| \leq \|a\| \cdot \|b\|$,
4. $\|aa^*\| = \|a\|^2$.

Hence $\mathcal{B}(H)$ is a Banach algebra with involution. (4) is the "C equality."*

Example

If $H = L^2(X, \mu)$ *and* $f\colon X \to \mathbb{C}$ *is bounded and measurable, then*

$$H \ni g \mapsto m_f(g) = fg \in H$$

is a bounded linear operator. We have $\|m_f\| = \|f\|_\infty$ *and*

$$m_f^* = m_{\bar{f}}.$$

Hence $m_f^* m_f = m_f m_f^* = m_{|f|^2}$.

An operator a is *normal* if $aa^* = a^*a$.
If $\Phi\colon H_1 \to H_2$ is an isomorphism between Hilbert spaces, then

$$a \mapsto \operatorname{Ad}\Phi(a) = \Phi a \Phi^{-1}$$

is an isomorphism between $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$.

Theorem (Spectral Theorem)

If a is a normal operator then there is a finite measure space (X, μ), a measurable function f on X, and a Hilbert space isomorphism $\Phi\colon L^2(X, \mu) \to H$ such that $\operatorname{Ad}\Phi m_f = a$.

An operator is *self-adjoint* if $a = a^*$. For any $b \in \mathcal{B}(H)$ we have

$$b = b_0 - ib_1,$$

with both $b_0 = (b + b^*)/2$ and $b_1 = i(b^* - b)/2$ self-adjoint.

Fact

a is self-adjoint iff $(a\xi|\xi)$ is real for all ξ.

Pf.

$$((a - a^*)\xi|\xi) = (a\xi|\xi) - (a^*\xi|\xi) = (a\xi|\xi) - (\xi|a\xi) = (a\xi|\xi) - \overline{(a\xi|\xi)}.$$

□

An operator b such that $(b\xi|\xi) \geq 0$ for all $\xi \in H$ is *positive*.

Example

$m_f \geq 0$ *iff* $\nu\{x \mid f(x) < 0\} = 0$.

For any self-adjoint $a \in \mathcal{B}(H)$ we have $a = a_0 - a_1$, with both a_0 and a_1 positive. (Hint: spectral theorem.)

Lemma

b is positive iff $b = a^* a$ for some (non-unique) a.

Proof.

($\Leftarrow$) $(a^* a\xi|\xi) = (a\xi|a\xi) \geq 0$.
($\Rightarrow$) If b is positive, by the spectral theorem we may assume $b = m_f$ for $f \geq 0$. Let $a = m_{\sqrt{f}}$. □

A $p \in \mathcal{B}(H)$ is a *projection* if $p^2 = p^* = p$.

Lemma

p is a projection iff it is an orthogonal projection to a closed subspace of H.

Pf. We have $p = m_f$ and $f = f^2 = \bar{f}$. Hence $f(x) \in \{0, 1\}$ for almost all x, and $m_f = \mathrm{proj}_{\{g \mid \overline{\mathrm{supp}}(g) \subseteq Y\}}$ with $Y = f^{-1}(\{1\})$. □

I is the identity operator on H.
An operator u is *unitary* if $uu^* = u^*u = I$.
An operator v is a *partial isometry* if

$$p = vv^* \text{ and } q = v^*v$$

are both projections.

Example

A partial isometry that is not a normal operator. Let (e_n) be the orthonormal basis of H. The unilateral shift S is defined by

$$S(e_n) = e_{n+1} \textit{ for all } n.$$

Then $S^*(e_{n+1}) = e_n$ and $S^*(e_0) = 0$.

$$S^*S = I \neq \text{proj}_{\overline{\text{Span}}\{e_n | n \geq 1\}} = SS^*.$$

We have an analogue of $z = re^{\theta}$ for complex numbers.

Theorem (Polar Decomposition)

Every a in $\mathcal{B}(H)$ can be written as

$$a = bv$$

where b is positive and v is a partial isometry.

This does not mean that understanding arbitrary operators reduces to understanding self-adjoints and partial isometries.

Problem

Does every $a \in \mathcal{B}(H)$ have a nontrivial closed invariant subspace?

The answer is easily positive for all normal operators and all partial isometries.

I is the identity operator on H.

Definition (Spectrum)

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda I \text{ is not invertible}\}.$$

Lemma

1. *$\sigma(a)$ is always a compact subset of $\mathbb{C}$.*
2. *$\sigma(a^*) = \{\overline{\lambda} \mid \lambda \in \sigma(a)\}$.*
3. *a is self-adjoint iff $\sigma(a) \subseteq \mathbb{R}$.*
4. *a is positive iff $\sigma(a) \subseteq [0, \infty)$.*

Concrete and abstract C* algebras

Definition (Concrete C* algebras)

If $X \subseteq \mathcal{B}(H)$ *let* $A = C^*(X)$ *be the smallest norm-closed subalgebra of* $\mathcal{B}(H)$*.*

Definition

A is an abstract C algebra if it is a Banach algebra with involution such that* $\|aa^*\| = \|a\|^2$ *for all* a*.*

Example

X is a locally compact Hausdorff space.

$$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and vanishes at } \infty\}.$$

$$f^* = \overline{f}.$$

$C_0(X)$ is abelian, in particular each operator is normal.

f is self-adjoint	*iff*	*the* $\mathrm{range}(f) \subseteq \mathbb{R}$.
f is positive	*iff*	$\mathrm{range}(f) \subseteq [0, \infty)$.
f is a projection	*iff*	$f^2(x) = f(x) = \overline{f(x)}$
	iff	$\mathrm{range}(f) \subseteq \{0, 1\}$
	iff	$f = \chi_U$ *for a clopen* $U \subseteq X$.

If X is compact then $C_0(X) = C(X)$ has the identity, and we have

$$\sigma(f) = \mathrm{range}(f).$$

Example

M_n: $n \times n$ complex matrices. $M_n \cong \mathcal{B}(\ell_2^n)$.

adjoint, unitary:	*the usual meaning.*
self-adjoint:	*hermitian.*
positive:	*positively definite.*
$\sigma(a)$:	*the set of eigenvalues.*
spectral theorem:	*spectral theorem.*
	(normal matrices are diagonalizable)

The algebra of compact operators,

$$\begin{aligned}\mathcal{K}(H) =& C^*(\{a \in \mathcal{B}(H) \mid a[H] \text{ is finite-dimensional}\}).\\ =& \{a \in \mathcal{B}(H) \mid a[\text{unit ball}] \text{ is compact}\}\end{aligned}$$

Fact

If $r_n = \mathrm{proj}_{\overline{\mathrm{Span}}\{e_j \mid j \le n\}}$ *TFAE*

1. $a \in \mathcal{K}(H)$,
2. $\lim_n \|a(I - r_n)\| = 0$,
3. $\lim_n \|(I - r_n)a\| = 0$.

□

Note: if a is self-adjoint then

$$\|a(I - r_n)\| = \|(a(I - r_n))^*\| = \|(I - r_n)a\|.$$

$\mathcal{K}(H)$ is an ideal of $\mathcal{B}(H)$ (closed, two-sided, self-adjoint ideal).
The quotient $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is the *Calkin algebra*.
$\pi\colon \mathcal{B}(H) \to \mathcal{C}(H)$ is the quotient map.
$\sigma(\pi(a)) = \sigma_e(a)$: the essential spectrum of a.
Here

$$\begin{aligned}\sigma_e(a) = \quad & \text{the set of all accumulation points of } \sigma(a) \\ & \text{plus all points of } \sigma(a) \text{ of infinite multiplicity}\end{aligned}$$

Direct (inductive) limits

If Ω is a directed set, A_i, $i \in \Omega$ are C* algebras and

$$\varphi_{i,j} \colon A_i \to A_j \qquad \text{for } i < j$$

is a commuting family of *-homomorphisms, define the *direct limit*

$$A = \varinjlim_i A_i.$$

For $a \in A_i$ let

$$\|a\| = \lim_i \|\varphi_{i,j}(a)\|_{A_j}$$

and take the completion.

Example

The CAR (Canonical Anticommutation Relations) algebra (aka the Fermion algebra, aka M_{2^∞} UHF algebra).

$$\Phi_n \colon M_{2^n} \to M_{2^{n+1}}$$

$$\Phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

FDD algebras

If (E_n) is an orthogonal decomposition of H into finite-dimensional subspaces then

$$\mathcal{D}[\vec{E}] = \{a \in \mathcal{B}(H) \mid \text{ each } E_n \text{ is } a\text{-invariant}\}.$$

If $\vec{E}$ refines $\vec{F}$, then $\mathcal{D}[\vec{E}] < \mathcal{D}[\vec{F}]$.

Fact

The unilateral shift S does not belong to $\mathcal{D}[\vec{E}]$ for any $\vec{E}$.

Pf. Some a is *Fredholm* if its *Fredholm index*

$$\text{index}(a) = \dim \ker(a) - \dim \ker(a^*)$$

is finite.
If $a \in \mathcal{D}[\vec{E}]$ is Fredholm then $\text{index}(a) = 0$.
However, $\text{index}(S) = -1$. □

Lemma

If $a \in \mathcal{B}(H)$ is normal then

$$C^*(a, I) \cong C(\sigma(a)).$$

For every $f : \sigma(a) \to \mathbb{C}$ we can define $f(a) \in C^*(a, I)$. □

For example:

$$a = \frac{|a| + a}{2} - \frac{|a| - a}{2}$$

If $a \geq 0$, then $\sqrt{a}$ is defined.

Unital algebras

A C* algebra is *unital* if it has a unit (multiplicative identity).

Lemma

Every C algebra A is contained in a unital C* algebra $\tilde{A} \cong A \oplus \mathbb{C}$.* □

We call $\tilde{A}$ the *unitization* of A.

If $A < B$ we say A is a *unital subalgebra* of B if both B is unital and its unit belongs to A.
If $a \in A$ and A is unital, one could define

$$\sigma_A(a) = \{\lambda \in \mathbb{C} \mid a - \lambda I \text{ is not invertible}\}.$$

Lemma

Assume A is a unital subalgebra of B and $a \in A$. Then $\sigma_A(a) = \sigma_B(a)$.

Lemma

*Every *-homomorphism Φ between C* algebras is continuous.*

Pf. We prove Φ is a contraction.

Note that $\sigma(\Phi(a)) \subseteq \sigma(a)$. Thus for a normal

$$\begin{aligned} \|a\| &= \sup\{|\lambda| \mid \lambda \in \sigma(a)\} \\ &\geq \sup\{|\lambda| \mid \lambda \in \sigma(\Phi(a))\} \\ &= \|\Phi(a) \end{aligned}$$

For general a we have

$$\|a\| = \sqrt{\|aa^*\|} \geq \sqrt{\|\Phi(aa^*)\|} = \|\Phi(a)\|.$$

□

Pure states and the GNS construction

Theorem (Gelfand–Naimark)

Every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. If it is moreover unital, then X can be chosen to be compact.

Theorem (Gelfand–Naimark–Segal)

Every C-algebra A is isomorphic to a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.*

A continuous linear functional $\varphi\colon A \to \mathbb{C}$ is *positive* if $\varphi(a) \geq 0$ for all positive a. It is a *state* if $\varphi(I) = 1$.
$\mathbb{S}(A)$ is the space of all states on A.

If ξ is a unit vector, define a functional ω_ξ on $\mathcal{B}(H)$ by

$$\omega_\xi(a) = (a\xi|\xi).$$

Then $\omega_\xi(a) \geq 0$ for a positive a and $\omega_\xi(I) = 1$; hence it is a state.
States form a weak*-compact convex subset of A^*.
Cauchy–Schwartz for states:

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

Theorem (GNS)

Assume φ is a state on A. There is a representation $\pi_\varphi \colon A \to \mathcal{B}(H_\varphi)$ and a unit vector $\xi = \xi_\varphi$ in H_φ such that

$$\varphi(a) = \omega_\xi(a)$$

for all a.

Proof.

On $A \times A$ let

$$(a|b) = \varphi(b^* a).$$

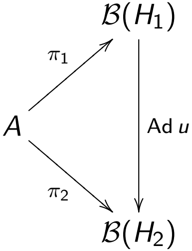
$$J_\varphi = \{a \mid \varphi(a^* a) = 0\}$$

$$H_\varphi = \widetilde{A/J}$$

$\pi_\varphi(a)$ sends $[b]_{J_\varphi}$ to $[ab]_{J_\varphi}$. □

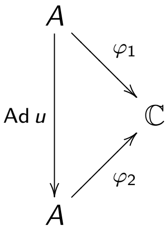
The space of states on A

$\pi_1 \sim \pi_2$ if $\exists u \colon H_1 \to H_2$ such that



$\mathrm{Ad}\, u(a) = uau^*$

$\varphi_1 \sim \varphi_2$ if and only if $\exists u \in A$ such that



Theorem

For φ_1, φ_2 in $\mathbb{S}(A)$ we have $\varphi_1 \sim \varphi_2 \Leftrightarrow \pi_{\varphi_1} \sim \pi_{\varphi_2}$. □

Lemma

If $\|\varphi\| = 1$ *then* φ *is a state iff* $\varphi(I) = 1$. □

A state φ is *pure* iff

$$\varphi = t\psi_0 + (1-t)\psi_1, \qquad 0 \le t \le 1$$

for some states ψ_0, ψ_1 implies $\varphi = \psi_0$ or $\varphi = \psi_1$.
$\mathbb{P}(A)$ is the space of all pure states of A.

Example

If $A = C(X)$*, then (by Riesz)* φ *is a state iff* $\varphi(f) = \int f \, d\mu$ *for some Borel probability measure* μ*.*

Lemma

For a state φ *of* $C(X)$ *TFAE:*

1. φ *is pure,*
2. *for some* $x_\varphi \in X$ *we have* $\varphi(f) = f(x_\varphi)$
3. $\varphi\colon C(X) \to \mathbb{C}$ *is a *-homomorphism.* □

If $\xi \in H$ is a unit vector, then

$$\omega_\xi(a) = (a\xi|\xi)$$

is a *vector state*. All vector states are pure.

Definition

Some $\varphi \in \mathbb{S}(\mathcal{B}(H)))$ *is singular if* $\varphi[\mathcal{K}(H)] = \{0\}$.

Theorem

Each state of $\mathcal{B}(H)$ *is a weak*-limit of vector states.*

Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then

$$\varphi_{\mathcal{U}}^{(\vec{e})}(a) = \lim_{n \to \mathcal{U}} (ae_n | e_n)$$

is a *singular state*.

A state of the form $\varphi_{\mathcal{U}}^{(\vec{\xi})}$ is *diagonalized*.

Theorem (Anderson, 1977)

Each $\varphi_{\mathcal{U}}^{(\vec{e})}$ is pure. □

Conjecture (Anderson, 1977)

Every pure state on $\mathcal{B}(H)$ can be diagonalized.

The lattice of projections

Let p, q be projections in $\mathcal{B}(H)$. Define $p \leq q$ if $pq = p$.

Fact

$pq = p$ iff $qp = p$.

Proof.

Since $p = p^*$, $pq = p$ implies $pq = (pq)^* = q^* p^* = qp$. □

Note that $pq = qp$ if and only if pq is a projection.
$p \wedge q$: the projection to $\text{range}(p) \cap \text{range}(q)$
$p \vee q$: the projection to $\overline{\text{Span}}(\text{range}(p) \cup \text{range}(q))$.

Lemma

The projections in $\mathcal{B}(H)$ *form a lattice with respect to* $\wedge, \vee, \leq, I, 0$. □

Lemma

$\mathcal{B}(H) = C^*(\mathcal{P}(\mathcal{B}(H)))$. *That is,* $\overline{\mathrm{Span}}\, \mathcal{P}(\mathcal{B}(H))$ *is norm-dense in* $\mathcal{B}(H)$. □

Lifting elements in the Calkin algebra

$\mathcal{K}(H)$ is a (self-adjoint, norm closed, two-sided) ideal of $\mathcal{B}(H)$.
$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is the *Calkin algebra*.
$\pi \colon \mathcal{B}(H) \to \mathcal{C}(H)$ is the quotient map.

Lemma

If $\mathbf{a}$ *is self-adjoint in* $\mathcal{C}(H)$*, then* $\mathbf{a} = \pi(a)$ *for a self-adjoint* a *in* $\mathcal{C}(H)$*.*

Pf. Fix any a_0 such that $\pi(a_0) = \mathbf{a}$. Let $a = (a_0 + a_0^*)/2$. $\square$

Lemma

If $\mathbf{p}$ *is a projection in* $\mathcal{C}(H)$*, then* $\mathbf{p} = \pi(p)$ *for a projection* p *in* $\mathcal{C}(H)$*.*

Pf. Fix a self-adjoint a such that $p = \pi(a)$. There are (X, μ) and $f \in L^\infty(X, \mu)$ and a Hilbert space isomorphism $\Phi\colon L^2(X, \mu) \to H$ such that $\Phi(m_f) = a$. Let

$$h(x) = \begin{cases} 1, & f(x) \geq 1/2 \\ 0, & f(x) < 1/2. \end{cases}$$

Then m_h is a projection and $\pi(m_h) = \pi(m_f)$. □

Lemma

There is a normal (even a unitary) operator in $\mathcal{C}(H)$ that is distinct from $\pi(v)$ for any normal v in $\mathcal{B}(H)$.

Pf. The image $\mathbf{S}$ of the unilateral shift is a unitary in $\mathcal{C}(H)$, since $\mathbf{S}^*\mathbf{S} = I = \mathbf{S}\mathbf{S}^*$.
If $v - S$ is compact then v is Fredholm, and index$(v) = -1$. □

General spectral theorem

Theorem (Spectral Theorem)

If A is an abelian C-subalgebra of $\mathcal{B}(H)$ then there is a finite measure space (X, μ), a subalgebra B of $L^\infty(X, \mu)$, and a Hilbert space isomorphism $\Phi\colon L^2(X, \mu) \to H$ such that $\Phi[B] = A$.*

The atomic masa

MASA: MAximal Self-Adjoint SubAlgebra.
Fix H and its orthonormal basis (e_n).

$$(\alpha_n) \in \ell^\infty$$

$$\sum_n \alpha_n P_{\mathbb{C}e_n} \in \mathcal{B}(H).$$

Lemma

$\mathcal{A}^{(\vec{e})} = \{\sum_n \alpha_n P_{\mathbb{C}e_n}\}$ is a masa in $\mathcal{B}(H)$. □

Embedding $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathcal{B}(H))$

$X \in \mathcal{P}(\mathbb{N})$
$P_X^{(\vec{e})} = P_X = \mathrm{proj}_{\overline{\mathrm{Span}}\{e_n | n \in X\}}$

$$\mathcal{P}(\mathbb{N}) \ni X \mapsto P_X \in \mathcal{P}(\mathcal{B}(H)).$$

Hence $\mathcal{P}(\mathbb{N})$ is a maximal Boolean subalgebra of $\mathcal{P}(\mathcal{B}(H))$.

Atomless masa

$L^\infty(\mu)$ is also a masa in $\mathcal{B}(L^2(\mu))$ for a diffused measure μ.

Fact

$\mathcal{P}(L^\infty(\mu))$ *is a maximal Boolean subalgebra of* $\mathcal{P}(\mathcal{B}(H))$ *isomorphic to the Lebesgue measure algebra, Borel/Null.*

Theorem (Johnson–Parrott)

If $\mathcal{A}$ is a masa in $\mathcal{B}(H)$ then $\pi[\mathcal{A}]$ is a masa in $\mathcal{C}(H)$. □

For the atomic masa $\mathcal{A}$ we have

$$\mathcal{A}/\mathcal{K}(H) \approx \ell^\infty/c_0.$$

$$\mathcal{P}(\mathbb{N})/\operatorname{Fin} \ni [A] \mapsto [P_A] \in \mathcal{P}(\ell^\infty/c_0).$$

Both $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and the Lebesgue measure algebra are maximal boolean subalgebras of $\mathcal{P}(\mathcal{C}(H))$.

Lemma

For projections p and q in $\mathcal{B}(H)$ TFAE

1. $\pi(p) \leq \pi(q)$,
2. $q(I-p)$ *is compact,*
3. $(\forall\varepsilon > 0)(\exists p_0 \leq I-p)$ p_0 *is finite-dimensional and* $\|q(I-p-p_0)\| < \varepsilon$.

We write $p \leq_{\mathcal{K}} q$ if the conditions of Lemma 23 are satisfied.

Corollary

The poset $(\mathcal{P}(\mathcal{C}(H)), \leq)$ is isomorphic to the quotient $(\mathcal{P}(\mathcal{B}(H)), \leq_{\mathcal{K}})$. □

Let's write $\dot{p} = \pi(p)$.

Proposition (Weaver)

$\mathcal{P}(\mathcal{C}(H))$ is not a lattice.

Proof.

Enumerate a basis of H as ξ_{mn}, η_{mn} for m, n in $\mathbb{N}$.

$$\zeta_{mn} = \frac{1}{n}\xi_{mn} + \frac{\sqrt{n-1}}{n}\eta_{mn}$$

$$K = \overline{\text{Span}}\{\xi_{mn} \mid m, n \in \mathbb{N}\}, \qquad p = \text{proj}_K$$

$$L = \overline{\text{Span}}\{\zeta_{mn} \mid m, n \in \mathbb{N}\}, \qquad q = \text{proj}_L$$

For $f \in \mathbb{N}^{\mathbb{N}}$ let $\quad M(f) = \overline{\text{Span}}\{\xi_{mn} \mid m \leq f(n)\}, \quad r(f) = \text{proj}_{M(f)}$.

Fact

1. *$r(f) \leq p$ for all f,*
2. *$r(f) \leq q$ for all f,*
3. *if $r \leq_{\mathcal{K}} p$ and $r \leq_{\mathcal{K}} q$ then $r \leq_{\mathcal{K}} r(f)$ for some f.* □

Cardinal invariants

Recall

$$\mathfrak{a} = \min\{|\mathbb{A}| \mid \mathbb{A} \text{ is a maximal infinite antichain in } \mathcal{P}(\mathbb{N})/\operatorname{Fin}\}.$$

Definition (Wofsey, 2006)

A family $\mathbb{A} \subseteq \mathcal{P}(\mathcal{B}(H))$ is almost orthogonal (aof) if pq is compact for $p \neq q$ in $\mathbb{A}$.

$$\mathfrak{a}^* = \min\{|\mathbb{A}| \mid \mathbb{A} \text{ is a maximal infinite aof}\}$$

Theorem (Wofsey, 2006)

1. *It is relatively consistent with ZFC that* $\aleph_1 = \mathfrak{a} = \mathfrak{a}^* < 2^{\aleph_0}$,
2. *MA implies* $\mathfrak{a}^* = 2^{\aleph_0}$.

Question

Is $\mathfrak{a} = \mathfrak{a}^*$*? Is* $\mathfrak{a} \geq \mathfrak{a}^*$*? Is* $\mathfrak{a}^* \geq \mathfrak{a}$*?*

It may seem obvious that $\mathfrak{a} \geq \mathfrak{a}^*$?

Definition/Theorem (Solecki, 1995)

An ideal J on $\mathbb{N}$ is an analytic P-ideal if there is a lower semicontinuous (lsc) submeasure φ on $\mathbb{N}$ such that

$$J = \{X \mid \limsup_n \varphi(X \setminus n) = 0\}.$$

Lemma (Steprāns, 2007)

Fix $a \in \mathcal{B}(H)$. Then

$$J_a = \{X \subseteq \mathbb{N} \mid aP_X^{(\vec{e})} \textit{ is compact}\}$$

is an analytic P-ideal.

Pf. Let $\varphi_a(X) = \|P_X a\|$. $P_X a$ is compact iff $\lim_n \varphi_a(X \setminus n) = 0$. $\square$

Proposition (Wofsey, 2006)

There is a mad family $\mathbb{A} \subseteq \mathcal{P}(\mathbb{N})$ *whose image in* $\mathcal{P}(\mathcal{B}(H))$ *is not a maof.*

Proof.

Let $\xi_n = 2^{-n/2} \sum_{j=2^n}^{2^{n+1}-1} e_j$ and $q = \mathrm{proj}_{\overline{\mathrm{Span}}\{\xi_n\}}$.
Then $\lim_n \|qe_n\| = 0$ hence J_q is a *dense* ideal: every infinite subset of $\mathbb{N}$ has an infinite subset in J_q.
Let $\mathbb{A}$ be a mad family contained in J_q.
Then q is almost orthogonal to all P_X, $X \in \mathbb{A}$. □

Let

$$\mathfrak{a}' = \min\{|\mathbb{A}| \mid \mathbb{A} \text{ is mad and } \mathbb{A} \not\subseteq J \text{ for any analytic P-ideal } J\}$$

Fact

$\mathfrak{a}' \geq \mathfrak{a}$, $\mathfrak{a}' \geq \mathfrak{a}^*$. □

One can define $\mathfrak{p}^*, \mathfrak{t}^*, \mathfrak{b}^*, \ldots$

Theorem (Hadwin, 1988)

CH implies that any two maximal chains of projections in $\mathcal{C}(H)$ are order-isomorphic.

Conjecture (Hadwin, 1988)

CH is equivalent to 'any two maximal chains in $\mathcal{P}(\mathcal{C}(H))$ are order-isomorphic.'

Theorem (Wofsey, 2006)

There is a forcing extension in which there are maximal chains in $\mathcal{P}(\mathcal{C}(H))$ of different cofinalities (and $2^{\aleph_0} = \aleph_2$). □

Theorem (essentially Shelah–Steprāns)

There is a model of $\neg CH$ in which all maximal chains in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are isomorphic. □

A twist of projections

Consider

$$\mathfrak{l} = \min\{|\mathbb{A}| \mid \mathbb{A} \text{ is a family of commuting projections in } \mathcal{C}(H)\}$$
$$\text{that cannot be lifted to a family of commuting projections of } \mathcal{B}(H)$$

Lemma

$\mathfrak{l} > \aleph_0$. □

Proposition (Farah, 2006)

$\mathfrak{l} = \aleph_1$: There are commuting projections p_ξ, $\xi < \omega_1$, in $\mathcal{C}(H)$ that cannot be lifted to commuting projections of $\mathcal{B}(H)$.

Pf. Construct p_ξ in $\mathcal{P}(\mathcal{B}(H))$ so that for $\xi \neq \eta$:

1. $p_\xi p_\eta$ is compact, and
2. $\|[p_\xi, p_\eta]\| > 1/4$

If (e_n) diagonalizes each p_ξ, fix $X(\xi) \subseteq \mathbb{N}$ such that

$$d_\xi = p_\xi - P^{(\vec{e})}_{X(\xi)}$$

is compact. Let

$$r_n = P^{(\vec{e})}_{\{0,1,\dots,n-1\}}.$$

Then a is compact iff $\lim_n \|a(I - r_n)\| = 0$.
Fix $\bar{n}$ such that $\|d_\xi(I - r_{\bar{n}})\| < 1/8$ for uncountably many ξ.
If $\|(d_\xi - d_\eta)\|r_{\bar{n}}\| < 1/8$, then

$$\|[p_\xi, p_\eta]\| \leq \|[P_{X(\xi)}, P_{X(\eta)}]\| + \frac{1}{4} = \frac{1}{4}$$

a contradiction. □

Automorphisms of C* algebras

$$\operatorname{Ad} u(a) = uau^*.$$

An automorphism Φ is *inner* if $\Phi = \operatorname{Ad} u$ for some unitary u.

Lemma

If A is abelian then id *is its only inner automorphisms.*
If $A = C(X)$ then each automorphism is of the form

$$f \mapsto f \circ \Psi$$

for an autohomeomorphism Ψ of X. □

Lemma

All automorphisms of $\mathcal{B}(H)$ are inner. Hence all automorphisms of any M_n are inner. $\square$

Lemma

The CAR algebra ($M_{2^\infty} = \bigotimes_n M_2$) has outer automorphisms.

Pf. $\Phi = \bigotimes_n \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is outer since $\bigotimes_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not in M_{2^∞}.

$\square$

Extending pure states

Lemma

If A is a unital subalgebra of B then

1. *The restriction of a state of B to A is a state of A.*
2. *Every (pure) state of A can be extended to a (pure) state of B.*

Pf. (2) By Hahn–Banach $\{\psi \in B^* \mid \psi \restriction A = \varphi, \|\psi\| = 1\}$ is nonempty and by Krein–Milman it has an extreme point. □

Example

Restriction of a pure state to a unital subalgebra need not be pure. If ω_ξ is a vector state of $\mathcal{B}(H)$ and $\mathcal{A}$ is the atomic masa diagonalized by (e_n), then $\omega_\xi \restriction \mathcal{A}$ is pure iff $|(\xi|e_n)| = 1$ for some n.

Proposition

Assume $A < B$ and B is abelian. If every pure state of A extends to the unique pure state of B, then $A = B$.

Proof.

$A < C(X)$ separates points of X. Use Stone–Weierstrass. □

Problem (Noncommutative Stone–Weierstrass problem)

Assume $A < B$ and A separates $\mathcal{P}(B) \cup \{0\}$. Does necessarily $A = B$?

A C* algebra is *simple* if and only if it has no (closed, two-sided, self-adjoint) nontrivial ideals.

Lemma (Akemann–Weaver)

Assume A is a simple separable unital C algebra and φ and ψ are its pure states. Then there is a simple separable unital $B > A$ such that*

1. *φ and ψ extend to pure states φ', ψ' of B in a unique way.*
2. *φ' and ψ' are equivalent.* □

Pure states on M_{2^∞}

.

On M_2:

$$\varphi_1 : \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{11}$$

$$\varphi_2 : \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{22}$$

For $f \in 2^{\mathbb{N}}$

$$\varphi_f = \bigotimes_n \varphi_{f(n)}$$

is in $\mathbb{P}(M_{2^\infty})$.
In M_{2^∞}, $\varphi_f \sim \varphi_g$ iff $\{n \mid f(n) \neq g(n)\}$ is finite.

Fact

If $f \neq g$ then $\|\varphi_f - \varphi_g\| = 2$. □

Type I algebras

Definition (Kaplansky)

A C algebra A is of type I if for every irreducible representation* $\pi : A \to \mathcal{B}(H)$ *we have* $\pi[A] \supseteq \mathcal{K}(H)$.

[Not to be confused with type I von Neumann algebras: $\mathcal{B}(H)$ is a type I von Neumann algebra and a non-type-I C* algebra.]
A C* algebra is *simple* if and only if it has no (closed, two-sided, self-adjoint) nontrivial ideals.

Lemma

A type I C algebra has only one irrep up to equivalence if and only if it is isomorphic to* $\mathcal{K}(H)$ *for some* H.

Theorem (Glimm)

If A is a non-type-I C algebra then there is* $B < A$ *that has a quotient isomorphic to* M_{2^∞}.

Corollary (Akemann–Weaver, 2002)

If A is non-type-I and has a dense subset of cardinality $< 2^{\aleph_0}$, then A has nonequivalent pure states.

Proof.

There are pure states φ_f, $f \in 2^{\mathbb{N}}$, such that if $f \neq g$ and $\mathrm{Ad}\, u\varphi_f = \mathrm{Ad}\, v\varphi_g$ then $\|u - v\| \geq 1$. □

Naimark's problem

Theorem (Naimark, 1948)

Any two irreps of $\mathcal{K}(H)$ are equivalent.

Question (Naimark, 1951)

Is the converse true?

Theorem (Akemann–Weaver, 2002)

Assume $\diamondsuit$. Then Naimark's problem has a negative solution.

Proof: $\diamondsuit$ and Naimark

Fix $h_\alpha \colon \alpha \to \omega_1$ such that for every $g \colon \omega_1 \to \omega_1$ the set $\{\alpha \mid g \restriction \alpha = h_\alpha\}$ is stationary.
Find an increasing chain of simple separable unital C* algebras A_α, $\alpha < \omega_1$ and pure state ψ_α of A_α so that

1. $\alpha < \beta$ implies $\psi_\beta \restriction A_\alpha = \psi_\alpha$,

For each A_α, let $\{\varphi_\alpha^\gamma \mid \gamma < \omega_1\}$ enumerate all of its pure states. If α is limit, let

$$A_\alpha = \varinjlim A_\beta.$$

Now we consider the successor ordinal case, $\beta = \alpha + 1$.
Assume there is $\varphi \in \mathbb{P}(A_\alpha)$ such that $\varphi \restriction A_\beta = \varphi_\beta^{h_\alpha(\beta)}$ for all $\beta < \alpha$.
Using lemma, let $A_{\alpha+1}$ be such that ψ_α and φ have unique extensions to $A_{\alpha+1}$ that are equivalent.
Since $A = A_{\omega_1}$ is unital and infinite-dimensional, $A \ncong \mathcal{K}(H')$.

Fix $\varphi \in \mathbb{P}(A)$.

Claim

$\{\alpha \mid \varphi \restriction A_\alpha \in \mathbb{P}(A_\alpha\}$ *contains a club.*

Proof.

For $x \in A_{\omega_1}$ and $m \in \mathbb{N}$

$$\{\alpha \mid \exists \psi_1, \psi_2 \in \mathbb{S}(A_\alpha), \varphi = \frac{1}{2}(\psi_1 + \psi_2) \text{ and } |\varphi(x) - \psi_1(x)| \geq \frac{1}{m}\}$$

is bounded in ω_1. □

Fix $h\colon \omega_1 \to \omega_1$ so that

$$\varphi \restriction A_\alpha = \varphi_\alpha^{h(\alpha)}$$

for all α.
Let α be such that $h \restriction \alpha = h_\alpha$. Then $\varphi \restriction A_{\alpha+1}$ is equivalent to $\psi_{\alpha+1}$. Since $\psi_{\alpha+1}$ has unique extension to A_{ω_1}, so does φ and they remain equivalent.

Kadison–Singer problem and Anderson's conjecture

Definition

A masa in $\mathcal{B}(H)$ has the extension property (EP) if each of its pure states extends uniquely to a pure state on $\mathcal{B}(H)$.

Every vector state has the unique extension to a pure state, hence this is a property of masas in the Calkin algebra.

1. Kadison–Singer, 1955: The atomless masa does not have the EP.
2. Anderson, 1974: CH implies there is a masa in the Calkin algebra with the EP.

Question (Kadison–Singer, 1955)

Does the atomic masa of $\mathcal{B}(H)$ have EP?

A positive answer is equivalent to an arithmetic statement, so let's go on.

Fix an orthonormal basis (e_n) of H, let $\mathcal{A}$ be the atomic masa diagonalized by (e_n). Each pure state of $\mathcal{A}$ is of the form

$$\varphi_{\mathcal{U}}(a) = \lim_{n \to \mathcal{U}} (ae_n | e_n)$$

for an ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

A state on $\mathcal{B}(H)$ of the form $\varphi_{\mathcal{U}}^{(\vec{\xi})}$ is *diagonalized (by $\mathcal{U}$, (e_n))*.

Conjecture (Anderson)

Every pure state φ of $\mathcal{B}(H)$ can be diagonalized.

Recall that on an abelian C* algebra a state is pure iff it is multiplicative.

Conjecture (Kadison–Singer)

For every pure state φ of $\mathcal{B}(H)$ there is an atomic masa $\mathcal{A}$ such that $\varphi \restriction \mathcal{A}$ is multiplicative.

If $\varphi \restriction \mathcal{A}$ is multiplicative, then there is an ultrafilter $\mathcal{U}$ such that φ and $\varphi_{\mathcal{U}}$ agree on $\mathcal{A}$. We can conclude that $\varphi = \pi_{\mathcal{U}}$ if the answer to the Kadison–Singer problem is positive.

Theorem (Akemann–Weaver, 2005)

CH implies there is a pure state φ on $\mathcal{B}(H)$ that is not multiplicative on any atomic masa.

States are coded by ‘noncommutative finitely additive measures.’

Theorem (Gleason)

Assume $\mu\colon \mathcal{P}(\mathcal{B}(H)) \to [0,1]$ *is such that* $\varphi(p+q) = \varphi(p) + \varphi(q)$ *whenever* $pq = 0$. *Then there is a unique state* φ *on* $\mathcal{B}(H)$ *that extends* μ. □

Lemma

If φ is a state on A and p is a projection such that $\varphi(p) = 1$, then $\varphi(a) = \varphi(pap)$ for all a.

Proof.

By Cauchy–Schwartz

$$|\varphi((I-p)a)| \leq \sqrt{\varphi(I-p)\varphi(a^*a)} = 0$$

since $a = pa + (I-p)a$ we have $\varphi(a) = \varphi(pa)$, similarly $\varphi(pa) = \varphi(pap)$. □

Definition

A family $\mathbb{F}$ of projections in a C algebra is a filter if*

1. *for p, q in $\mathbb{F}$ there is $r \in \mathbb{F}$ such that $r \leq p$ and $r \leq q$.*
2. *for $p \in \mathbb{F}$ and $r \geq p$ we have $r \in \mathbb{F}$.*

A filter generated by $\mathbb{X} \subseteq \mathcal{P}(A)$ is the intersection of all filters containing $\mathbb{X}$.

A filter $\mathcal{F}$ in $\mathcal{P}(\mathcal{C}(H))$ *lifts* if there is a commuting family $\mathbb{X}$ in $\mathcal{P}(\mathcal{B}(H)))$ that generates a filter $\mathbb{F}$ such that $\pi[\mathbb{F}] = \mathcal{F}$.
Note: If $\mathcal{F}$ is a filter in $\mathcal{C}(H)$, then

$$\widetilde{\mathcal{F}} = \{p \in \mathcal{P}(\mathcal{B}(H)) \mid \pi[p] \in \mathcal{F}\}$$

is not necessarily a filter.

Question

Does every maximal filter $\mathcal{F}$ in $\mathcal{P}(\mathcal{C}(H))$ lift?

Theorem (Anderson)

There are a singular pure state φ of $\mathcal{B}(H)$, an atomic masa $\mathcal{A}_1$, and an atomless masa $\mathcal{A}_2$ such that $\varphi \restriction \mathcal{A}_j$ is multiplicative for $j = 1, 2$.

Lemma (Weaver, 2007)

For $\tilde{\mathcal{F}}$ in $\mathcal{P}(\mathcal{B}(H))$ TFAE:

(A) $\|\mathbf{p}_1\mathbf{p}_2\ldots\mathbf{p}_n\| = 1$ *for any n-tuple of projections in* $\tilde{\mathcal{F}}$ *and* $\mathcal{F}$ *is maximal with respect to this property.*

(B) $(\forall\varepsilon > 0)$ *for all finite* $F \subseteq \widetilde{\mathcal{F}}$ *there is a unit vector* ξ *such that*

$$\|p\xi\| > 1 - \varepsilon$$

for all $p \in F$.

Definition

A family $\tilde{\mathcal{F}}$ *in* $\mathcal{P}(\mathcal{B}(H))$ *is a quantum filter if the conditions of Lemma 37 hold.*

Theorem (Farah–Weaver, 2007)

Assume $\mathcal{F} \subseteq \mathcal{P}(\mathcal{C}(H))$. TFAE:

1. *$\mathcal{F}$ is a maximal quantum filter,*
2. *$\mathcal{F} = \mathcal{F}_\varphi = \{\mathbf{p} \mid \varphi(p) = 1\}$ for some pure state φ.*

Proof.

(1) implies (2). For a finite $F \subseteq \mathcal{F}$ and $\varepsilon > 0$ let

$$\mathcal{X}_{F,\varepsilon} = \{\varphi \in \mathbb{S}(\mathcal{B}(H)) \mid \varphi(p) \geq 1 - \varepsilon \text{ for all } p \in F\}.$$

If ξ is as in (B) then $\omega_\xi \in \mathcal{X}_{F,\varepsilon}$.
Since $\mathcal{X}_{F,\varepsilon}$ is weak*-compact $\bigcap_{(F,\varepsilon)} \mathcal{X}_{F,\varepsilon} \neq \emptyset$. Any extreme point is a pure state.
(It can be proved that this intersection is a singleton.)
(2) implies (1). If $\varphi(p_j) = 1$ for $j = 1, \ldots, k$, then $\varphi(p_1 p_2 \ldots p_k) = 1$, hence (A) follows. □

Lemma

Let (ξ_n) *be an orthonormal basis. If for some* n *we have* $\mathbb{N} = \bigcup_{j=1}^{n} A_j$ *and there is* $q \in \widetilde{\mathcal{F}}$ *such that*

$$\|P_{A_j}^{(\vec{\xi})} q\| < 1$$

for all j*, then* $\mathcal{F}$ *is not diagonalized by* (ξ_n)*.*

Lemma

Assume (ξ_n) is an orthonormal basic sequence. There is a partition of $\mathbb{N}$ into finite intervals (J_n) such that for all k

$$\xi_k \in \overline{\mathrm{Span}}\{e_i \mid i \in J_n \cup J_{n+1}\}$$

(modulo a small perturbation of ξ_k) for some $n = n(k)$.

For (J_n) as in Lemma 39 let

$$\mathbb{D}_{\vec{J}} = \{q \mid \|P^{(\vec{e})}_{J_n \cup J_{n+1}} q\| < 1/2 \text{ for all } n\}$$

Lemma

Each $\mathbb{D}_{\vec{J}}$ is dense in $\mathcal{P}(\mathcal{B}(H))$.

$$\mathfrak{d} = \min\{|\mathbb{F}| \mid \mathbb{F} \subseteq \mathbb{N}^{\mathbb{N}} \text{ is } \leq\text{-cofinal}\}.$$

$$\mathfrak{t}^* = \min\{|\mathbb{T}| \mid \mathbb{T} \subseteq \mathcal{P}(\mathcal{C}(H)) \setminus 0$$
$$\mathbb{T} \text{ is a maximal decreasing well-ordered chain}\}$$

Theorem (Farah–Weaver)

Assume $\mathfrak{d} \leq \mathfrak{t}^*$.[1] *Then there exists a maximal proper filter in $\mathcal{P}(\mathcal{C}(H))$ that is not diagonalized by any atomic masa.*

[1] CH would do; $\mathfrak{d} <$ 'the Novák number of $\mathcal{P}(\mathcal{C}(H))$' is best if it makes sense

Pf. By $\mathfrak{d} \leq \mathfrak{t}^*$, we may choose $\mathcal{F}$ so that $\mathcal{F} \cap \mathbb{D}_{\vec{J}} \neq \emptyset$ for all $(\vec{J})$. Given (ξ_k), pick (J_n) such that $\xi_k \in J_{n(k)} \cup J_{n(k)+1}$ for all k. Let

$$A_i = \{k \mid n(k) \mod 4 = i\}$$

for $0 \leq i < 4$.

If $q \in \mathcal{F} \cap \mathbb{D}_{\vec{J}}$, then $\|P_{A_i}^{(\vec{\xi})} q\| < 1$ for $0 \leq i < 4$. □

Corollary (Akemann-Weaver, 2006)

CH implies there is a pure state that is not multiplicative on any atomic masa. □

An extra: Reid's theorem

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a *Q-point* if every partition of $\mathbb{N}$ into finite intervals has a transversal in $\mathcal{U}$.
Recall $P_X^{(\vec{e})} = P_X = \text{proj}_{\overline{\text{Span}}\{e_n | n \in X\}}$.

Theorem (Reid)

If $\mathcal{U}$ is a Q-point then $\varphi_{\mathcal{U}} \restriction \mathcal{A}^{(\vec{e})}$ has the unique extension to a pure state of $\mathcal{B}(H)$.

Proof of Reid's theorem

Fix a pure state extension φ of $\varphi_{\mathcal{U}} \restriction \mathcal{A}^{(\vec{e})}$ and $a \in \mathcal{B}(H)$.
Fix finite intervals (J_i) such that $\mathbb{N} = \bigcup_n J_n$ and

$$\|P_{J_m} a P_{J_n}\| < 2^{-m-n}$$

whenever $|m - n| \geq 2$ and let $X \in \mathcal{U}$ be such that

$$X \cap (J_{2i} \cup J_{2i+1}) = \{n(i)\}$$

for all i.

Then with $Q_i = P_{n(i)}$ and $f_i = e_{n(i)}$ we have $\varphi(\sum_i Q_i) = 1$ and

$$QaQ = \sum_i Q_i a \sum_i Q_i = \sum_i Q_i a Q_i + \sum_{i \neq j} Q_i a Q_j.$$

The second summand is compact, and

$$Q_i a Q_i = (af_i | f_i) f_i$$

therefore if $\alpha = \lim_{i \to \mathcal{U}} (af_i | f_i)$ we have

$$\lim_{X \to \mathcal{U}} (P_X a P_X - \alpha P_X) = 0$$

and $\varphi(a) = \alpha$.
Hence $\varphi(a) = \varphi_{\mathcal{U}}(a)$ for all a.