Wrinkling of a Twisted Ribbon

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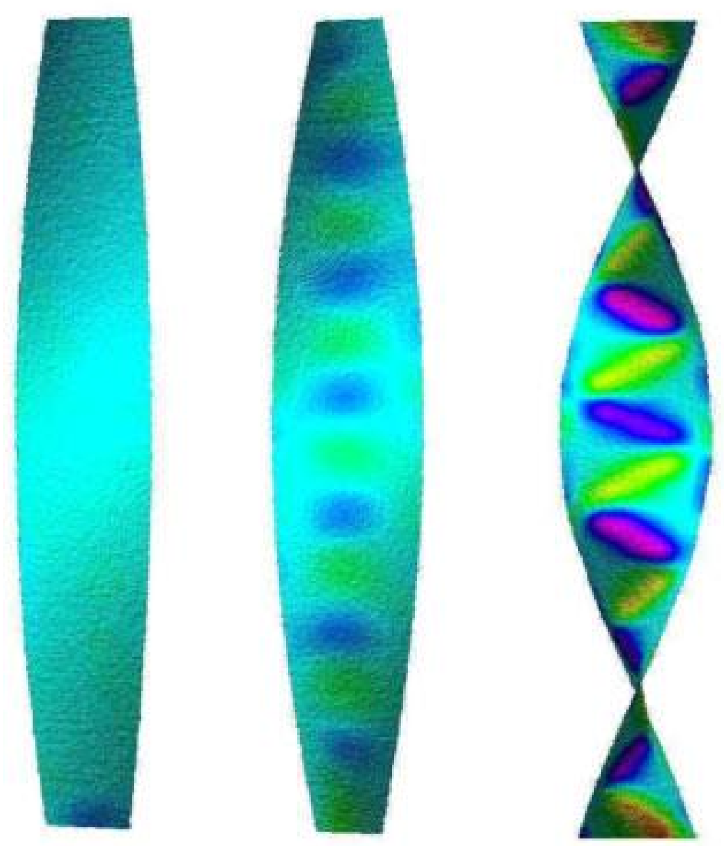
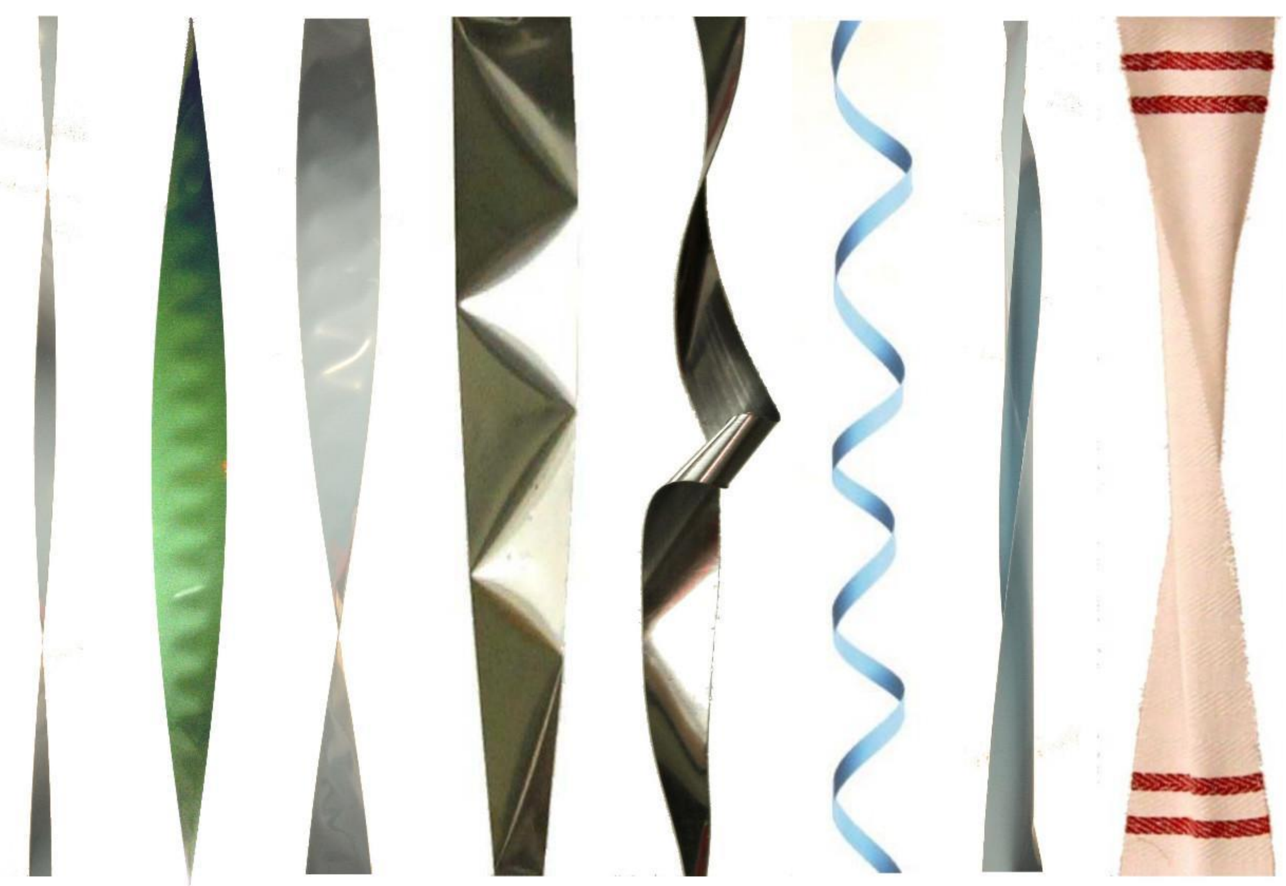
An elastic ribbon, when twisted and allowed to compress slightly, will form wrinkles along the center. The physical intuition is simple: in a compressed helicoid the outer edges are stretched and the inner strip is compressed. A thin ribbon will wrinkle to relax the compression.

The relaxed problem, which amounts to finding where the wrinkles form but not the structure of the wrinkles, was solved in[3]. We find upper and lower bounds for the energy in the limit where the non-dimensionalized thickness h is small.





Experiment with mylar ribbons[4]. The left figure[4] uses colors to show deviation from a helicoid. The right figure[3] shows that a range of morphologies are possible, but only the three leftmost ribbons are relevant to this work.

Mathematical Model

Variables:

- (r, s): reference coordinates.
- h: non-dimensionalized thickness.
- ω: twist rate of the elastic strip.
- (u_1, u_2, v): displacement from a helicoid.
- ξ: compression.

Energy Functional: We start with a physically reasonable energy

$$E^{(h)}_{\text{phys}} = \int \frac{1}{2} \left| I - \begin{pmatrix} 1 & 0 \\ 0 & 1-\xi \end{pmatrix} \right|^2 + h^2 \left| II \right|^2$$

and consider small deflections from a helicoid to formally derive a von Kármán like energy $E^{(h)}$.

Main Result

The energy functional

$$E^{(h)}(u,v) = \int \frac{1}{2} \left| e(u) + 1/2 \begin{pmatrix} v_{,r} \\ v_{,s} + \omega r \end{pmatrix} \otimes \begin{pmatrix} v_{,r} \\ v_{,s} + \omega r \end{pmatrix} - \begin{pmatrix} 0 & \omega v/2 \\ \omega v/2 & \xi \end{pmatrix} \right|^2 + h^2 \left| \nabla\nabla v + \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} \right|^2$$

satisfies $E_0 + Ch^{4/3} \leq \min E^{(h)} \leq E_0 + C'h^{4/3}$ for all $u, v = 0$ on the top and bottom.

References

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[3] J. Chopin, V. Démery, and B. Davidovitch. Roadmap to the morphological instabilities of a stretched twisted ribbon. *J. Elasticity*, 119(1-2):137–189, 2015.

[4] J. Chopin and A. Kudrolli. Helicoids, wrinkles, and loops in twisted ribbons. *Phys. Rev. Lett.*, 111:174302, Oct 2013.

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