Descriptive set theory is the study of separable complete metric space, which are called Polish spaces. For example, \( \mathbb{R} \) is a Polish space! The definable subsets of Polish spaces are classified according to the complexity of their definitions. Properties are proved by induction along the complexity hierarchy. The Borel subsets of \( \mathbb{R}^n \) form an initial segment of the hierarchy. However, the notation commonly used by analysts, namely \( \mathcal{G}, \mathcal{F}, \mathcal{F}_\sigma, \mathcal{G}_\delta, \mathcal{G}_\delta^\sigma, \mathcal{F}_\sigma^\delta \), etc., is inadequate because the Borel hierarchy has uncountably many levels. Instead, we will use the notation \( \Sigma^0_\alpha \) and \( \Pi^0_\alpha \) for the pair of classes at the \( \alpha \)'th level of the Borel hierarchy, where the index \( \alpha \) varies over countable ordinals. Beyond the Borel hierarchy is the projective hierarchy, which is defined as follows. If there is a Borel set \( B \subseteq \mathbb{R}^{n+1} \) such that

\[
A = \{ x \in \mathbb{R}^n \mid \text{there exists } y \in \mathbb{R} \text{ such that } (x, y) \in B \},
\]

then \( A \) is called analytic or \( \Sigma^1_1 \). In general, if \( k, n \in \mathbb{N} \) and \( A \subseteq \mathbb{R}^n \), then

- \( A \) is \( \Pi^1_k \iff \mathbb{R}^n - A \) is \( \Sigma^1_k \)
- \( A \) is \( \Sigma^1_{k+1} \iff \) there is a \( \Pi^1_k \) set \( B \subseteq \mathbb{R}^{n+1} \) such that

\[
A = \{ x \in \mathbb{R}^n \mid \text{there exists } y \in \mathbb{R} \text{ such that } (x, y) \in B \}.
\]
Recommended texts:

- Alexander S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag Graduate Texts in Mathematics 156

- Yiannis N. Moschovakis, *Descriptive Set Theory*, North-Holland Studies in Logic 100