Operators on a complex Hilbert space

Operator algebras and set theory

Ilijas Farah

York University

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$H$: a complex, infinite-dimensional Hilbert space

$(e_n)$: an orthonormal basis of $H$

$(\xi|\eta)$: the inner product on $H$

$\|\xi\| = \sqrt{(\xi|\xi)}$

$a: H \to H$: a linear operator

$\|a\| = \sup\{\|a\xi\| \mid \xi \in H, \|\xi\| = 1\}$

$a$ is bounded if $\|a\| < \infty$.

$(\mathcal{B}(H), +, \cdot, *, \| \cdot \|)$: the algebra of all bounded operators on $H$.

The adjoint, $a^*$, is defined implicitly by

$$(a^*\xi|\eta) = (\xi|a\eta)$$

for all $\xi, \eta$ in $H$. 
Lemma

For all $a, b$ we have

1. $(a^*)^* = a$,
2. $\|a\| = \|a^*\|$,
3. $\|ab\| \leq \|a\| \cdot \|b\|$, 
4. $\|aa^*\| = \|a\|^2$.

Hence $\mathcal{B}(H)$ is a Banach algebra with involution. (4) is the “$C^*$ equality.”
Example

If $H = L^2(X, \mu)$ and $f : X \to \mathbb{C}$ is bounded and measurable, then

$$H \ni g \mapsto m_f(g) = fg \in H$$

is a bounded linear operator. We have $\|m_f\| = \|f\|_\infty$ and

$$m_f^* = m_f^*.$$

Hence $m_f^* m_f = m_f m_f^* = m|f|^2.$
An operator $a$ is normal if $aa^* = a^*a$.

If $\Phi: H_1 \to H_2$ is an isomorphism between Hilbert spaces, then

$$a \mapsto \text{Ad } \Phi(a) = \Phi a \Phi^{-1}$$

is an isomorphism between $B(H_1)$ and $B(H_2)$.

**Theorem (Spectral Theorem)**

If $a$ is a normal operator then there is a finite measure space $(X, \mu)$, a measurable function $f$ on $X$, and a Hilbert space isomorphism $\Phi: L^2(X, \mu) \to H$ such that $\text{Ad } \Phi m_f = a$. 
An operator is *self-adjoint* if \( a = a^* \). For any \( b \in \mathcal{B}(H) \) we have

\[
b = b_0 - ib_1,
\]

with both \( b_0 = (b + b^*)/2 \) and \( b_1 = i(b^* - b)/2 \) self-adjoint.

**Fact**

*a is self-adjoint iff \((a\xi|\xi)\) is real for all \( \xi \).*

**Pf.**

\[
((a - a^*)\xi|\xi) = (a\xi|\xi) - (a^*\xi|\xi) = (a\xi|\xi) - (\xi|a\xi) = (a\xi|\xi) - (\overline{a\xi|\xi}).
\]

\[\square\]
An operator $b$ such that $(b\xi|\xi) \geq 0$ for all $\xi \in H$ is \textit{positive}.

Example

$m_f \geq 0$ \textit{iff} $\nu\{x \mid f(x) < 0\} = 0$. 
For any self-adjoint \( a \in \mathcal{B}(H) \) we have \( a = a_0 - a_1 \), with both \( a_0 \) and \( a_1 \) positive. (Hint: spectral theorem.)

**Lemma**

\( b \) is positive iff \( b = a^* a \) for some (non-unique) \( a \).

**Proof.**

\((\Leftarrow)\) \( (a^* a \xi | \xi) = (a \xi | a \xi) \geq 0 \).

\((\Rightarrow)\) If \( b \) is positive, by the spectral theorem we may assume \( b = m_f \) for \( f \geq 0 \). Let \( a = m \sqrt{f} \).

\(\square\)
A $p \in \mathcal{B}(H)$ is a projection if $p^2 = p^* = p$.

**Lemma**

$p$ is a projection iff it is an orthogonal projection to a closed subspace of $H$.

**Pf.** We have $p = m_f$ and $f = f^2 = \bar{f}$. Hence $f(x) \in \{0, 1\}$ for almost all $x$, and $m_f = \text{proj}_{\{g | \text{supp}(g) \subseteq Y\}}$ with $Y = f^{-1}(\{1\})$. □
$I$ is the identity operator on $H$. 
An operator $u$ is unitary if $uu^* = u^*u = I$.
An operator $v$ is a partial isometry if

$$p = vv^* \text{ and } q = v^*v$$

are both projections.

**Example**

A partial isometry that is not a normal operator. Let $(e_n)$ be the orthonormal basis of $H$. The unilateral shift $S$ is defined by

$$S(e_n) = e_{n+1} \text{ for all } n.$$ 

Then $S^*(e_{n+1}) = e_n$ and $S^*(e_0) = 0$.

$$S^*S = I \neq \text{proj}_{\text{Span}\{e_n | n \geq 1\}} = SS^*.$$
We have an analogue of $z = re^{\theta}$ for complex numbers.

**Theorem (Polar Decomposition)**

*Every $a$ in $\mathcal{B}(H)$ can be written as*

$$a = bv$$

*where $b$ is positive and $v$ is a partial isometry.*

This does not mean that understanding arbitrary operators reduces to understanding self-adjoints and partial isometries.

**Problem**

*Does every $a \in \mathcal{B}(H)$ have a nontrivial closed invariant subspace?*

The answer is easily positive for all normal operators and all partial isometries.
$I$ is the identity operator on $H$.

**Definition (Spectrum)**

$$\sigma(a) = \{ \lambda \in \mathbb{C} \mid a - \lambda I \text{ is not invertible}\}.$$ 

**Lemma**

1. $\sigma(a)$ is always a compact subset of $\mathbb{C}$.
2. $\sigma(a^*) = \{ \overline{\lambda} \mid \lambda \in \sigma(a) \}$.
3. $a$ is self-adjoint iff $\sigma(a) \subseteq \mathbb{R}$.
4. $a$ is positive iff $\sigma(a) \subseteq [0, \infty)$. 
Concrete and abstract C* algebras

Definition (Concrete C* algebras)
If $X \subseteq \mathcal{B}(H)$ let $A = C^*(X)$ be the smallest norm-closed subalgebra of $\mathcal{B}(H)$.

Definition
$A$ is an abstract C* algebra if it is a Banach algebra with involution such that $\|aa^*\| = \|a\|^2$ for all $a$. 
Example

$X$ is a locally compact Hausdorff space.

$$C_0(X) = \{ f : X \to \mathbb{C} \mid f \text{ is continuous and vanishes at } \infty \}.$$ 

$f^* = \overline{f}$.

$C_0(X)$ is abelian, in particular each operator is normal.

- $f$ is self-adjoint iff the range($f$) $\subseteq \mathbb{R}$.
- $f$ is positive iff range($f$) $\subseteq [0, \infty)$.
- $f$ is a projection iff $f^2(x) = f(x) = \overline{f(x)}$
  iff range($f$) $\subseteq \{0, 1\}$
  iff $f = \chi_U$ for a clopen $U \subseteq X$.

If $X$ is compact then $C_0(X) = C(X)$ has the identity, and we have

$$\sigma(f) = \text{range}(f).$$
Example

$M_n$: $n \times n$ complex matrices. $M_n \cong B(\ell^n_2).

- adjoint, unitary: the usual meaning.
- self-adjoint: hermitian.
- positive: positively definite.
- $\sigma(a)$: the set of eigenvalues.

(normal matrices are diagonalizable)
The algebra of compact operators,

\[ \mathcal{K}(H) = C^*(\{a \in \mathcal{B}(H) \mid a[H] \text{ is finite-dimensional}\}) = \{a \in \mathcal{B}(H) \mid a[\text{unit ball}] \text{ is compact}\} \]

**Fact**

If \( r_n = \text{proj}_{\text{Span}\{e_j \mid j \leq n\}} \), TFAE

1. \( a \in \mathcal{K}(H) \),
2. \( \lim_n \|a(I - r_n)\| = 0 \),
3. \( \lim_n \|(I - r_n)a\| = 0 \).
Note: if $a$ is self-adjoint then

$$\|a(I - r_n)\| = \|(a(I - r_n))^*\| = \|(I - r_n)a\|.$$  

$\mathcal{K}(H)$ is an ideal of $\mathcal{B}(H)$ (closed, two-sided, self-adjoint ideal). The quotient $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is the Calkin algebra. $\pi: \mathcal{B}(H) \rightarrow \mathcal{C}(H)$ is the quotient map.

$\sigma(\pi(a)) = \sigma_e(a):$ the essential spectrum of $a.$

Here

$\sigma_e(a) =$ the set of all accumulation points of $\sigma(a)$ plus all points of $\sigma(a)$ of infinite multiplicity
Direct (inductive) limits

If $\Omega$ is a directed set, $A_i$, $i \in \Omega$ are C*-algebras and 

$$\varphi_{i,j} : A_i \rightarrow A_j \quad \text{for } i < j$$

is a commuting family of *-homomorphisms, define the direct limit

$$A = \lim_{\rightarrow} A_i.$$ 

For $a \in A_i$ let

$$\|a\| = \lim_i \|\varphi_{i,j}(a)\|_{A_j}$$

and take the completion.

Example

The CAR (Canonical Anticommutation Relations) algebra (aka the Fermion algebra, aka $M_{2^\infty}$ UHF algebra).

$$\Phi_n : M_{2^n} \rightarrow M_{2^{n+1}}$$

$$\Phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$
If \((E_n)\) is an orthogonal decomposition of \(H\) into finite-dimensional subspaces then

\[
D[\vec{E}] = \{ a \in B(H) \mid \text{each } E_n \text{ is } a\text{-invariant} \}.
\]

If \(\vec{E}\) refines \(\vec{F}\), then \(D[\vec{E}] \subset D[\vec{F}]\).
Fact
The unilateral shift $S$ does not belong to $\mathcal{D}[\vec{E}]$ for any $\vec{E}$.

Pf. Some $a$ is Fredholm if its Fredholm index

$$\text{index}(a) = \dim \ker(a) - \dim \ker(a^*)$$

is finite.
If $a \in \mathcal{D}[\vec{E}]$ is Fredholm then $\text{index}(a) = 0$.
However, $\text{index}(S) = -1$. □
Lemma

If $a \in \mathcal{B}(H)$ is normal then

$$C^*(a, l) \cong C(\sigma(a)).$$

For every $f : \sigma(a) \to \mathbb{C}$ we can define $f(a) \in C^*(a, l)$.

For example:

$$a = \frac{|a| + a}{2} - \frac{|a| - a}{2}$$

If $a \geq 0$, then $\sqrt{a}$ is defined.
A C* algebra is *unital* if it has a unit (multiplicative identity).

**Lemma**

*Every C* algebra $A$ is contained in a unital C* algebra $	ilde{A} \cong A \oplus \mathbb{C}$.*

We call $\tilde{A}$ the *unitization* of $A$. 
If $A < B$ we say $A$ is a *unital subalgebra* of $B$ if both $B$ is unital and its unit belongs to $A$.

If $a \in A$ and $A$ is unital, one could define

$$
\sigma_A(a) = \{ \lambda \in \mathbb{C} \mid a - \lambda I \text{ is not invertible} \}.
$$

**Lemma**

*Assume $A$ is a unital subalgebra of $B$ and $a \in A$. Then $\sigma_A(a) = \sigma_B(a)$.***
Lemma

Every \(*\)-homomorphism \(\Phi\) between \(C^*\) algebras is continuous.

\(\text{Pf.}\) We prove \(\Phi\) is a contraction.

Note that \(\sigma(\Phi(a)) \subseteq \sigma(a)\). Thus for a normal

\[
\| a \| = \sup \{| \lambda | \mid \lambda \in \sigma(a) \}
\geq \sup \{| \lambda | \mid \lambda \in \sigma(\Phi(a))\}
= \| \Phi(a) \|
\]

For general \(a\) we have

\[
\| a \| = \sqrt{\| aa^* \|} \geq \sqrt{\| \Phi(aa^*) \|} = \| \Phi(a) \|.
\]
Pure states and the GNS construction

Theorem (Gelfand–Naimark)

Every commutative $C^*$-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space $X$. If it is moreover unital, then $X$ can be chosen to be compact.

Theorem (Gelfand–Naimark–Segal)

Every $C^*$-algebra $A$ is isomorphic to a closed subalgebra of $B(H)$ for some Hilbert space $H$.

A continuous linear functional $\varphi: A \to \mathbb{C}$ is positive if $\varphi(a) \geq 0$ for all positive $a$. It is a state if $\varphi(I) = 1$.

$\mathcal{S}(A)$ is the space of all states on $A$. 
If $\xi$ is a unit vector, define a functional $\omega_\xi$ on $B(H)$ by

$$\omega_\xi(a) = (a\xi|\xi).$$

Then $\omega_\xi(a) \geq 0$ for a positive $a$ and $\omega_\xi(I) = 1$; hence it is a state. States form a weak*-compact convex subset of $A^*$. Cauchy–Schwartz for states:

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$
Theorem (GNS)

Assume $\varphi$ is a state on $A$. There is a representation $\pi_\varphi: A \to \mathcal{B}(H_\varphi)$ and a unit vector $\xi = \xi_\varphi$ in $H_\varphi$ such that

$$\varphi(a) = \omega_\xi(a)$$

for all $a$.

Proof.
On $A \times A$ let

$$(a|b) = \varphi(b^*a).$$

$J_\varphi = \{a \mid \varphi(a^*a) = 0\}$

$H_\varphi = \widetilde{A}/J$

$\pi_\varphi(a)$ sends $[b]_{J_\varphi}$ to $[ab]_{J_\varphi}$. 

$\blacksquare$
The space of states on $A$

$\pi_1 \sim \pi_2$ if $\exists u : H_1 \to H_2$ such that

\[
\pi_1 \sim \pi_2 \quad \text{if} \quad \exists u : H_1 \to H_2 \quad \text{such that} \\
\pi_1 \sim \pi_2 \quad \text{if} \quad \exists u : H_1 \to H_2 \quad \text{such that} \\
B(H_1) \quad \pi_1 \\
\downarrow \quad \pi_1 \\
A \quad \text{Ad} u \\
\downarrow \quad \text{Ad} u \\
B(H_2) \quad \pi_2 \\
\downarrow \quad \pi_2 \\
B(H_2) \\
\text{Ad} u(a) = uau^*$
\( \varphi_1 \sim \varphi_2 \) if and only if \( \exists u \in A \) such that

\[ A \xrightarrow{\text{Ad} \ u} C \xleftarrow{\varphi_2} A \]

**Theorem**

*For \( \varphi_1, \varphi_2 \) in \( \mathbb{S}(A) \) we have \( \varphi_1 \sim \varphi_2 \iff \pi \varphi_1 \sim \pi \varphi_2 \).*
Lemma

If $\|\varphi\| = 1$ then $\varphi$ is a state iff $\varphi(I) = 1$.

A state $\varphi$ is pure iff

$$\varphi = t\psi_0 + (1 - t)\psi_1, \quad 0 \leq t \leq 1$$

for some states $\psi_0, \psi_1$ implies $\varphi = \psi_0$ or $\varphi = \psi_1$.

$\mathbb{P}(A)$ is the space of all pure states of $A$. 

Example

If $A = C(X)$, then (by Riesz) $\varphi$ is a state iff $\varphi(f) = \int f \, d\mu$ for some Borel probability measure $\mu$.

Lemma

For a state $\varphi$ of $C(X)$ TFAE:

1. $\varphi$ is pure,
2. for some $x_\varphi \in X$ we have $\varphi(f) = f(x_\varphi)$
3. $\varphi : C(X) \to \mathbb{C}$ is a $\ast$-homomorphism.
If $\xi \in H$ is a unit vector, then 

$$\omega_\xi(a) = (a\xi|\xi)$$

is a vector state. All vector states are pure.

**Definition**

Some $\varphi \in S(\mathcal{B}(H))$ is singular if $\varphi[K(H)] = \{0\}$.

**Theorem**

Each state of $\mathcal{B}(H)$ is a weak*-limit of vector states.
Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then

$$\varphi^{(\vec{e})}_{\mathcal{U}}(a) = \lim_{n \to \mathcal{U}} (ae_n | e_n)$$

is a singular state.

A state of the form $\varphi^{(\vec{\xi})}_{\mathcal{U}}$ is diagonalized.

**Theorem (Anderson, 1977)**

Each $\varphi^{(\vec{e})}_{\mathcal{U}}$ is pure.

**Conjecture (Anderson, 1977)**

Every pure state on $\mathcal{B}(H)$ can be diagonalized.
Let $p, q$ be projections in $\mathcal{B}(H)$. Define $p \leq q$ if $pq = p$.

**Fact**

$pq = p$ iff $qp = p$.

**Proof.**

Since $p = p^*$, $pq = p$ implies $pq = (pq)^* = q^*p^* = qp$.

Note that $pq = qp$ if and only if $pq$ is a projection.

$p \land q$: the projection to $\text{range}(p) \cap \text{range}(q)$

$p \lor q$: the projection to $\text{Span}(\text{range}(p) \cup \text{range}(q))$. 
Lemma

The projections in $\mathcal{B}(H)$ form a lattice with respect to $\wedge$, $\vee$, $\leq$, $I$, 0.

Lemma

$\mathcal{B}(H) = C^*(\mathcal{P}(\mathcal{B}(H)))$. That is, $\overline{\text{Span} \mathcal{P}(\mathcal{B}(H))}$ is norm-dense in $\mathcal{B}(H)$. 

\[\square\]
$\mathcal{K}(H)$ is a (self-adjoint, norm closed, two-sided) ideal of $\mathcal{B}(H)$. 
$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is the Calkin algebra.
$\pi : \mathcal{B}(H) \rightarrow \mathcal{C}(H)$ is the quotient map.
Lemma
If $a$ is self-adjoint in $C(H)$, then $a = \pi(a)$ for a self-adjoint $a$ in $C(H)$.

Pf. Fix any $a_0$ such that $\pi(a_0) = a$. Let $a = (a_0 + a_0^*)/2$. □
Lemma

*If* \( p \) *is a projection in* \( \mathcal{C}(H) \), *then* \( p = \pi(p) \) *for a projection* \( p \) *in* \( \mathcal{C}(H) \).

**Pf.** Fix a self-adjoint \( a \) such that \( p = \pi(a) \). There are \((X, \mu)\) and \( f \in L^\infty(X, \mu) \) and a Hilbert space isomorphism \( \Phi: L^2(X, \mu) \to H \) such that \( \Phi(m_f) = a \). Let

\[
h(x) = \begin{cases} 
1, & f(x) \geq 1/2 \\
0, & f(x) < 1/2.
\end{cases}
\]

Then \( m_h \) is a projection and \( \pi(m_h) = \pi(m_f) \). \( \square \)
Lemma

There is a normal (even a unitary) operator in $\mathcal{C}(H)$ that is distinct from $\pi(v)$ for any normal $v$ in $\mathcal{B}(H)$.

Pf. The image $S$ of the unilateral shift is a unitary in $\mathcal{C}(H)$, since $S^*S = I = SS^*$. If $v - S$ is compact then $v$ is Fredholm, and $\text{index}(v) = -1$. □
General spectral theorem

Theorem (Spectral Theorem)

If $A$ is an abelian $C^*$-subalgebra of $\mathcal{B}(H)$ then there is a finite measure space $(X, \mu)$, a subalgebra $B$ of $L^\infty(X, \mu)$, and a Hilbert space isomorphism $\Phi: L^2(X, \mu) \rightarrow H$ such that $\Phi[B] = A$. 
The atomic masa

MASA: MAximal Self-Adjoint SubAlgebra.
Fix $H$ and its orthonormal basis $(e_n)$.

$$(\alpha_n) \in \ell^\infty$$

$$\sum_n \alpha_n P_{\mathbb{C}e_n} \in \mathcal{B}(H).$$

Lemma

$A^{(\vec{\alpha})} = \{ \sum_n \alpha_n P_{\mathbb{C}e_n} \}$ is a masa in $\mathcal{B}(H)$. \qed
Embedding $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathcal{B}(H))$

\[ X \in \mathcal{P}(\mathbb{N}) \]
\[ P_X^{(e)} = P_X = \text{proj}_{\text{Span}\{e_n|n \in X\}} \]

\[ \mathcal{P}(\mathbb{N}) \ni X \mapsto P_X \in \mathcal{P}(\mathcal{B}(H)). \]

Hence $\mathcal{P}(\mathbb{N})$ is a maximal Boolean subalgebra of $\mathcal{P}(\mathcal{B}(H))$. 
Atomless masa

$L^\infty(\mu)$ is also a masa in $\mathcal{B}(L^2(\mu))$ for a diffused measure $\mu$.

Fact

$\mathcal{P}(L^\infty(\mu))$ is a maximal Boolean subalgebra of $\mathcal{P}(\mathcal{B}(H))$ isomorphic to the Lebesgue measure algebra, Borel/Null.
Theorem (Johnson–Parrott)

If $A$ is a masa in $B(H)$ then $\pi[A]$ is a masa in $C(H)$.

For the atomic masa $A$ we have

$$A/K(H) \approx \ell^\infty / c_0.$$ 

$$\mathcal{P}(\mathbb{N})/\text{Fin} \ni [A] \mapsto [P_A] \in \mathcal{P}(\ell^\infty / c_0).$$

Both $\mathcal{P}(\mathbb{N})/\text{Fin}$ and the Lebesgue measure algebra are maximal boolean subalgebras of $\mathcal{P}(C(H))$. 

Lemma
For projections $p$ and $q$ in $\mathcal{B}(H)$ TFAE

1. $\pi(p) \leq \pi(q)$,
2. $q(I - p)$ is compact,
3. $(\forall \varepsilon > 0)(\exists p_0 \leq I - p) \ p_0$ is finite-dimensional and $\|q(I - p - p_0)\| < \varepsilon$.

We write $p \leq_K q$ if the conditions of Lemma 23 are satisfied.

Corollary
The poset $(\mathcal{P}(\mathcal{C}(H)), \leq)$ is isomorphic to the quotient $(\mathcal{P}^{\mathcal{B}}(H)), \leq_K)$. Let’s write $\dot{p} = \pi(p)$. 

□
Proposition (Weaver)
\( \mathcal{P}(C(H)) \) is not a lattice.

Proof.
Enumerate a basis of \( H \) as \( \xi_{mn}, \eta_{mn} \) for \( m, n \) in \( \mathbb{N} \).

\[ \zeta_{mn} = \frac{1}{n} \xi_{mn} + \frac{\sqrt{n-1}}{n} \eta_{mn} \]

\[ K = \text{Span}\{\xi_{mn} \mid m, n \in \mathbb{N}\}, \quad p = \text{proj}_K \]

\[ L = \text{Span}\{\zeta_{mn} \mid m, n \in \mathbb{N}\}, \quad q = \text{proj}_L \]

For \( f \in \mathbb{N}^\mathbb{N} \) let \( M(f) = \text{Span}\{\xi_{mn} \mid m \leq f(n)\}, \quad r(f) = \text{proj}_{M(f)} \).

Fact

1. \( r(f) \leq p \) for all \( f \),
2. \( r(f) \leq q \) for all \( f \),
3. if \( r \leq_K p \) and \( r \leq_K q \) then \( r \leq_K r(f) \) for some \( f \).
Recall

\[ \alpha = \min \{|A| \mid A \text{ is a maximal infinite antichain in } \mathcal{P}(\mathbb{N})/\text{Fin}\}. \]

**Definition (Wofsey, 2006)**

A family \( A \subseteq \mathcal{P}(\mathcal{B}(H)) \) is almost orthogonal (aof) if \( pq \) is compact for \( p \neq q \) in \( A \).

\[ \alpha^* = \min \{|A| \mid A \text{ is a maximal infinite aof}\} \]
Theorem (Wofsey, 2006)

1. *It is relatively consistent with ZFC that* $\aleph_1 = \alpha = \alpha^* < 2^{\aleph_0}$,
2. *MA implies* $\alpha^* = 2^{\aleph_0}$.

Question

*Is* $\alpha = \alpha^*$? *Is* $\alpha \geq \alpha^*$? *Is* $\alpha^* \geq \alpha$?

It may seem obvious that $\alpha \geq \alpha^*$?
Definition/Theorem (Solecki, 1995)

An ideal \( J \) on \( \mathbb{N} \) is an analytic \( P \)-ideal if there is a lower semicontinuous (lsc) submeasure \( \varphi \) on \( \mathbb{N} \) such that

\[
J = \{ X \mid \limsup_{n} \varphi(X \setminus n) = 0 \}.
\]

Lemma (Steprāns, 2007)

Fix \( a \in \mathcal{B}(H) \). Then

\[
J_a = \{ X \subseteq \mathbb{N} \mid aP^{(\vec{e})}_X \text{ is compact} \}
\]

is an analytic \( P \)-ideal.

Pf. Let \( \varphi_a(X) = \| P_X a \| \). \( P_X a \) is compact iff \( \lim_n \varphi_a(X \setminus n) = 0 \). \( \square \)
Proposition (Wofsey, 2006)

There is a mad family $A \subseteq \mathcal{P}(\mathbb{N})$ whose image in $\mathcal{P}(\mathcal{B}(H))$ is not a maof.

Proof.

Let $\xi_n = 2^{-n/2} \sum_{j=2^n}^{2^{n+1}-1} e_j$ and $q = \text{proj}_{\text{Span}\{\xi_n\}}$. Then $\lim_n \|qe_n\| = 0$ hence $J_q$ is a dense ideal: every infinite subset of $\mathbb{N}$ has an infinite subset in $J_q$.

Let $A$ be a mad family contained in $J_q$.

Then $q$ is almost orthogonal to all $P_X$, $X \in A$. \qed
Let

\[ a' = \min\{|A| \mid A \text{ is mad and } A \not\subseteq J \}
\text{ for any analytic P-ideal } J \]
Theorem (Wofsey, 2006)

There is a forcing extension in which there are maximal chains in $\mathcal{P}(\mathcal{C}(H))$ of different cofinalities (and $2^{\aleph_0} = \aleph_2$).

Theorem (essentially Shelah–Steprāns)

There is a model of $\neg CH$ in which all maximal chains in $\mathcal{P}(\mathbb{N})/\text{Fin}$ are isomorphic.
A twist of projections

Consider

\[ I = \min\{|A| \mid A \text{ is a family of commuting projections in } \mathcal{C}(H)\} \]

that cannot be lifted to a family of commuting projections of \( \mathcal{B}(H) \).

Lemma

\[ I > \aleph_0. \]

Proposition (Farah, 2006)

\[ I = \aleph_1: \text{There are commuting projections } p_\xi, \xi < \omega_1, \text{ in } \mathcal{C}(H) \text{ that cannot be lifted to commuting projections of } \mathcal{B}(H). \]
Pf. Construct $p_\xi$ in $\mathcal{P}(\mathcal{B}(H))$ so that for $\xi \neq \eta$:

1. $p_\xi p_\eta$ is compact, and
2. $\|[p_\xi, p_\eta]\| > 1/4$

If $(e_n)$ diagonalizes each $p_\xi$, fix $X(\xi) \subseteq \mathbb{N}$ such that

$$d_\xi = p_\xi - P_{X(\xi)}^{(\bar{e})}$$

is compact. Let

$$r_n = P_{\{0,1,\ldots,n-1\}}^{(\bar{e})}.$$

Then $a$ is compact iff $\lim_n \|a(I - r_n)\| = 0$.

Fix $\tilde{n}$ such that $\|d_\xi(I - r_{\tilde{n}})\| < 1/8$ for uncountably many $\xi$.

If $\|(d_\xi - d_\eta)\|_{r_{\tilde{n}}} < 1/8$, then

$$\|[p_\xi, p_\eta]\| \leq \|[P_{X(\xi)}, P_{X(\eta)}]\| + \frac{1}{4} = \frac{1}{4}$$

a contradiction. \qed
Automorphisms of C* algebras

\[ \text{Ad } u(a) = uau^*. \]

An automorphism \( \Phi \) is \textit{inner} if \( \Phi = \text{Ad } u \) for some unitary \( u \).

**Lemma**

If \( A \) is abelian then \( \text{id} \) is its only inner automorphisms.

If \( A = C(X) \) then each automorphism is of the form

\[ f \mapsto f \circ \Psi \]

for an autohomeomorphism \( \Psi \) of \( X \).
Lemma
All automorphisms of $\mathcal{B}(H)$ are inner. Hence all automorphisms of any $M_n$ are inner.

Lemma
The CAR algebra $(M_{2^\infty} = \bigotimes_n M_2)$ has outer automorphisms.

Pf. $\Phi = \bigotimes_n \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is outer since $\bigotimes_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not in $M_{2^\infty}$. 

Lemma
If $A$ is a unital subalgebra of $B$ then

1. The restriction of a state of $B$ to $A$ is a state of $A$.

2. Every (pure) state of $A$ can be extended to a (pure) state of $B$.

$\text{Pf.}$ (2) By Hahn–Banach $\{\psi \in B^* \mid \psi \upharpoonright A = \varphi, \|\psi\| = 1\}$ is nonempty and by Krein–Milman it has an extreme point. $\square$
Example

Restriction of a pure state to a unital subalgebra need not be pure. If $\omega_\xi$ is a vector state of $B(H)$ and $A$ is the atomic masa diagonalized by $(e_n)$, then $\omega_\xi \upharpoonright A$ is pure iff $|\langle \xi | e_n \rangle| = 1$ for some $n$. 
Proposition
Assume $A < B$ and $B$ is abelian. If every pure state of $A$ extends to the unique pure state of $B$, then $A = B$.

Proof.
$A < C(X)$ separates points of $X$. Use Stone–Weierstrass. \hfill \square

Problem (Noncommutative Stone–Weierstrass problem)
Assume $A < B$ and $A$ separates $\mathcal{P}(B) \cup \{0\}$. Does necessarily $A = B$?
A C* algebra is *simple* if and only if it has no (closed, two-sided, self-adjoint) nontrivial ideals.

**Lemma (Akemann–Weaver)**

Assume $A$ is a simple separable unital C* algebra and $\varphi$ and $\psi$ are its pure states. Then there is a simple separable unital $B > A$ such that

1. $\varphi$ and $\psi$ extend to pure states $\varphi'$, $\psi'$ of $B$ in a unique way.
2. $\varphi'$ and $\psi'$ are equivalent.
Pure states on $M_{2\infty}$

On $M_2$:

\[ \varphi_1 : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{11} \]

\[ \varphi_2 : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{22} \]

For $f \in 2^\mathbb{N}$

\[ \varphi_f = \bigotimes_n \varphi_{f(n)} \]

is in $\mathbb{P}(M_{2\infty})$.

In $M_{2\infty}$, $\varphi_f \sim \varphi_g$ iff $\{ n \mid f(n) \neq g(n) \}$ is finite.

Fact

If $f \neq g$ then $\| \varphi_f - \varphi_g \| = 2$. \qed
Type I algebras

Definition (Kaplansky)

A C* algebra $A$ is of type I if for every irreducible representation \(\pi: A \to \mathcal{B}(H)\) we have \(\pi[A] \supseteq \mathcal{K}(H)\).

[Not to be confused with type I von Neumann algebras: \(\mathcal{B}(H)\) is a type I von Neumann algebra and a non-type-I C* algebra.]

A C* algebra is simple if and only if it has no (closed, two-sided, self-adjoint) nontrivial ideals.

Lemma

A type I C* algebra has only one irrep up to equivalence if and only if it is isomorphic to \(\mathcal{K}(H)\) for some $H$.

Theorem (Glimm)

If $A$ is a non-type-I C* algebra then there is $B < A$ that has a quotient isomorphic to $M_{2^\infty}$. 
Corollary (Akemann–Weaver, 2002)

If $A$ is non-type-I and has a dense subset of cardinality $< 2^{\aleph_0}$, then $A$ has nonequivalent pure states.

Proof.
There are pure states $\varphi_f$, $f \in 2^N$, such that if $f \neq g$ and $\text{Ad } u \varphi_f = \text{Ad } v \varphi_g$ then $\|u - v\| \geq 1$. □
Theorem (Naimark, 1948)
Any two irreps of $\mathcal{K}(H)$ are equivalent.

Question (Naimark, 1951)
Is the converse true?

Theorem (Akemann–Weaver, 2002)
Assume $\diamondsuit$. Then Naimark’s problem has a negative solution.
Fix $h_\alpha : \alpha \to \omega_1$ such that for every $g : \omega_1 \to \omega_1$ the set
\[ \{ \alpha \mid g \upharpoonright \alpha = h_\alpha \} \] is stationary.
Find an increasing chain of simple separable unital C* algebras $A_\alpha$, $\alpha < \omega_1$ and pure state $\psi_\alpha$ of $A_\alpha$ so that

1. $\alpha < \beta$ implies $\psi_\beta \upharpoonright A_\alpha = \psi_\alpha$.

For each $A_\alpha$, let $\{ \phi_\alpha^\gamma \mid \gamma < \omega_1 \}$ enumerate all of its pure states. If $\alpha$ is limit, let

\[ A_\alpha = \lim_{\rightarrow} A_\beta. \]
Now we consider the successor ordinal case, $\beta = \alpha + 1$. Assume there is $\varphi \in \mathcal{P}(A_\alpha)$ such that $\varphi \upharpoonright A_\beta = \varphi^{h_\alpha(\beta)}_\beta$ for all $\beta < \alpha$.

Using lemma, let $A_{\alpha+1}$ be such that $\psi_\alpha$ and $\varphi$ have unique extensions to $A_{\alpha+1}$ that are equivalent. Since $A = A_{\omega_1}$ is unital and infinite-dimensional, $A \ncong \mathcal{K}(H')$. 
Fix $\varphi \in \mathbb{P}(A)$.

**Claim**

$\{ \alpha \mid \varphi \upharpoonright A_\alpha \in \mathbb{P}(A_\alpha) \}$ contains a club.

**Proof.**

For $x \in A_{\omega_1}$ and $m \in \mathbb{N}$

$$\{ \alpha \mid \exists \psi_1, \psi_2 \in \mathbb{S}(A_\alpha), \varphi = \frac{1}{2}(\psi_1 + \psi_2) \text{ and } |\varphi(x) - \psi_1(x)| \geq \frac{1}{m} \}$$

is bounded in $\omega_1$. □
Fix $h: \omega_1 \to \omega_1$ so that

$$\varphi \upharpoonright A_\alpha = \varphi_{\alpha}^{h(\alpha)}$$

for all $\alpha$.

Let $\alpha$ be such that $h \upharpoonright \alpha = h_{\alpha}$. Then $\varphi \upharpoonright A_{\alpha+1}$ is equivalent to $\psi_{\alpha+1}$. Since $\psi_{\alpha+1}$ has unique extension to $A_{\omega_1}$, so does $\varphi$ and they remain equivalent.
Definition

A masa in $\mathcal{B}(H)$ has the extension property (EP) if each of its pure states extends uniquely to a pure state on $\mathcal{B}(H)$.

Every vector state has the unique extension to a pure state, hence this is a property of masas in the Calkin algebra.

1. Kadison–Singer, 1955: The atomless masa does not have the EP.

2. Anderson, 1974: CH implies there is a masa in the Calkin algebra with the EP.
Question (Kadison–Singer, 1955)

*Does the atomic masa of \( \mathcal{B}(H) \) have EP?*

A positive answer is equivalent to an arithmetic statement, so let’s go on.
Fix an orthonormal basis \((e_n)\) of \(H\), let \(A\) be the atomic masa diagonalized by \((e_n)\). Each pure state of \(A\) is of the form

\[
\varphi_U(a) = \lim_{n \to U} (ae_n|e_n)
\]

for an ultrafilter \(U\) on \(\mathbb{N}\).

A state on \(B(H)\) of the form \(\varphi^{(\xi)}_U\) is diagonalized (by \(U, (e_n)\)).

**Conjecture (Anderson)**

*Every pure state \(\varphi\) of \(B(H)\) can be diagonalized.*
Recall that on an abelian $C^*$ algebra a state is pure iff it is multiplicative.

**Conjecture (Kadison–Singer)**

For every pure state $\varphi$ of $B(H)$ there is an atomic masa $\mathcal{A}$ such that $\varphi \restriction \mathcal{A}$ is multiplicative.

If $\varphi \restriction \mathcal{A}$ is multiplicative, then there is an ultrafilter $\mathcal{U}$ such that $\varphi$ and $\varphi_{\mathcal{U}}$ agree on $\mathcal{A}$. We can conclude that $\varphi = \pi_{\mathcal{U}}$ if the answer to the Kadison–Singer problem is positive.
Theorem (Akemann–Weaver, 2005)

*CH implies there is a pure state \( \varphi \) on \( \mathcal{B}(H) \) that is not multiplicative on any atomic masa.*
States are coded by ‘noncommutative finitely additive measures.’

**Theorem (Gleason)**

Assume $\mu: \mathcal{P}(\mathcal{B}(H)) \to [0, 1]$ is such that $\varphi(p + q) = \varphi(p) + \varphi(q)$ whenever $pq = 0$. Then there is a unique state $\varphi$ on $\mathcal{B}(H)$ that extends $\mu$. \qed
Lemma
If $\varphi$ is a state on $A$ and $p$ is a projection such that $\varphi(p) = 1$, then $\varphi(a) = \varphi(pap)$ for all $a$.

Proof.
By Cauchy–Schwartz

$$|\varphi((I - p)a)| \leq \sqrt{\varphi(I - p)\varphi(a^*a)} = 0$$

since $a = pa + (I - p)a$ we have $\varphi(a) = \varphi(pa)$, similarly $\varphi(pa) = \varphi(pap)$.  \qed
Definition

A family $\mathcal{F}$ of projections in a $C^*$ algebra is a filter if

1. for $p, q$ in $\mathcal{F}$ there is $r \in \mathcal{F}$ such that $r \leq p$ and $r \leq q$.

2. for $p \in \mathcal{F}$ and $r \geq p$ we have $r \in \mathcal{F}$.

A filter generated by $\mathcal{X} \subseteq \mathcal{P}(A)$ is the intersection of all filters containing $\mathcal{X}$.
A filter $\mathcal{F}$ in $\mathcal{P}(\mathcal{C}(H))$ lifts if there is a commuting family $\mathcal{X}$ in $\mathcal{P}(\mathcal{B}(H)))$ that generates a filter $\mathcal{F}$ such that $\pi[\mathcal{F}] = \mathcal{F}$.

Note: If $\mathcal{F}$ is a filter in $\mathcal{C}(H)$, then

$$\tilde{\mathcal{F}} = \{ p \in \mathcal{P}(\mathcal{B}(H)) \mid \pi[p] \in \mathcal{F} \}$$

is not necessarily a filter.
Question

Does every maximal filter $\mathcal{F}$ in $P(C(H))$ lift?

Theorem (Anderson)

There are a singular pure state $\varphi$ of $B(H)$, an atomic masa $A_1$, and an atomless masa $A_2$ such that $\varphi \upharpoonright A_j$ is multiplicative for $j = 1, 2$. 
Lemma (Weaver, 2007)

For $\tilde{F}$ in $\mathcal{P}(B(H))$ TFAE:

(A) $\|p_1 p_2 \ldots p_n\| = 1$ for any $n$-tuple of projections in $\tilde{F}$ and $F$ is maximal with respect to this property.

(B) $(\forall \varepsilon > 0)$ for all finite $F \subseteq \tilde{F}$ there is a unit vector $\xi$ such that

$$\|p\xi\| > 1 - \varepsilon$$

for all $p \in F$.

Definition

A family $\tilde{F}$ in $\mathcal{P}(B(H))$ is a quantum filter if the conditions of Lemma 37 hold.
Theorem (Farah–Weaver, 2007)

Assume $\mathcal{F} \subseteq \mathcal{P}(\mathcal{C}(H))$. TFAE:

1. $\mathcal{F}$ is a maximal quantum filter,
2. $\mathcal{F} = \mathcal{F}_\varphi = \{ p \mid \varphi(p) = 1 \}$ for some pure state $\varphi$.

Proof.

(1) implies (2). For a finite $F \subseteq \mathcal{F}$ and $\epsilon > 0$ let

$$X_{F,\epsilon} = \{ \varphi \in \mathcal{S}(\mathcal{B}(H)) \mid \varphi(p) \geq 1 - \epsilon \text{ for all } p \in F \}.$$

If $\xi$ is as in (B) then $\omega_\xi \in X_{F,\epsilon}$.

Since $X_{F,\epsilon}$ is weak*-compact $\bigcap_{(F,\epsilon)} X_{F,\epsilon} \neq \emptyset$. Any extreme point is a pure state.

(It can be proved that this intersection is a singleton.)

(2) implies (1). If $\varphi(p_j) = 1$ for $j = 1, \ldots, k$, then $\varphi(p_1 p_2 \ldots p_k) = 1$, hence (A) follows.

$\square$
Lemma

Let \((\xi_n)\) be an orthonormal basis. If for some \(n\) we have \(\mathbb{N} = \bigcup_{j=1}^{n} A_j\) and there is \(q \in \tilde{F}\) such that

\[
\|P_{A_j}^{(\xi)} q\| < 1
\]

for all \(j\), then \(F\) is not diagonalized by \((\xi_n)\).
Lemma
Assume \((\xi_n)\) is an orthonormal basic sequence. There is a partition of \(\mathbb{N}\) into finite intervals \((J_n)\) such that for all \(k\)

\[ \xi_k \in \overline{\text{Span}} \{ e_i \mid i \in J_n \cup J_{n+1} \} \]

(modulo a small perturbation of \(\xi_k\)) for some \(n = n(k)\).

For \((J_n)\) as in Lemma 39 let

\[ \mathbb{D}_j = \{ q \mid \| P_{J_n \cup J_{n+1}} \bar{e} \| < 1/2 \text{ for all } n \} \]

Lemma
Each \(\mathbb{D}_j\) is dense in \(\mathcal{P}(\mathcal{B}(H))\).
\[ \vartheta = \min\{|F| \mid F \subseteq \mathbb{N}^\mathbb{N} \text{ is } \leq\text{-cofinal}\}. \]

\[ t^* = \min\{|T| \mid T \subseteq \mathcal{P}(\mathcal{C}(H)) \setminus 0 \quad \text{\(T\) is a maximal decreasing well-ordered chain}\} \]

**Theorem (Farah–Weaver)**

Assume \( \vartheta \leq t^* \). Then there exists a maximal proper filter in \( \mathcal{P}(\mathcal{C}(H)) \) that is not diagonalized by any atomic masa.

---

1CH would do; \( \vartheta \) <‘the Novák number of \( \mathcal{P}(\mathcal{C}(H)) \)' is best if it makes sense
Pf. By $d \leq t^*$, we may choose $\mathcal{F}$ so that $\mathcal{F} \cap \mathbb{D}_j \neq \emptyset$ for all $(\vec{J})$. Given $(\xi_k)$, pick $(J_n)$ such that $\xi_k \in J_{n(k)} \cup J_{n(k)+1}$ for all $k$. Let

$$A_i = \{k \mid n(k) \mod 4 = i\}$$

for $0 \leq i < 4$.

If $q \in \mathcal{F} \cap \mathbb{D}_j$, then $\|P_{A_i}^{(\xi)} q\| < 1$ for $0 \leq i < 4$. □

Corollary (Akemann-Weaver, 2006)

CH implies there is a pure state that is not multiplicative on any atomic masa.
An extra: Reid’s theorem

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a $Q$-point if every partition of $\mathbb{N}$ into finite intervals has a transversal in $\mathcal{U}$. Recall $P^{(\vec{e})}_X = P_X = \text{proj}_{\text{Span}\{e_n|n \in X\}}$. 

**Theorem (Reid)**

If $\mathcal{U}$ is a $Q$-point then $\varphi_{\mathcal{U}} \upharpoonright \mathcal{A}^{(\vec{e})}$ has the unique extension to a pure state of $\mathcal{B}(H)$.
Proof of Reid’s theorem

Fix a pure state extension \( \varphi \) of \( \varphi_U \upharpoonright A(\vec{e}) \) and \( a \in \mathcal{B}(H) \). Fix finite intervals \((J_i)\) such that \( \mathbb{N} = \bigcup_n J_n \) and

\[
\| P_{J_m} a P_{J_n} \| < 2^{-m-n}
\]

whenever \( |m - n| \geq 2 \) and let \( X \in \mathcal{U} \) be such that

\[
X \cap (J_{2i} \cup J_{2i+1}) = \{n(i)\}
\]

for all \( i \).
Then with \( Q_i = P_{n(i)} \) and \( f_i = e_{n(i)} \) we have \( \varphi(\sum_i Q_i) = 1 \) and

\[
Q a Q = \sum_i Q_i a \sum_i Q_i = \sum_i Q_i a Q_i + \sum_{i \neq j} Q_i a Q_j.
\]

The second summand is compact, and

\[
Q_i a Q_i = (af_i | f_i) f_i
\]

therefore if \( \alpha = \lim_{i \to \mathcal{U}} (af_i | f_i) \) we have

\[
\lim_{X \to \mathcal{U}} (P_X a P_X - \alpha P_X) = 0
\]

and \( \varphi(a) = \alpha \).

Hence \( \varphi(a) = \varphi_{\mathcal{U}}(a) \) for all \( a \).