Not UBH, not for circulation
–ES

Woodin proved that if there is a cardinal κ that is 2^{2^κ} supercompact, then there is an iteration tree on V with two branches, both of which are wellfounded. These notes record explanations by Woodin and Steel. [1]

We will need the following definitions and facts.

- If S is a tree on $\omega \times \lambda$, then S is κ *homogeneous* iff there is a family $\langle \mu_u \mid u \in {}^{<\omega}\omega \rangle$ of κ complete ultrafilters such that
$$\mu_u\left(S_u\right) = 1$$
$$\mu_u \text{ projects to } \mu_{u \restriction n} \text{ for all } n < |u|$$
and, if $x \in \operatorname{proj}\left([S]\right)$, then $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is a wellfounded tower.
- If $x \notin \operatorname{proj}\left([S]\right)$, then $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is is an illfounded tower. This is because if
$$i : V \longrightarrow \operatorname{ult}\left(V, \langle \mu_{x \restriction n} \mid n < \omega \rangle\right),$$
then
$$\left\langle [\mathrm{id}]_{\mu_{x \restriction n}} \mid n < \omega \right\rangle$$
is an infinite branch through $i\left(S_x\right)$.
- A tower $\langle \mu_n \mid n < \omega \rangle$ is *countably complete* iff for all $\langle X_n \mid n < \omega \rangle$, if
$$\mu_n\left(X_n\right) = 1$$
for all $n < \omega$, then there exists a function $f : \omega \longrightarrow \lambda$ such that
$$f \restriction n \in X_n$$
for all $n < \omega$

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[1] I made these notes while lecturing to James Cummings, Katie Thompson, Uri Abraham and Menachem Magidor on Steel's simplification of Woodin's theorem. I ended up handing out only an initial segment of these notes. Good thing because there are some things to fix up at the end! (See subsequent footnotes.)

- A tower is wellfounded iff it is countably complete.

The following is an important connection between homogeneous trees and iteration trees.

Theorem 1 (Martin-Steel [1]) *Assume that δ is a Woodin cardinal. Let S be a δ^+ homogeneous tree on $\omega \times \lambda$. Then there exists a Lipschitz continuous function*

$$x \mapsto \mathcal{A}(x)$$

such that for all $x \in {}^{\omega}\omega$, $\mathcal{A}(x)$ is an alternating chain on V using extenders in V_δ. Moreover:

- *If $x \in \mathrm{proj}([S])$, then $M_{\mathrm{even}}^{\mathcal{A}(x)}$ is illfounded and $M_{\mathrm{odd}}^{\mathcal{A}(x)}$ is wellfounded.*
- *If $x \notin \mathrm{proj}([S])$, then $M_{\mathrm{odd}}^{\mathcal{A}(x)}$ is illfounded and $M_{\mathrm{even}}^{\mathcal{A}(x)}$ is wellfounded.*

Proof from Martin-Steel [1]. Let $(u, v) \in S$. Say $\ell = |u| = |v|$. We will begin by constructing a finite alternating chain $\mathcal{A}(u, v)$ of length 2ℓ. Here is a picture of $\mathcal{A}(u, v)$.

$$\begin{tikzcd}
 & M_1 \arrow[r, "i_{1,3}", "E_2"'] & M_3 \arrow[r, "i_{3,5}", "E_4"'] & M_5 & \cdots & M_{2\ell-1} \\
M_0 \arrow[ur, "i_{0,1}", "E_0"'] \arrow[drr, "E_1", "i_{0,2}"'] & & & & & \\
 & & M_2 \arrow[r, "E_3", "i_{2,4}"'] & M_4 & \cdots & M_{2\ell-2}
\end{tikzcd}$$

Pick $c_2 > c_1 > c_0 > \lambda$ such that each c_i is a strong limit cardinal,

$$\mathrm{cf}(c_i) > \delta$$

and

$$\mathrm{Th}^{V_{c_2}}\left(V_{\lambda+1} \cup \{c_0\}\right) = \mathrm{Th}^{V_{c_2}}\left(V_{\lambda+1} \cup \{c_1\}\right).$$

Define

$$M_0 = V,$$

$$A_0 = \mathrm{Th}^{\left(V_{c_0+1}\,,\ \delta\,,\ S\,,\ v_0\right)}\left(V_\delta\right)$$

and

$$B_0 = \mathrm{Th}^{\left(V_{c_0}\,,\ \delta\,,\ S\,,\ v_0\right)}\left(V_\delta\right).$$

Let κ_0 be $< \delta$ - A_0 - strong.

Pick E_0 such that

$$\mathrm{crit}(E_0) = \kappa_0$$

and

$$i_{E_0}(A_0) \cap V_{\kappa_0+1} = A_0 \cap V_{\kappa_0+1}.$$

We may assume that

$$\mathrm{lh}(E_0) = 2^{\kappa_0}.$$

Define

$$M_1 = \mathrm{ult}(M_0, E_0)$$

and

$$i_{0,1} = i_{E_0}.$$

Because the beginning of the construction is slightly degenerate, we define

$$\kappa_1 = \kappa_0.$$

Then

$$i_{0,1}(A_0) \cap V^{M_1}_{\kappa_1+1} = A_0 \cap V_{\kappa_1+1}$$

so

$$\mathrm{Th}^{\left(V^{M_1}_{c_0+1}\,,\ i_{0,1}(S)\,,\ i_{0,1}(v_0)\right)}\left(V^{M_1}_{\kappa_1+1} \cup \{\delta\}\right) = \mathrm{Th}^{\left(V_{c_0+1}\,,\ S\,,\ v_0\right)}\left(V_{\kappa_1+1} \cup \{\delta\}\right).$$

The following $\mathcal{L}(\dot{S})$ sentence is an element of this theory.

> There exist d and w such that d is the largest ordinal and, if
>
> $$B = \mathrm{Th}^{\left(V_d\,,\ \delta\,,\ \dot{S}\,,\ w\right)}\left(V_\delta\right),$$
>
> then
>
> $$B \cap V_{\kappa_1} = B_0 \cap V_{\kappa_1}$$
>
> and κ_1 is $< \delta$ - B - strong.

To witness this sentence in V_{c_0+1}, we must have $d = c_0$.

Let (c_0, w_0) be the least witness to this sentence in $\big(V^{M_1}_{c_0+1}\,,\ i_{0,1}(S)\big)$.

Then

$$w_0 \in \mathrm{Def}^{V^{M_1}_{c_2}}\left(V_{\kappa_1+1} \cup \{\delta\,,\ i_{0,1}(S)\,,\ c_0\}\right).$$

Define

$$B_1 = \mathrm{Th}^{\left(V^{M_1}_{c_0}\,,\ \delta\,,\ i_{0,1}(S)\,,\ w_0\right)}\left(V^{M_1}_\delta\right).$$

Then

$$B_1 \cap V_{\kappa_1} = B_0 \cap V_{\kappa_1}$$

and κ_1 is $< \delta$ - B_1 - strong in M_1.

Define

$$A_1 = \mathrm{Th}^{\left(V^{M_1}_{c_0+1}\,,\ \delta\,,\ i_{0,1}(S)\,,\ w_0\right)}\left(V^{M_1}_\delta\right).$$

Pick $\kappa_2 > \kappa_1$ such that κ_2 is $< \delta$ - A_1 - strong in M_1.

Pick $E_1 \in M_1$ such that

$$\mathrm{crit}(E_1) = \kappa_1$$

and

$$i^{M_1}_{E_1}(B_1) \cap V_{\kappa_2+1} = B_1 \cap V_{\kappa_2+1}.$$

We may assume that

$$\mathrm{lh}(E_1) = \left(2^{\kappa_2}\right)^{M_1}$$

Define

$$M_2 = \mathrm{ult}(M_0, E_1) = \mathrm{ult}(V, E_1)$$

and

$$i_{0,2} = i_{E_1}.$$

Then

$$i_{0,2}(B_0) \cap V_{\kappa_2+1} = B_1 \cap V_{\kappa_2+1}$$

so

$$\mathrm{Th}^{\left(V_{c_0}^{M_2}\,,\ i_{0,2}(S)\,,\ i_{0,2}(v_0)\right)}\left(V_{\kappa_2+1}^{M_2} \cup \{\delta\}\right) = \mathrm{Th}^{\left(V_{c_0}^{M_1}\,,\ i_{0,1}(S)\,,\ w_0\right)}\left(V_{\kappa_2+1}^{M_1} \cup \{\delta\}\right).$$

The following $\mathcal{L}(\dot{S}, \dot{z}_0)$ sentence is an element of this theory.

> There exist d such that, if
>
> $$A = \mathrm{Th}^{\left(V_{d+1}\,,\ \delta\,,\ \dot{S}\,,\ \dot{z}_0\right)}\left(V_\delta\right),$$
>
> then
>
> $$A \cap V_{\kappa_2} = A_1 \cap V_{\kappa_2}^{M_1}$$
>
> and κ_2 is $< \delta$ - A - strong.

To see this, use the indiscernibility of c_0 and c_1.

Let d_2 be a witness to this sentence in $\left(V_{c_0}^{M_2}\,,\ \delta\,,\ i_{0,2}(S), i_{0,2}(v_0)\right)$.

Define $d_0 = c_0$.

Then

$$i_{0,2}(d_0) = i_{0,2}(c_0) = c_0 > d_2.$$

Define

$$A_2 = \mathrm{Th}^{\left(V_{d_2+1}^{M_2}\,,\ \delta\,,\ i_{0,2}(S)\,,\ i_{0,2}(v_0)\right)}\left(V_\delta^{M_2}\right).$$

Then

$$A_2 \cap V_{\kappa_2}^{M_2} = A_1 \cap V_{\kappa_2}^{M_1}$$

and κ_2 is $< \delta$ - A_2 - strong in M_2.

Define

$$B_2 = \mathrm{Th}^{\left(V^{M_2}_{d_2}\,,\ \delta\,,\ i_{0,2}(S)\,,\ i_{0,2}(v_0)\,,\ i_{0,2}(v_1)\right)}\left(V^{M_2}_{\delta}\right).$$

Pick $\kappa_3 > \kappa_2$ such that κ_3 is $< \delta$ - B_2 - strong in M_2.

Pick $E_2 \in M_2$ such that

$$\mathrm{crit}(E_2) = \kappa_2$$

and

$$i^{M_2}_{E_2}(A_2) \cap V^{M_2}_{\kappa_3+1} = A_2 \cap V^{M_2}_{\kappa_3+1}.$$

We may assume that

$$\mathrm{lh}(E_2) = (2^{\kappa_3})^{M_2}$$

Define

$$M_3 = \mathrm{ult}(M_1, E_2)$$

and

$$i_{1,3} = i^{M_1}_{E_2}.$$

Then

$$\mathrm{Th}^{\left(V^{M_3}_{c_0+1}\,,\ i_{0,3}(S)\,,\ i_{1,3}(w_0)\right)}\left(V^{M_3}_{\kappa_3+1} \cup \{\delta\}\right) = \mathrm{Th}^{\left(V^{M_2}_{d_2+1}\,,\ i_{0,2}(S)\,,\ i_{0,2}(v_0)\right)}\left(V^{M_2}_{\kappa_3+1} \cup \{\delta\}\right)$$

The following $\mathcal{L}(\dot{S}, \dot{z}_0)$ sentence is an element of this theory.

> There exists d and w such that d is the largest ordinal and if
>
> $$B = \mathrm{Th}^{\left(V_d\,,\ \delta\,,\ \dot{S}\,,\ \dot{z}_0\,,\ w\right)}\left(V_\delta\right),$$
>
> then
>
> $$B \cap V_{\kappa_3} = B_2 \cap V_{\kappa_3}$$
>
> and κ_3 is $< \delta$ - B - strong.

To witness this sentence in V_{c_0+1}, we must have $d = c_0$.

Let (c_0, w_1) be the least witness to this sentence in $\left(V^{M_3}_{c_0+1}\,,\ i_{0,3}(S)\,,\ i_{1,3}(w_0)\right)$.

Then

$$w_1 \in \mathrm{Def}^{V_{c_2}^{M_3}} \left(V_{\kappa_3+1} \cup \{\delta\,,\ i_{0,3}(S)\,,\ c_0\}\right).$$

Define

$$B_3 = \mathrm{Th}^{\left(V_{c_0}^{M_3}\,,\ \delta\,,\ i_{0,3}(S)\,,\ i_{1,3}(w_0)\,,\ w_1\right)} \left(V_\delta\right).$$

Then κ_3 is $< \delta$ - B_3 - strong in M_3.

Define

$$A_3 = \mathrm{Th}^{\left(V_{c_0+1}^{M_3}\,,\ \delta\,,\ i_{0,3}(S)\,,\ i_{1,3}(w_0)\,,\ w_1\right)} \left(V_\delta\right).$$

Let $\kappa_4 > \kappa_3$ such that κ_4 is $< \delta$ - A_3 - strong in M_3.

Pick $E_3 \in M_3$ such that

$$\mathrm{crit}(E_3) = \kappa_3$$

and

$$i_{E_3}^{M_3}(B_3) \cap V_{\kappa_4+1}^{M_3} = B_3 \cap V_{\kappa_4+1}^{M_3}.$$

We may assume that

$$\mathrm{lh}(E_3) = \left(2^{\kappa_4}\right)^{M_3}.$$

Define

$$M_4 = \mathrm{ult}(M_2, E_3)$$

and

$$i_{2,4} = i_{E_3}^{M_2}.$$

Then

$$\begin{aligned} &\mathrm{Th}^{\left(V_{i_{2,4}(d_2)}^{M_4}\,,\ i_{0,4}(S)\,,\ i_{0,4}(v_0)\,,\ i_{0,4}(v_1)\right)} \left(V_{\kappa_4+1}^{M_2} \cup \{\delta\}\right) \\ &= \mathrm{Th}^{\left(V_{c_0}^{M_3}\,,\ i_{0,3}(S)\,,\ i_{1,3}(w_0)\,,\ w_1\right)} \left(V_{\kappa_4+1}^{M_3} \cup \{\delta\}\right). \end{aligned}$$

The following $\mathcal{L}(\dot S, \dot z_0, \dot z_1)$ sentence is an element of this theory.

There exist d such that, if

$$A = \mathrm{Th}^{\left(V_{d+1}\,,\ \delta\,,\ \dot S\,,\ \dot z_0\,,\ \dot z_1\right)} \left(V_\delta\right),$$

then

$$A \cap V_{\kappa_4} = A_3 \cap V_{\kappa_4}^{M_3}$$

and κ_4 is $< \delta$ - A - strong.

To see this, use the indiscernibility of c_0 and c_1.

Let d_4 witness this sentence in $\Big(V^{M_4}_{i_{2,4}(d_2)}\,,\ i_{0,4}(S)\,,\ i_{0,4}(v_0)\,,\ i_{0,4}(v_1)\Big)$.

Then

$$i_{2,4}(d_2) > d_4.$$

Define

$$A_4 = \mathrm{Th}^{\left(V^{M_4}_{d_4+1}\,,\ \delta,\ i_{0,4}(S)\,,\ i_{0,4}(v_0)\,,\ i_{0,4}(v_1)\right)}(V_\delta)\,.$$

Then

$$A_4 \cap V_{\kappa_4}^{M_4} = A_3 \cap V_{\kappa_4}^{M_3}$$

and κ_4 is $< \delta$ - A_4 - strong in M_4.

Define

$$B_4 = \mathrm{Th}^{\left(V^{M_4}_{d_4}\,,\ \delta,\ i_{0,4}(S)\,,\ i_{0,4}(v_0)\,,\ i_{0,4}(v_1)\,,\ i_{0,4}(v_2)\right)}(V_\delta)\,.$$

Pick $\kappa_5 > \kappa_4$ so that κ_5 is $< \delta$ - B_4 - strong in M_4.

Pick $E_4 \in M_4$ such that

$$\mathrm{crit}(E_4) = \kappa_4$$

and

$$i^{M_4}_{E_4}(A_4) \cap V^{M_4}_{\kappa_5+1} = A_4 \cap V^{M_4}_{\kappa_5+1}.$$

Keep going this way to define $\mathcal{A}(u, v)$.

Observe that on the even side we have

$$(u, \langle i_{0,2\ell-2}(v_0)\,,\ i_{0,2\ell}(v_1)\,,\ \ldots\,,\ i_{0,2\ell-2}(v_{\ell-1})\rangle) \in i_{0,2\ell-2}(S).$$

Also,

$$i_{0,2\ell-1}(d_0) > i_{2,2\ell-1}(d_2) > \cdots > d_{2\ell-2}.$$

On the odd side we have

$$(u, \langle i_{1,2\ell-1}(w_0)\,,\ i_{3,2\ell-1}(w_1)\,,\ \dots\,,\ w_{\ell-1}\rangle) \in i_{0,2\ell-1}(S).$$

And, for all $k < \ell$,

$$w_k \in \mathrm{Def}^{V_{c_2}}\left(V_\delta \cup \{\delta\,,\ S\,,\ c_0\}\right).$$

We need to keep track of objects that come up in the construction of $\mathcal{A}(u,v)$.

$$M_n(u,v) = M_n^{\mathcal{A}(u,v)}$$

$$E_n(u,v) = E_n^{\mathcal{A}(u,v)}$$

$$d_{2k}(u,v)$$

$$w_{2k+1}(u,v)$$

Let $\langle \mu_u \mid u \in {}^{<\omega}\omega\rangle$ be a δ^+ homogeneity system for S.

Consider an arbitrary $u \in {}^{<\omega}\omega$.

Let

$$S_u = \{v \in {}^{<\omega}\lambda \mid (u,v) \in S\}$$

Then

$$\mu_u(S_u) = 1.$$

Consider the function

$$v \mapsto \langle E_n(u,v) \mid n < 2\ell\rangle$$

from S_u to V_δ.

By δ^+ completeness, there exists $R_u \subseteq S_u$ such that

$$\mu_u(R_u) = 1$$

and there exists $\langle E_n(u) \mid n < 2\ell \rangle$ such that

$$\langle E_n(u, v) \mid n < 2\ell \rangle = \langle E_n(u) \mid n < 2\ell \rangle$$

for all $u \in {}^{<\omega}\omega$ and $v \in R_u$.

The sets in V_δ from which $w_k(u, v)$ can be defined can be frozen in the same way, and therefore $w_k(u, v)$ can be frozen.

So we may assume that

$$\langle w_k(u, v) \mid k < \ell \rangle = \langle w_k(u) \mid k < \ell \rangle$$

for all $v \in R_u$.

It is clear from our construction that if $(u, v) = (u' \restriction \ell, v' \restriction \ell) \in S_u$, then

$$\langle E_n(u, v) \mid n < 2\ell \rangle = \langle E_n(u', v') \mid n < 2\ell \rangle$$

$$\langle w_k(u, v) \mid k < \ell \rangle = \langle w_k(u', v') \mid k < \ell \rangle .$$

and

$$\langle d_{2k}(u, v) \mid k < \ell \rangle = \langle d_{2k}(u', v') \mid k < \ell \rangle .$$

If $(x, f) \in [S]$, we let $\mathcal{A}(x, f)$ be the iteration tree on V with extenders

$$E_n(x, f) = E_n(u, v)$$

where $(u, v) = (x \restriction \ell, f \restriction \ell)$ for any $n < 2\ell$.

We also let

$$d_{2k}(x, f) = d_{2k}(u, v)$$

where $(u, v) = (x \restriction \ell, f \restriction \ell)$ for any $k < \ell$.

If $x \in {}^{\omega}\omega$, we let $\mathcal{A}(x)$ be the iteration tree on V with extenders

$$E_n(x) = E_n(u)$$

where $u = x \restriction \ell$ for any $n < 2\ell$.

We also let

$$w_k(x) = w_k(u)$$

where $u = x \restriction \ell$ for any $k < \ell$.

Now we show that $\mathcal{A}(x)$ has the desired properties.

First suppose that $x \in \mathrm{proj}([S])$.

Then $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is a countably complete tower.

Pick $f : \omega \longrightarrow \lambda$ such that $(x \restriction n, f \restriction n) \in R_{x \restriction n}$ for all $n < \omega$.

Then

$$\left\langle i^{\mathcal{A}(x,f)}_{2k,\mathrm{even}}(d_{2k}(x,f)) \mid k < \omega \right\rangle = \left\langle i^{\mathcal{A}(x)}_{2k,\mathrm{even}}(d_{2k}(x,f)) \mid k < \omega \right\rangle$$

is an infinite descending sequence in $M^{\mathcal{A}(x)}_{\mathrm{even}} = M^{\mathcal{A}(x,f)}_{\mathrm{even}}$

Thus $M^{\mathcal{A}(x)}_{\mathrm{even}}$ is illfounded.

By the maximal wellfounded branch theorem, $M^{\mathcal{A}(x)}_{\mathrm{odd}}$ is wellfounded

Finally suppose that $x \notin \mathrm{proj}([S])$.

For contradiction, suppose that $M^{\mathcal{A}(x)}_{\mathrm{odd}}$ is wellfounded.

Then $i^{\mathcal{A}(x)}_{0,\mathrm{odd}}(S_x)$ is wellfounded.

But

$$\left\langle i^{\mathcal{A}(x)}_{2k+1,\mathrm{odd}}(w_k(x)) \mid k < \omega \right\rangle$$

is a branch through $i^{\mathcal{A}(x)}_{0,\mathrm{odd}}(S_x)$. Contradiction.

Therefore $M^{\mathcal{A}(x)}_{\mathrm{odd}}$ is illfounded.

By the maximal wellfounded branch theorem, $M^{\mathcal{A}(x)}_{\mathrm{even}}$ is wellfounded.

That completes the proof of Theorem 1.

Theorem 2 (Woodin) *Assume τ is a family of κ^+ complete measures over ${}^{<\omega}\lambda$ such that $|\tau| = \kappa$. Let*

$$T = \{\mu \in {}^{\omega}\tau \mid \mu \textit{ is a tower}\}$$

and

$$W = \{\mu \in T \mid \mu \textit{ is wellfounded}\}.$$

Then there is a κ^+-homogeneous tree S on $\tau \times \lambda$ such that

$$W = \mathrm{proj}([S]).$$

Proof taken from Steel [3]. For each $\mu \in T - W$, pick $\langle X^\mu_\ell \mid \ell < \omega\rangle$ so that

$$\mu_\ell(X^\mu_\ell) = 1$$

and

$$X^\mu_\ell \subseteq \left\{v \in {}^{\ell}\lambda \mid v \restriction k \in X^\mu_k\right\}$$

for all $k < \ell < \omega$ but there is $\underline{\text{no}}$ $f : \omega \longrightarrow \lambda$ such that

$$f \restriction \ell \in X^\mu_\ell$$

for all $\ell < \omega$.

Define

$$F = \{\varphi \in {}^{<\omega}\tau \mid \varphi \text{ is a tower}\}.$$

We may assume that for all $\varphi \in F$, there are both wellfounded and illfounded towers from τ extending φ.

Let

$$S_{\langle\rangle} = X^\mu_0 = \{\langle\rangle\}.$$

If $\varphi = \langle\varphi_k \mid k \leq \ell\rangle \in F$, then let

$$S_\varphi = \bigcap_{\substack{\mu \in T - W \\ \mu \restriction \ell + 1 = \varphi}} X^\mu_\ell$$

and, if $v \in {}^{\ell+1}\lambda$, then write

$$(\varphi, v) \in S \iff v \in S_\varphi$$

Notice that if $(\varphi, v) \in S$, then

$$\varphi_k \left(S_{\varphi \restriction k+1}\right) = 1$$

for all $k \leq \ell$.

This is because each measure in τ is $(\kappa^{\aleph_0})^+$ complete since its completeness is a measurable cardinal strictly greater than κ.

Define

$$\nu_{\langle\rangle}\left(\{\langle\rangle\}\right) = 1$$
$$\nu_{\langle\rangle}\left(\emptyset\right) = 0$$

and

$$\nu_{\langle \varphi_k \mid k \leq \ell\rangle} = \varphi_\ell.$$

So if $\varphi = \langle \varphi_k \mid k \leq \ell\rangle \in F$, then

$$\nu_\varphi\left(S_\varphi\right) = 1.$$

Also, if $\mu \in T$, then

$$\mu = \left\langle \nu_{\langle \mu_0, \ldots, \mu_\ell\rangle} \mid \ell < \omega \right\rangle.$$

It is now straightforward to verify that for all $\mu \in T$,

$$\mu \notin \operatorname{proj}\left([S]\right) \iff \mu \notin W$$

and that

$$\langle \nu_\varphi \mid \varphi \in F\rangle$$

is a κ homogeneous system for S.

That completes the proof of Theorem 2.

Corollary 3 *Suppose that* $j : V \longrightarrow N$ *witnesses that* κ *is* 2^{2^κ} *supercompact. Let* σ *be the set of measures over* ${}^{<\omega}\kappa$ *and* $\tau = j[\sigma]$*. Let*

$$T = \{\mu \in {}^{\omega}\tau \mid \mu \textit{ is a tower}\}$$

and

$$W = \{\mu \in T \mid \mu \textit{ is wellfounded}\}\,.$$

Let $\delta < j(\kappa)$ *be a Woodin cardinal in* N*. Then there is a Lipschitz continuous function*

$$\mu \mapsto \mathcal{A}(\mu)$$

such that for all $\mu \in T$*,* $\mathcal{A}(x)$ *is an alternating chain on* N *using extenders in* V_δ*. Moreover:*

- *If* $\mu \in W$*, then* $M^{\mathcal{A}(x)}_{\mathrm{even}}$ *is illfounded and* $M^{\mathcal{A}(x)}_{\mathrm{odd}}$ *is wellfounded.*
- *If* $x \notin W$*, then* $M^{\mathcal{A}(x)}_{\mathrm{odd}}$ *is illfounded and* $M^{\mathcal{A}(x)}_{\mathrm{even}}$ *is wellfounded.*

Sketch of proof. First apply the Theorem 2 in N with $\lambda = j(\kappa)$.

The rest would be a direct application of Theorem 1 except that the first coordinates of S are finite towers from τ instead of finite sequences of natural numbers.

Since τ has cardinality $2^{2^\kappa} < \delta$, we may construe S as a tree on $2^{2^\kappa} \times \lambda$.

In the proof of Theorem 1, we may add the requirement that all the critical points of extenders used in the iteration trees constructed there have critical point strictly greater than 2^{2^κ}. In fact, Martin and Steel's original formulation of Theorem 1 included this flexibility.

That is all we will say about the proof of Corollary 3.

Theorem 4 (Woodin) *Suppose that* κ *is a* 2^{2^κ} *supercompact cardinal. Then there is an iteration tree* $\mathcal{T}$ *of length* ω *on* V *with structure:*

$$\begin{tikzcd}
 & & M_2^{\mathcal{T}} \arrow[rr] & & M_4^{\mathcal{T}} \arrow[rr] & & M_6^{\mathcal{T}} & & \cdots \\
M_0^{\mathcal{T}} \arrow[r] & M_1^{\mathcal{T}} \arrow[ru] \arrow[rrd] & & & & & & & \\
 & & & M_3^{\mathcal{T}} \arrow[rr] & & M_5^{\mathcal{T}} & & \cdots &
\end{tikzcd}$$

such that both branches $\{0, 1, 2, 4, 6, \ldots\}$ *and* $\{0, 1, 3, 5, 7, \ldots\}$ *are wellfounded.*

Proof as explained by Steel. We are looking for an extender E and an alternating chain $\mathcal{D}$ on

$$M_0^{\mathcal{D}} = \mathrm{ult}\,(V, E)$$

such that both $M_{\text{odd}}^{\mathcal{D}}$ and $M_{\text{even}}^{\mathcal{D}}$ are wellfounded. In pictures:

$$\begin{tikzcd}
 & & M_1^{\mathcal{D}} \arrow[rr] & & M_3^{\mathcal{D}} \arrow[rr] & & M_5^{\mathcal{D}} & & \cdots & & M_{\text{odd}}^{\mathcal{D}} \\
V \arrow[r] & M_0^{\mathcal{D}} \arrow[ru] \arrow[rrd] & & & & & & & & & \\
 & & & M_2^{\mathcal{D}} \arrow[rr] & & M_4^{\mathcal{D}} & & \cdots & & M_{\text{even}}^{\mathcal{D}} &
\end{tikzcd}$$

Let $j : V \longrightarrow N$ witness that κ is 2^{2^κ} supercompact.

Let σ be the set of measures on ${}^{<\omega}\kappa$ and $\tau = j[\sigma]$.

Let $F = \{\varphi \in {}^{<\omega}\tau \mid \varphi \text{ is a tower}\}$ and $T = \{\mu \in {}^{\omega}\tau \mid \varphi \text{ is a tower}\}$.

Let $\mu \mapsto \mathcal{A}(\mu)$ be the function from Corollary 3.

For simplicity, we work in second order set theory.

For each $\varphi \in F$, pick $X(\varphi) \prec V$ with

$$\kappa \cup \mathcal{P}(\kappa) \subseteq X(\varphi)$$

and $|X(\varphi)| = 2^\kappa$.

Pick these so that, if $\varphi = \langle \varphi_k \mid k < \ell \rangle$, then $\langle X(\varphi \restriction k) \mid k < \ell \rangle$ is an internally approachable chain.

If $\mu \in T$, then let

$$X(\mu) = \bigcup_{\ell < \omega} X(\mu \restriction \ell)$$

and

$$\pi(\mu) : M(\mu) \simeq X(\mu)$$

with $M(\mu)$ transitive.

Say that under $\pi(\mu)$, j and $\mathcal{A}(\mu)$ collapse to $k(\mu)$ and $\mathcal{B}(\mu)$ respectively to give the following commutative diagram.

Let $E(\mu)$ be the superstrong extender derived from $k(\mu)$ and

$$P(\mu) = \mathrm{ult}(M(\mu), E(\mu)).$$

The factor map from $P(\mu)$ to $N(\mu)$ has critical point $k(\mu)(\kappa)$, which is greater than the length of any extender used on $\mathcal{B}(\mu)$.

Let $\mathcal{C}(\mu)$ be the alternating chain on $P(\mu)$ that uses the same extenders as $\mathcal{B}(\mu)$.

$$\begin{tikzcd}
 & & M^{\mathcal{A}(\mu)}_{\text{odd}} \\
V \arrow[r, "j"] & N \arrow[ur] \arrow[dr] & \\
 & & M^{\mathcal{A}(\mu)}_{\text{even}} \\
 & & M^{\mathcal{B}(\mu)}_{\text{odd}} \arrow[uuu, bend right] \\
M(\mu) \arrow[uuu, "\pi(\mu)"] \arrow[r, "k(\mu)"] \arrow[dddr, "i^{M(\mu)}_{E(\mu)}"'] & N(\mu) \arrow[uuu] \arrow[ur] \arrow[dr] & \\
 & & M^{\mathcal{B}(\mu)}_{\text{even}} \arrow[uuu, bend right] \\
 & & M^{\mathcal{C}(\mu)}_{\text{odd}} \arrow[uuu, bend right] \\
 & P(\mu) \arrow[uuu] \arrow[ur] \arrow[dr] & \\
 & & M^{\mathcal{C}(\mu)}_{\text{even}} \arrow[uuu, bend right]
\end{tikzcd}$$

Let $G(\mu)$ be the superstrong extender derived from $i^{M(\mu)}_{E(\mu)} \circ i^{\mathcal{C}(\mu)}_{\text{even}}$. [2]

Both $E(\mu)$ and $G(\mu)$ are extenders over V because $\mathcal{P}(\kappa) \subseteq X(\mu)$.

Here is another description of $G(\mu)$.

Let $Q(\mu) = \text{ult}(V, E(\mu))$.

Let $\mathcal{D}(\mu)$ be $\mathcal{C}(\mu)$ construed as an alternating chain on $Q(\mu)$.

---

[2] It must be explained what is meant by this if $M^{\mathcal{C}}_{\text{even}}(\mu)$ is illfounded. In a sense, the meaning is implicit in Lemma 4.1

Then $G(\mu)$ is the superstrong extender derived from $i^V_{E(\mu)} \circ i^{\mathcal{D}(\mu)}_{\text{even}}$.

$$\begin{tikzcd}
 & & M^{\mathcal{A}(\mu)}_{\text{odd}} & \\
 & N \arrow[ur] \arrow[dr] & & \\
 & & M^{\mathcal{A}(\mu)}_{\text{even}} & \\
V \arrow[uur, "j"] \arrow[ddr, "i^V_{E(\mu)}"'] & & & \\
 & & M^{\mathcal{D}(\mu)}_{\text{odd}} \arrow[uuuu, bend right] & \\
 & Q(\mu) \arrow[uuuu] \arrow[ur] \arrow[dr] & & \\
 & & M^{\mathcal{D}(\mu)}_{\text{even}} \arrow[uuuu, bend right] & = \mathrm{ult}(V, G(\mu))
\end{tikzcd}$$

Lemma 4.1 *There is a function*

$$\mu \mapsto \overline{\mu}$$

such that for all $\mu \in T$, $\overline{\mu}$ *is a tower of measures on* ${}^{<\omega}\kappa$ *and*

$$\mathrm{ult}(V, G(\mu)) = \mathrm{ult}(V, \overline{\mu}).$$

Moreover, $\overline{\mu}_\ell$ *is determined from* $\mu \restriction \ell$ *for all* $\ell < \omega$.

Assume this Lemma 4.1 for now. [3]

Lemma 4.2 *There exists* $\mu \in T$ *such that*

$$j(\overline{\mu}) = \mu.$$

---

[3] I never got to typing up the proof for the audience.

Proof. Consider the following run of a certain Lipschitz game.

Player I plays $\mu \restriction 0 = \langle\rangle$.

Player II uses Lemma 4.1 to compute $\overline{\mu}_0$.

Player I plays $\mu \restriction 1 = \langle j\left(\overline{\mu}_0\right)\rangle$.

Player II uses Lemma 4.1 to compute $\overline{\mu}_1$.

Player I plays $\mu \restriction 1 = \langle j\left(\overline{\mu}_0\right), j\left(\overline{\mu}_1\right)\rangle$.

Continuing in this way gives the desired fixed point of $\mu \mapsto j(\overline{\mu})$.

For the rest of the proof of Theorem 4, fix μ as in Lemma 4.2.

Lemma 4.3 *μ is a wellfounded tower on N.*

Proof. Suppose otherwise.

Then $M^{\mathcal{A}(\mu)}_{\text{even}}$ is wellfounded.

$M^{\mathcal{D}(\mu)}_{\text{even}}$ embeds into $M^{\mathcal{A}(\mu)}_{\text{even}}$, hence $M^{\mathcal{D}(\mu)}_{\text{even}}$ is also wellfounded.

So $\overline{\mu}$ is a wellfounded tower on V.

Thus $\mu = j(\overline{\mu})$ is a wellfounded tower on N, which is a contradiction.

Lemma 4.4 *Both $M^{\mathcal{D}(\mu)}_{\text{odd}}$ and $M^{\mathcal{D}(\mu)}_{\text{even}}$ are wellfounded.*

From Lemma 4.3 it follows that $M^{\mathcal{A}(\mu)}_{\text{even}}$ is illfounded and $M^{\mathcal{A}(\mu)}_{\text{odd}}$ is wellfounded.

$M^{\mathcal{D}(\mu)}_{\text{odd}}$ embeds into $M^{\mathcal{A}(\mu)}_{\text{odd}}$, so $M^{\mathcal{D}(\mu)}_{\text{odd}}$ is wellfounded.

μ is wellfounded and $\mu = j(\overline{\mu})$, so $\overline{\mu}$ is wellfounded.

Hence
$$M^{\mathcal{D}(\mu)}_{\text{even}} = \text{ult}(V, G(\mu)) = \text{ult}(V, \overline{\mu})$$
is wellfounded.

Modulo Lemma 4.1, that completes the proof of Theorem 4.

References

[1] Martin, D. A. and Steel, J. R., *A proof of projective determinacy*, Journal of the American Mathematical Society, Volume 2, Number 1, January 1989.

[2] Martin, D. A. and Steel, J. R., *Iteration trees*, Journal of the American Mathematical Society, Volume 7, Number 1, January 1994.

[3] Steel, J.R., *The derived model theorem*, preprint.