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Incompressible Fluids with Vorticity in Besov Spaces

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Incompressible Fluids with Vorticity in Besov Spaces

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Incompressible Fluids with Vorticity in Besov Spaces

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In this thesis, we consider solutions to the two-dimensional Euler equations with uniformly continuous initial vorticity in a critical or subcritical Besov space. We use paradifferential calculus to show that the solution will lose an arbitrarily small amount of smoothness over any fixed finite time interval. This result is motivated by a theorem of Bahouri and Chemin which states that the Sobolev exponent of a solution to the two-dimensional Euler equations in a critical or subcritical Sobolev space may decay exponentially with time. To prove our result, one can use methods similar to those used by Bahouri and Chemin for initial vorticity in a Besov space with Besov exponent between 0 and 1; however, we use different methods to prove a result which applies for any Sobolev exponent between 0 and 2.

The remainder of this thesis is based on joint work with J. Kelliher. We study the vanishing viscosity limit of solutions of the Navier-Stokes equations to solutions of the Euler equations in the plane assuming initial vorticity is in a variant Besov space introduced by Vishik. Our methods allow us to extend a global in time uniqueness result established by Vishik for the two-dimensional Euler equations in this space.

Table of Contents

Acknowledgments **iv**

Abstract **vi**

Chapter 1. Introduction **1**

1.1 Chapter summaries 2

1.2 Notational conventions 5

Chapter 2. Background **6**

2.1 Littlewood-Paley theory 6

2.2 Function spaces 8

2.3 Properties of ideal incompressible fluids 11

2.3.1 The Biot-Savart law 12

2.3.2 The vorticity of an ideal incompressible fluid 13

2.3.3 A few technical lemmas 19

Chapter 3. An initial value problem for the Euler equations in the plane **28**

3.1 Introduction and history 28

3.2 A proof of Theorem 3.1.4 33

3.3 Paradifferential estimates for the vorticity equation 38

3.4 Statement and proof of the main results 48

3.4.1 Besov regularity of the vorticity 48

3.4.2 Besov regularity of the flow 52

Chapter 4. Vanishing viscosity in the plane **55**

4.1 Introduction and history 55

4.2 Statement of the main results 56

4.2.1 A comparison of rates of convergence 61

4.3 Proof of the main theorem 61
4.3.1 Convergence in Equation (4.3.5) 64

Bibliography **68**

Vita **72**

Chapter 1

Introduction

We begin by introducing the Navier-Stokes equations:

$$(NS)\quad \begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v = -\nabla p, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v^0. \end{cases}$$

Here $v : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ denotes the velocity vector field, and $p : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ denotes the pressure. The Navier-Stokes equations model incompressible fluid flow in $\mathbb{R}^n$, $n \geq 2$, with constant density and constant viscosity, denoted by ν. When $\nu = 0$, the Navier-Stokes equations reduce to the Euler equations:

$$(E)\quad \begin{cases} \partial_t v + v\cdot\nabla v = -\nabla p, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v^0. \end{cases}$$

In this thesis, we study properties of solutions to (NS) and (E) in the plane. Specifically, we investigate regularity of solutions to (E) in the plane and the limit of solutions of (NS) to solutions of (E) in the energy norm as ν goes to zero.

It is known that when initial data v^0 for (E) is in the supercritical Sobolev space $W^{s+1,p}(\mathbb{R}^2)$, where $sp > 2$, the solution does not lose any regularity as time evolves (see, for example, [11], [12]). Much less is known when initial data belongs to the critical Sobolev spaces, where $sp = 2$, or to the subcritical Sobolev spaces, where $sp < 2$. We therefore restrict our attention to these two cases. In addition, we assume that the initial vorticity is uniformly continuous.

We also study the vanishing viscosity limit of solutions of the Navier-Stokes equations to solutions of the Euler equations in the plane assuming initial vorticity is in a variant Besov space introduced by Vishik in [21]. Our methods allow us to extend a uniqueness result proved by Vishik.

In the chapters that follow, we begin with an introduction to known results in each of the above areas. We then establish our results. Before we begin, we give a brief summary of each chapter.

1.1 Chapter summaries

Chapter 2: In this chapter we discuss tools which we utilize throughout Chapters 3 and 4. In particular, we discuss Littlewood-Paley theory and Bony's paraproduct decomposition. We also define useful function spaces. Finally, we state some properties of incompressible fluids in the plane.

Chapter 3: In this chapter we study regularity of solutions to (E) with initial data in a critical or subcritical Sobolev space.

Let $v = (v_1, v_2)$ be a solution to the two-dimensional Euler equations with vorticity $\omega(v) = \partial_1 v_2 - \partial_2 v_1$. Assume v satisfies $\omega(v^0) = \omega^0 \in W^{s,p}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, where $sp \leq 2$ and $s \in (0, 1]$. Under these assumptions, Bahouri and Chemin find in [1] a lower bound for the Sobolev exponent of $\omega(t)$ which decays exponentially with time. They show that for all $\tilde{s} < s$, $\omega(t)$ belongs to $W^{\tilde{s}\alpha(t),p}(\mathbb{R}^2)$, where $\alpha(t) = \exp(-\int_0^t ||v(\tau)||_{LL} d\tau)$ is the Holder exponent of the flow $g(t)$, and LL denotes the space of log-Lipschitz functions. The authors find this lower bound by studying what happens to the initial vorticity under composition with $g(t)$.

Now let v be the unique solution to the two-dimensional Euler equations such that $\omega(v^0) = \omega^0 \in UC(\mathbb{R}^2) \cap W^{s,p}(\mathbb{R}^2)$, for $sp \leq 2$ with $p \in (1, \infty)$ and $s \in (0, 2)$, where $UC(\mathbb{R}^2)$ denotes the space of uniformly continuous functions on $\mathbb{R}^2$. We show that under these assumptions, the Holder exponent of $g(t)$ can be made arbitrarily close to 1 over any fixed finite time interval. After a slight modification of the proof in [1], we are able to show that when $\omega^0 \in UC(\mathbb{R}^2) \cap W^{s,p}(\mathbb{R}^2)$ and $s \in (0, 1]$, $\omega(t)$ loses an arbitrarily small amount of smoothness in finite time.

Since the methods used by Bahouri and Chemin only apply to the case $s \in (0, 1]$, the modification of their proof fails for the case $s \in (1, 2)$. To handle this case, we construct a different proof which shows an arbitrarily small loss in smoothness in the appropriate critical and subcritical Besov spaces for *all* $s \in (0, 2)$. By embedding properties, one can conclude from this result an arbitrarily small loss of smoothness in Sobolev spaces when vorticity is uniformly continuous.

To prove the result, we consider the localized vorticity equation given by

$$\begin{aligned} &\partial_t \Delta_q \omega + v \cdot \nabla \Delta_q \omega = [v \cdot \nabla, \Delta_q]\omega, \\ &\Delta_q \omega|_{t=0} = \Delta_q \omega^0, \end{aligned} \tag{1.1.1}$$

where Δ_q is the Littlewood-Paley operator which projects in Fourier space onto an annulus with inner and outer radii of order 2^q. We then use Bony's paraproduct decomposition given in [3] to bound the commutator on the right hand side of (1.1.1). This estimate, combined with a Gronwall type of argument, implies the result. As $h(t,x) = g(t)^{-1}(x) - x$ satisfies a similar equation, given by

$$\begin{aligned}
&\partial_t \Delta_q h + v \cdot \nabla \Delta_q h = -\Delta_q v + [v \cdot \nabla, \Delta_q] h, \\
&\Delta_q h|_{t=0} = 0,
\end{aligned} \tag{1.1.2}$$

we apply similar techniques, combined with the regularity result for the velocity v, to prove an analogous result for the flow. Specifically, we show that h belongs to $L^\infty_{loc}(\mathbb{R}^+; W^{s+1-\delta}_p(\mathbb{R}^2))$ for any fixed $\delta > 0$.

Chapter 4: The content of this chapter stems from joint work with J. Kelliher. We consider a uniqueness class established by Vishik in [21]. This class is a variant of a critical Besov space. Let $\Gamma : \mathbb{R} \to [1, \infty)$ be a locally Lipschitz continuous nondecreasing function with $\Gamma = 1$ on the interval $(-\infty, -1]$ and $\lim_{N\to\infty} \Gamma(N) = \infty$. ($\Gamma$ also satisfies two minor technical conditions which can be found on p. 771 of [21].) Define $B_\Gamma(\mathbb{R}^2) = \{f \in S'(\mathbb{R}^2) : \sum_{j=-1}^{N} ||\Delta_j f||_{L^\infty} \leq C(\Gamma(N))\}$. In [21], Vishik shows that a solution v to the two-dimensional Euler equations with $\omega(v) \in L^\infty([0,T]; L^{p_0}(\mathbb{R}^2) \cap B_\Gamma(\mathbb{R}^2))$ for $p_0 < 2$ is unique if $\Gamma(N)$ grows no faster than $N \log N$. His result implies, among many things, that if one can construct a solution to the Euler equations in the space $bmo(\mathbb{R}^2)$, then the solution must be unique. He proves global in time existence of solutions in his uniqueness class when initial vorticity belongs to $B_\Gamma(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2) \cap L^{p_1}(\mathbb{R}^2)$, where $\Gamma(N) = \log^\kappa N$ with $0 < \kappa \leq \frac{1}{2}$, and $p_0 < 2 < p_1$; he proves short time existence of solutions in his uniqueness class when $\frac{1}{2} < \kappa \leq 1$.

We prove that there exists a unique global in time solution v to the Euler equations in $L^\infty([0,\infty); H^1(\mathbb{R}^2))$ when initial vorticity belongs to $B_\Gamma(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, and $\Gamma(N) = \log^\kappa N$, $0 < \kappa < 1$. We prove this result *without* showing that the vorticity remains in Vishik's uniqueness class globally in time. We then prove that with this initial data the solutions of the Navier-Stokes equations converge to the unique solution of the Euler equations in the energy norm as viscosity tends to 0.

1.2 Notational conventions

We use C to denote a constant which may differ in value on two sides of an inequality. The constant C may or may not be absolute; for example, C will sometimes depend on the initial data. We only clarify this dependence when it is important to the discussion. To demonstrate the dependence of C on some variable, we often write $C(*)$, where $*$ denotes the variable. If necessary, we distinguish between two unequal constants by writing C_1 and C_2 or C and $\tilde{C}$.

Chapter 2

Background

2.1 Littlewood-Paley theory

We begin by defining the Littlewood-Paley operators. These operators will play an important role in the proofs of the main results. We start with the following lemma:

Lemma 2.1.1. *There exist two radial functions* $\chi \in S(\mathbb{R}^2)$ *and* $\varphi \in S(\mathbb{R}^2)$ *satisfying the following properties:*

(i) *supp* $\chi \subset \{\xi \in \mathbb{R}^2 : 0 \leq |\xi| \leq \frac{4}{3}\}$,

(ii) *supp* $\varphi \subset \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$,

(iii) $\chi(\xi) + \sum_{j=0}^{\infty} \varphi_j(\xi) = 1$,

where $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ *(so* $\check{\varphi}_j(x) = 2^{jn}\check{\varphi}(2^j x)$*).*

Proof. This is classical. See for example [19]. □

Observe that, if $|j - j'| \geq 2$, then supp $\varphi_j \cap$ supp $\varphi_{j'} = \emptyset$, and, if $j \geq 1$, then supp $\varphi_j \cap$ supp $\chi = \emptyset$.

Let $f \in S'(\mathbb{R}^2)$. We define, for any integer j,

$$\Delta_j f = \begin{cases} 0, & j < -1, \\ \chi(D)f = \check{\chi} * f, & j = -1, \\ \varphi(D)f = \check{\varphi}_j * f, & j > -1, \end{cases}$$

and

$$S_j f = \sum_{k=-\infty}^{j-1} \Delta_k f = \chi(2^{-j}D)f.$$

An important tool in chapter 3 will be a decomposition introduced by J.-M. Bony in [3]. We recall the definition of the paraproduct and remainder used in this decomposition.

Definition 2.1.1. Define the paraproduct of two functions f and g by

$$T_f g = \sum_{\substack{i,j \\ i \leq j-2}} \Delta_i f \Delta_j g = \sum_{j=1}^{\infty} S_{j-1} f \Delta_j g.$$

We use $R(f,g)$ to denote the remainder. $R(f,g)$ is given by the following bilinear operator:

$$R(f,g) = \sum_{\substack{i,j \\ |i-j| \leq 1}} \Delta_i f \Delta_j g.$$

Bony's decomposition gives

$$fg = T_f g + T_g f + R(f,g).$$

Finally, we state Bernstein's inequality. For a proof of this inequality, see [6].

Lemma 2.1.2. *(Bernstein's inequality) Let* r_1 *and* r_2 *satisfy* $0 < r_1 < r_2 < \infty$, *and let* p *and* q *satisfy* $1 \leq p \leq q \leq \infty$. *There exists a positive constant* C

such that for every integer k , if u belongs to $L^p(\mathbb{R}^n)$, and supp $\hat{u} \subset B(0, r_1\lambda)$, then

$$\sup_{|\alpha|=k} ||\partial_\alpha u||_{L^q} \leq C^k \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} ||u||_{L^p}. \tag{2.1.1}$$

Furthermore, if supp $\hat{u} \subset C(0, r_1\lambda, r_2\lambda)$, then

$$C^{-k}\lambda^k ||u||_{L^p} \leq \sup_{|\alpha|=k} ||\partial^\alpha u||_{L^p} \leq C^k \lambda^k ||u||_{L^p}. \tag{2.1.2}$$

2.2 Function spaces

We now define several useful function spaces.

Definition 2.2.1. Assume $s \in (0, 1)$. We define the Holder space $C^s(\mathbb{R}^n)$ to be the space of bounded functions f on $\mathbb{R}^n$ such that there exists a constant C with

$$|f(x) - f(y)| \leq C|x - y|^s \tag{2.2.1}$$

for all x and y in $\mathbb{R}^n$. For s a non-integer number greater than 1, we define C^s to be the set of functions f such that, for all multi-indices α with $\alpha \leq [s]$, $\partial^\alpha f$ belongs to $C^{s-[s]}$. We define the C^s-norm as follows:

$$||f||_{C^s} = \sum_{\substack{\alpha \\ |\alpha| \leq [s]}} \left(||\partial^\alpha f||_{L^\infty} + \sup_{x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^{s-[s]}} \right). \tag{2.2.2}$$

Definition 2.2.2. Let $s \in \mathbb{R}$. We define the Zygmund space $C^s_*(\mathbb{R}^n)$ to be the space of tempered distributions f on $(\mathbb{R}^n)$ such that

$$||f||_{C^s_*} := \sup_{q \geq -1} 2^{qs} ||\Delta_q f||_{L^\infty} < \infty.$$

It is well known that the norm on C^s_* is equivalent to the classical C^s-norm when s is not an integer and $s > 0$. For a proof of this, see [6], Proposition 2.3.1.

Definition 2.2.3. Let $s \in \mathbb{R}$, $p, q \in [1, \infty]$. We define the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^n)$ to be the space of tempered distributions f on $(\mathbb{R}^n)$ such that

$$||f||_{B^s_{p,q}} := \left(\sum_{j=-1}^{\infty} 2^{jqs} ||\Delta_j f||^q_{L^p} \right)^{\frac{1}{q}} < \infty.$$

When $q = \infty$, write

$$||f||_{B^s_{p,\infty}} := \sup_{j \geq -1} 2^{js} ||\Delta_j f||_{L^p}.$$

Remark 2.2.1. Note that $B^s_{\infty,\infty} = C^s_*$.

Definition 2.2.4. Let $s \in \mathbb{R}$, $p, q \in [1, \infty]$. We define the inhomogeneous Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ to be the space of tempered distributions f on $\mathbb{R}^n$ such that

$$||f||_{F^s_{p,q}} := \left\| \left(\sum_{j=-1}^{\infty} 2^{jqs} |\Delta_j f(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p} < \infty.$$

When $q = \infty$, write

$$||f||_{F^s_{p,\infty}} := \| \sup_{j \geq -1} 2^{js} |\Delta_j f(\cdot)| \|_{L^p}.$$

It is well known that the space $F^s_{p,2}(\mathbb{R}^n)$ coincides with the classical Sobolev space $W^{s,p}(\mathbb{R}^n)$ whenever $1 < p < \infty$ (see, for example, [7]).

Definition 2.2.5. The space of log-Lipschitz functions, denoted by $LL(\mathbb{R}^n)$, is the space of bounded functions f on $\mathbb{R}^n$ such that

$$||f||_{LL} := ||f||_{L^\infty} + \sup_{|x-y|\leq 1} \frac{|f(x) - f(y)|}{|x-y|(1-\log|x-y|)} < \infty.$$

We now define $BMO(\mathbb{R}^n)$, a space first introduced by John and Nirenberg in [10].

Definition 2.2.6. Assume f is a function defined on $\mathbb{R}^n$. Then f has bounded mean oscillation (or, equivalently, belongs to the space $BMO(\mathbb{R}^n)$) if there exists a positive constant C such that for all balls B in $\mathbb{R}^n$,

$$\frac{1}{m(B)} \int_B |f(x) - f_B| dx \leq C, \tag{2.2.3}$$

where f_B denotes the average of f over B. The BMO-seminorm of f is defined to be the smallest constant C satisfying (2.2.3).

It is easy to see that all bounded functions on $(\mathbb{R}^n)$ are also in $BMO(\mathbb{R}^n)$, and $||f||_{BMO} \leq 2||f||_{L^\infty}$.

We will also make use of the space $bmo(\mathbb{R}^n)$ which is a local version of the space $BMO(\mathbb{R}^n)$.

Definition 2.2.7. The space $bmo(\mathbb{R}^n)$ is the set of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\sup_{|B|\leq 1} \frac{1}{m(B)} \int_B |f(x) - f_B| dx + \sup_{|B|=1} \int_B |f(x)| dx \leq C. \tag{2.2.4}$$

It is clear from the definition that $L^p(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ continuously embeds into $bmo(\mathbb{R}^n)$. Moreover, the space $bmo(\mathbb{R}^n)$ is equivalent to the Triebel-Lizorkin space $F^0_{\infty,2}(\mathbb{R}^n)$ (see [19]).

We define $VMO(\mathbb{R}^n)$ and $vmo(\mathbb{R}^n)$, first introduced by Sarason in [15].

Definition 2.2.8. The space $VMO(\mathbb{R}^n)$ is the closure of the space $C_c^\infty(\mathbb{R}^n)$ in the BMO-norm. Similarly, $vmo(\mathbb{R}^n)$ is the closure of the space $C_c^\infty(\mathbb{R}^n)$ in the bmo-norm.

We now state several important embeddings which we use in what follows. For proofs of these embeddings, we refer the reader to [7] or [19].

Proposition 2.2.1. *The following embeddings hold:*

(i) $B^s_{p,q} \hookrightarrow F^s_{p,q}$ *for* $s \in \mathbb{R}$, $0 < q < p \leq \infty$.

(ii) $F^s_{p,q} \hookrightarrow B^s_{p,q}$ *for* $s \in \mathbb{R}$, $0 < p < q \leq \infty$.

(iii) $B^s_{p,q_1} \hookrightarrow B^s_{p,q_2}$ *for* $s \in \mathbb{R}$, $q_1 < q_2$.

(iv) $F^s_{p,q_1} \hookrightarrow F^s_{p,q_2}$ *for* $s \in \mathbb{R}$, $q_1 < q_2$.

(v) $B^s_{p,q_1} \hookrightarrow B^t_{p,q_2}$ *for* $t < s$ *and for all* q_1, q_2.

(vi) $F^s_{p,q_1} \hookrightarrow F^t_{p,q_2}$ *for* $t < s$ *and for all* q_1, q_2.

2.3 Properties of ideal incompressible fluids

In this section we state some properties of ideal incompressible fluids. We will make use of these properties in what follows. We begin with the

Biot-Savart law.

2.3.1 The Biot-Savart law

The Biot-Savart law states that the velocity of an ideal incompressible fluid can be uniquely determined by the vorticity as long as the vorticity has sufficient decay at infinity:

Theorem 2.3.1. *(Biot-Savart law) Assume ω belongs to $L^a(\mathbb{R}^2)$ for some $a < 2$. If $b > \frac{2a}{2-a}$, then there exists a unique divergence-free velocity vector field v in $L^a(\mathbb{R}^2) + L^b(\mathbb{R}^2)$ with vorticity equal to ω. If E_2 denotes the fundamental solution of the Laplacian in dimension 2, (that is, $E_2 = (2\pi)^{-1} \log |x|$), then*

$$v = (-\partial_2 E_2 * \omega, \partial_1 E_2 * \omega). \tag{2.3.1}$$

Proof. Write $\tilde{v}^i = (-1)^i \partial_j E_2 * \omega(v)$. We will show that $\omega(\tilde{v}) = \omega(v)$ and div $\tilde{v} = 0$. This will imply by Lemma 2.3.5 in Section 2.3.3 that $\tilde{v} = v$. To show that $\omega(\tilde{v}) = \omega(v)$, we write

$$\begin{aligned} \omega(\tilde{v}) &= \partial_1 \partial_1 E_2 * \omega(v) + \partial_2 \partial_2 E_2 * \omega(v) \\ &= \Delta E_2 * \omega(v) = \omega(v). \end{aligned} \tag{2.3.2}$$

Moreover, div $\tilde{v} = -\partial_1 \partial_2 E_2 * \omega + \partial_2 \partial_1 E_2 * \omega = 0$. Therefore, $v = \tilde{v} + \tilde{p}$, where the coefficients of $\tilde{p}$ are harmonic polynomials. Since ω belongs to $L^a(\mathbb{R}^2)$ for $a < 2$, and $|\partial_i E_2(x)| \leq \chi \frac{1}{|x|} + (1 - \chi) \frac{1}{|x|}$ belongs to $L^1(\mathbb{R}^2) + L^s(\mathbb{R}^2)$ for $s > 2$, $\tilde{v}$ belongs to $L^a(\mathbb{R}^2) + L^b(\mathbb{R}^2)$ by Young's inequality. This implies that $\tilde{p} = 0$ and $\tilde{v} = v$. This completes the proof. □

If we take the gradient of each side of (2.3.1), we get

$$\nabla v = (-\nabla \partial_2 E_2 * \omega, \nabla \partial_1 E_2 * \omega), \tag{2.3.3}$$

and we see that ∇v can be written as a Calderon-Zygmund operator acting on ω. We can therefore use mapping properties of Calderon-Zygmund operators to determine information about ∇v from our assumptions on ω. For example, it is well known that Calderon-Zygmund operators map $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. These operators are not bounded from $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$; however, they continuously map $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$ (see [18] or [16] for details).

2.3.2 The vorticity of an ideal incompressible fluid

The goal of this section is to introduce properties of weak solutions to the two-dimensional Euler equations. We use these properties in Chapter 3.

We begin this section by taking the curl of (E) and using the definition of vorticity in $\mathbb{R}^2$ given by $\omega = \partial_1 v^2 - \partial_2 v^1$ to get the two-dimensional vorticity equation:

$$(V) \quad \begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \\ \omega|_{t=0} = \omega^0. \end{cases}$$

When v is smooth, we can easily conclude from (V) that the L^p-norm of vorticity is conserved in two dimensions for all $p \in [1, \infty)$. To see this, we multiply (V) by $|\omega|^{r-2}\omega$ and observe that

$$\partial_t |\omega|^r = r|\omega|^{r-2}\omega(\partial_t \omega) = -r|\omega|^{r-2}\omega(u \cdot \nabla \omega). \tag{2.3.4}$$

Integrating both sides of (2.3.4), we get

$$\partial_t \int_{\mathbb{R}^2} |\omega|^r dx = -\int_{\mathbb{R}^2} u \cdot r|\omega|^{r-2}\omega\nabla\omega$$
$$= \int_{\mathbb{R}^2} (\operatorname{div} u)|\omega|^r dx = 0. \tag{2.3.5}$$

We can draw the same conclusions using the flow $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The flow is defined by the following ordinary differential equation:

$$\partial_t g(t,x) = v(t, g(t,x)),$$
$$g(0,x) = x. \tag{2.3.6}$$

When v is smooth, one can infer from the divergence-free condition on v that the flow is measure-preserving (see, for example, chapter 1 of [2]). Moreover, we can rewrite (V) using (2.3.6) and the chain rule as

$$\partial_t \left(\omega(t, g(t,x))\right) = 0, \tag{2.3.7}$$

which implies that the vorticity is conserved along flow lines and once again that the L^p-norms of ω are conserved over time.

In this section we discuss why these properties hold in a weaker setting. Throughout the discussion, we assume only that ω^0 belongs to the space $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. We closely follow chapter 8 of [2], and we refer the reader to [2] for further details. We will outline the proof of the following theorem:

Theorem 2.3.2. *Given $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, there exists a unique weak solution to (V) in the sense that there exists a unique pair (v,ω) satisfying:*

(i) ω belongs to $L^\infty([0,T]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$,

(ii) $\omega = \omega(v)$ and $v = K * \omega$, where K is the Biot-Savart kernal,

(iii) for all $\varphi \in C^1([0,T]; C_0^1(\mathbb{R}^2))$,

$$\int_{\mathbb{R}^2} \varphi(T,x)\omega(T,x)\,dx - \int_{\mathbb{R}^2} \varphi(0,x)\omega(0,x)\,dx = \int_0^T \int_{\mathbb{R}^2} (\partial_t \varphi + v \cdot \nabla \varphi)\omega\,dx\,dt. \tag{2.3.8}$$

In fact, uniqueness was proved by Yudovich in [23]. In what follows, we outline a proof for existence of such solutions. From this existence proof, we are able to show that several of the properties satisfied by classical solutions carry over to the weak solution (v,ω).

Proof. We begin by mollifying the initial vorticity. We consider a sequence of solutions (ω_n) to (V) with initial data $S_n\omega^0$, and with velocity v_n determined uniquely by the Biot-Savart law. We infer from the discussion above that

$$\omega_n(t) = S_n\omega^0(g_n(t)^{-1}(x)), \tag{2.3.9}$$

where $g_n(t)$ satisfies

$$\begin{aligned} &\partial_t g_n(t,x) = v_n(t, g_n(t,x)), \\ &g_n(0,x) = x. \end{aligned} \tag{2.3.10}$$

Moreover, by Lemma 2.3.9 in Section 2.3.3 and (2.3.10), we have

$$|g_n(t)^{-1}(x) - x| = |\int_0^t v_n(\tau, g_n(\tau,x))d\tau| \leq CT. \tag{2.3.11}$$

This uniform bound, combined with Holder estimates on ${g_n}^{-1}$ (see Lemma 8.2 of [2]), imply that the sequence $({g_n}^{-1})$ is equicontinuous on $K \times [0,T]$ for $K \subseteq \mathbb{R}^2$ compact. We apply the Arzela-Ascoli theorem and conclude that there exists a subsequence converging uniformly on $K \times [0,T]$ to a function which we denote by g^{-1}.

We now claim that $\omega_n^0(g_n(t)^{-1})$ converges to $\omega^0(g(t)^{-1})$ in $L^1(\mathbb{R}^2)$, and that $\omega^0(g(t)^{-1})$ solves the weak formulation of (V). To see this, we first need to establish that $g(t)^{-1}$ is indeed a measure-preserving map for each t in $[0,T]$.

Lemma 2.3.3. *For every t in $[0,T]$, the map $g(t)^{-1}$ established above is measure-preserving on $\mathbb{R}^2$.*

Proof. We follow the approach in [2] and prove that $\int_{\mathbb{R}^2} f(g(t)^{-1}(x))dx = \int_{\mathbb{R}^2} f(x)dx$ for all f in $L^1(\mathbb{R}^2)$. We assume that f is continuous with compact support and use density of $C_c(\mathbb{R}^2)$ in $L^1(\mathbb{R}^2)$. Because f is in $C_c(\mathbb{R}^2)$, and $g_n(t)^{-1}$ converges uniformly to $g(t)^{-1}$ on compact sets, we have that $f(g_n(t)^{-1})$ converges to $f(g(t)^{-1})$ pointwise. We apply Lebesgue's dominated convergence theorem and conclude that

$$\int_{\mathbb{R}^2} f(g(t)^{-1})dx = \lim_{n\to\infty} \int_{\mathbb{R}^2} f(g_n(t)^{-1})dx = \int_{\mathbb{R}^2} f dx. \tag{2.3.12}$$

We use the Riesz representation theorem and (2.3.12) to conclude that $|B|$ and $|g(t)^{-1}(B)|$ are equivalent on Borel sets. Therefore, $g(t)^{-1}$ is measure-preserving. □

We now use Lemma 2.3.3 to show that $\omega_n^0(g_n(t)^{-1})$ converges to $\omega^0(g(t)^{-1})$ in $L^1(\mathbb{R}^2)$. We write

$$\begin{aligned} ||\omega_n^0(g_n(t)^{-1}) - \omega^0(g(t)^{-1})||_{L^1} &\leq ||\omega_n^0(g_n(t)^{-1}) - \omega^0(g_n(t)^{-1})||_{L^1} \\ &+ ||\omega^0(g_n(t)^{-1}) - \omega^0(g(t)^{-1})||_{L^1}. \end{aligned} \tag{2.3.13}$$

The first term on the right hand side of (2.3.13) is equal to $||\omega_n^0 - \omega^0||_{L^1}$, which converges to 0 as n approaches infinity by our construction of the smooth initial data. We break the second term on the right hand side of (2.3.13) down as in [2]; we let ω_m^0 be a continuous function with compact support on $\mathbb{R}^2$ such that $||\omega^0 - \omega_m^0||_{L^1} < \frac{1}{m}$, and we write

$$||\omega^0(g_n(t)^{-1}) - \omega^0(g(t)^{-1})||_{L^1} \leq ||\omega^0(g_n(t)^{-1}) - \omega_m^0(g_n(t)^{-1})||_{L^1} + ||\omega_m^0(g_n(t)^{-1}) - \omega_m^0(g(t)^{-1})||_{L^1} + ||\omega_m^0(g(t)^{-1}) - \omega^0(g(t)^{-1})||_{L^1}. \tag{2.3.14}$$

Since $g_n(t)^{-1}$ and $g(t)^{-1}$ are measure-preserving, we can choose m sufficiently large to make the first and third terms on the right hand side of (2.3.14) small. For the middle term, for fixed, sufficiently large m, we use sequential continuity of ω_m^0 to apply Lebesgue's dominated convergence theorem. This completes the argument that $\omega_n^0(g_n(t)^{-1})$ converges to $\omega^0(g(t)^{-1})$ in $L^1(\mathbb{R}^2)$.

Using this convergence, we are also able to show that $v_n(t)$ converges to $v(t)$ uniformly on compact sets for fixed $t \in [0, T]$, where $v_n(t)$ and $v(t)$ and determined uniquely from ω_n and ω using the Biot-Savart law. The strategy of the proof is to write the difference of the two velocities as a convolution of the Biot-Savart kernel with the difference of the two vorticities, to apply Young's inequality to the convolution, and to utilize convergence of the sequence (ω_n) to ω in $L^1(\mathbb{R}^2)$. We refer the reader to [2] for a detailed proof.

Finally, we use convergence of (ω_n) to ω in $L^1(\mathbb{R}^2)$ and uniform convergence of $v_n(t)$ to $v(t)$ on compact sets to show that $\omega(t) = \omega^0(g(t)^{-1})$ solves our weak formulation. We recall the definition of a weak solution:

Definition 2.3.1. Given $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, the pair (v,ω) is a weak solution to (V) if the following three conditions hold:

(i) ω belongs to $L^\infty([0,T]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$,

(ii) $\omega = \omega(v)$ and $v = K * \omega$, where K is the Biot-Savart kernal,

(iii) for all $\varphi \in C^1([0,T]; C_0^1(\mathbb{R}^2))$,

$$\int_{\mathbb{R}^2} \varphi(T,x)\omega(T,x)\,dx - \int_{\mathbb{R}^2} \varphi(0,x)\omega(0,x)\,dx = \int_0^T \int_{\mathbb{R}^2} (\partial_t \varphi + v \cdot \nabla \varphi)\omega\,dx\,dt. \tag{2.3.15}$$

A straightforward computation (see Proposition 8.1 of [2]) shows that every smooth solution of (V) solves the weak formulation. We must show that $\omega^0(g(t)^{-1})$ satisfies (iii) as well. We will show that

$$\begin{aligned} &\int_{\mathbb{R}^2} \varphi(T,x)\omega_n(T,x)\,dx - \int_{\mathbb{R}^2} \varphi(0,x)\omega_n(0,x)\,dx \\ &\qquad = \int_0^T \int_{\mathbb{R}^2} (\partial_t \varphi + v_n \cdot \nabla \varphi)\omega_n\,dx\,dt \end{aligned} \tag{2.3.16}$$

converges to

$$\int_{\mathbb{R}^2} \varphi(T,x)\omega(T,x)\,dx - \int_{\mathbb{R}^2} \varphi(0,x)\omega(0,x)\,dx = \int_0^T \int_{\mathbb{R}^2} (\partial_t \varphi + v \cdot \nabla \varphi)\omega\,dx\,dt \tag{2.3.17}$$

as n approaches infinity. The term $\int_{\mathbb{R}^2} \varphi(T,x)\omega_n(T,x)\,dx$ clearly converges to $\int_{\mathbb{R}^2} \varphi(T,x)\omega(T,x)\,dx$ because φ is in $C_0^1(\mathbb{R}^2)$ and ω_n converges to ω in $L^1(\mathbb{R}^2)$; similarly $\int_{\mathbb{R}^2} \varphi(0,x)\omega_n(0,x)\,dx$ converges to $\int_{\mathbb{R}^2} \varphi(0,x)\omega(0,x)\,dx$, and $\int_0^T \int_{\mathbb{R}^2} \partial_t\varphi\omega_n\,dx\,dt$ converges to $\int_0^T \int_{\mathbb{R}^2} \partial_t\varphi\omega\,dx\,dt$. To handle the nonlinearity, we observe that ω_n is uniformly bounded independent of n by $C||\omega^0||_{L^\infty}$, and

$||v_n||_{L^\infty} \leq ||v||_{L^\infty} + \epsilon$ for large n. Furthermore, there exists a subsequence of $(v_n \cdot \nabla\varphi)\omega_n$ converging pointwise to $(v \cdot \nabla\varphi)\omega$. We apply Lebesgue's dominated convergence theorem, and we conclude that ω is a weak solution to (V). $\square$

Remark 2.3.1. From this discussion we draw several conclusions. Most importantly, we are able to conclude that for a solution ω to the weak formulation given by Definition 2.3.1, the solution possesses a flow g such that $g(t)$ and $g(t)^{-1}$ are measure-preserving on $\mathbb{R}^2$. Moreover, the weak solution ω satisfies $\omega(t) = \omega^0(g(t)^{-1})$ by construction. In fact, Yudovich shows in [23] that these properties hold for our solution if we assume only that ω^0 belongs to $L^p(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for some $p \in [1, \infty)$. One can prove this using an approximation argument similar to that given above. We refer the reader to [23] for details.

2.3.3 A few technical lemmas

In this section we state and prove a few technical lemmas. We frequently make use of these lemmas in what follows.

We begin with Osgood's Lemma. We refer the reader to [5], pg. 92, for a detailed proof.

Lemma 2.3.4. *(Osgood's Lemma) Let ρ be a measurable positive function, γ a positive locally integrable function, and μ a continuous, increasing function. Assume that a is some positive real number and that*

$$\rho(t) \leq a + \int_{t_0}^{t} \gamma(s)\mu(\rho(s))ds. \tag{2.3.18}$$

If we define $M(x) = \int_x^1 \frac{dr}{\mu(r)}$*, then we have*

$$-M(\rho(t)) + M(a) \leq \int_{t_0}^{t} \gamma(s) ds. \tag{2.3.19}$$

Lemma 2.3.5. *Two vector fields whose coefficients are tempered distributions and with the same vorticity and divergence are equal up to a vector field with harmonic polynomials as coefficients.*

Proof. We have

$$\partial_i v^j = \partial_j v^i + \omega(v), \tag{2.3.20}$$

with $i \neq j$. Taking the partial derivative with respect to the *ith* variable of both sides and summing over $i = 1, 2$ gives

$$\Delta v^i = \partial_j \operatorname{div} v + \sum_i \partial_i \omega(v). \tag{2.3.21}$$

Therefore, if div $v =$ div $\tilde{v}$ and $\omega(v) = \omega(\tilde{v})$, then $\Delta(v - \tilde{v}) = 0$, and $v - \tilde{v}$ is a harmonic polynomial. This completes the proof. □

Lemma 2.3.6. *If* v *is a divergence-free vector field in* $L^2(\mathbb{R}^2)$ *with vorticity* ω *in* $L^2(\mathbb{R}^2)$ *then* v *belongs to* $H^1(\mathbb{R}^2)$ *and* $||\nabla v||_{L^2} = ||\omega||_{L^2}$.

Proof. Because v belongs to $L^2(\mathbb{R}^2)$, the Fourier transform of v belongs to $L^2(\mathbb{R}^2)$, and $\widehat{\partial_j v^i}(\xi) = \xi_j \hat{v}^i(\xi)$ is defined pointwise almost everywhere. Therefore, we can write

$$\begin{aligned}
|\widehat{\nabla v}(\xi)|^2 &= {\xi_1}^2(\hat{v}^1(\xi))^2 + {\xi_1}^2(\hat{v}^1(\xi))^2 + {\xi_2}^2(\hat{v}^1(\xi))^2 + {\xi_2}^2(\hat{v}^2(\xi))^2 \\
&= {\xi_1}^2(\hat{v}^1(\xi))^2 + {\xi_2}^2(\hat{v}^2(\xi))^2 + 2\xi_1\xi_2\hat{v}^1(\xi)\hat{v}^2(\xi) \\
&+ {\xi_1}^2(\hat{v}^2(\xi))^2 + {\xi_2}^2(\hat{v}^1(\xi))^2 - 2\xi_1\xi_2\hat{v}^1(\xi)\hat{v}^2(\xi) \qquad (2.3.22) \\
&= (\xi_1\hat{v}^1(\xi) + \xi_2\hat{v}^2(\xi))^2 + (\xi_1\hat{v}^2(\xi) - \xi_2\hat{v}^1(\xi))^2 \\
&= \left(\widehat{\operatorname{div} v(\xi)}\right)^2 + (\hat{\omega}(\xi))^2 = (\hat{\omega}(\xi))^2
\end{aligned}$$

for almost every $\xi \in \mathbb{R}^2$. Thus $||\nabla v||_{L^2} = ||\widehat{\nabla v}||_{L^2} = ||\widehat{\omega}||_{L^2} = ||\omega||_{L^2}$. This completes the proof. □

In Section 2.3.1, we showed that whenever ω belongs to $L^p(\mathbb{R}^2)$ for $p < 2$, the velocity v can be uniquely determined from ω using the Biot-Savart law. If we only assume ω belongs to $L^p(\mathbb{R}^2)$ for some $p \geq 2$, we cannot *determine* v from ω; however, if v is a divergence-free vector field in $L^p(\mathbb{R}^2)$ for some $p < \infty$ with vorticity $\omega \in L^q(\mathbb{R}^2)$ for $q \in [2, \infty)$, we can still write v as the Biot-Savart kernal convolved with ω. We have the following lemma.

Lemma 2.3.7. *Let $p \in (1, \infty)$ and $q \in [2, \infty)$, and assume v is a divergence-free vector field in $L^p(\mathbb{R}^2)$ with $\omega = \omega(v)$ in $L^q(\mathbb{R}^2)$. Then $v = (-\partial_2 E_2 * \omega, \partial_1 E_2 * \omega)$.*

Proof. Let ψ be a tempered distribution on $\mathbb{R}^2$ satisfying $v = (-\partial_2\psi, \partial_1\psi)$, so that $\Delta\psi = \omega$. (To see that such a tempered distribution exists, we refer the reader to Corollary 1.2.2 on page 7 of [5].) Let $\psi' = \Delta^{-1}\omega$. Then, if

$v' = (-\partial_2\psi', \partial_1\psi')$, we have div $v' = 0$ and $\omega(v') = \Delta\psi' = \omega$. Therefore, v and v' have the same divergence and vorticity. By Lemma 2.3.5, we can write

$$v - v' = q, \tag{2.3.23}$$

where q is a vector field with polynomials as coefficients. Using the equalities $v' = \nabla^\perp\psi = \nabla^\perp\Delta^{-1}\omega$, we see that for $r \in [\max\{p, q\}, \infty)$,

$$\begin{aligned}
||v'||_{L^r} &= ||\Delta_{-1}v'||_{L^r} + \sum_{j\geq 0} ||\Delta_j v'||_{L^r} \\
&\leq C||\Delta_{-1}v'||_{L^p} + C\sum_{j\geq 0} 2^{2j(\frac{1}{q}-\frac{1}{r})-j}||\Delta_j \nabla v'||_{L^q} \\
&= C||\Delta_{-1}\nabla^\perp\Delta^{-1}\omega||_{L^p} + C\sum_{j\geq 0} 2^{2j(\frac{1}{q}-\frac{1}{r})-j}||\Delta_j \nabla\nabla^\perp\Delta^{-1}\omega||_{L^q} \\
&\leq C||\Delta_{-1}\nabla^\perp\Delta^{-1}\nabla v||_{L^p} + C||\omega||_{L^q} \\
&\leq C||v||_{L^p} + C||\omega||_{L^q},
\end{aligned} \tag{2.3.24}$$

where we repeatedly used Bernstein's inequality, and we used boundedness of Calderon-Zygmund operators on $L^s(\mathbb{R}^2)$ for $s \in (1, \infty)$ to get the second-to-last and last inequalities. Since $v \in L^p(\mathbb{R}^2)$ and $v' \in L^r(\mathbb{R}^2)$, it follows that $q = 0$ in (2.3.23). Therefore, $v = v' = \nabla^\perp\Delta^{-1}\omega$ This completes the proof. □

Lemma 2.3.8. *Let v be a divergence-free vector field in $L^2(\mathbb{R}^2)$ with vorticity ω in $L^2(\mathbb{R}^2)$. There exists an absolute constant C such that for all $q \geq 0$ (that is, avoiding the low frequencies),*

$$\left\|\Delta_q \nabla v\right\|_{L^\infty} \leq C \left\|\Delta_q \omega\right\|_{L^\infty}.$$

Proof. Since v is a divergence-free vector field in $L^2(\mathbb{R}^2)$, and $\omega \in L^2(\mathbb{R}^2)$, it follows by Lemma 2.3.7 that $\nabla\Delta_q v = \nabla\nabla^\perp\Delta^{-1}\Delta_q\omega$. We can then write

$$\|\Delta_q\nabla v\|_{L^\infty} \le C\sup_{i,j}\|\Delta_q\partial_i\partial_j\Delta^{-1}\omega\|_{L^\infty}.$$

But,

$$\begin{aligned}\|\Delta_q\partial_i\partial_j\Delta^{-1}\omega\|_{L^\infty} &= \|\mathcal{F}^{-1}(\varphi_q(\xi)\frac{\xi_i\xi_j}{|\xi|^2}\widehat{\omega}(\xi))\|_{L^\infty}\\ &= \|\mathcal{F}^{-1}(\varphi_q(\xi)h_q(\xi)\widehat{\omega}(\xi))\|_{L^\infty} = \|\Delta_q(\check{h}_q(\xi)*\omega)\|_{L^\infty},\end{aligned}$$

where

$$h_q(\xi) = \chi(2^{-3-q}\xi)(1-\chi(2^{-q+1}\xi))\frac{\xi_i\xi_j}{|\xi|^2},$$

χ and φ being defined in Lemma 2.1.1. Observe that $h_q = 1$ on the support of φ_q. Because $h_q(\xi) = h(2^{-q}\xi)$, where

$$h(\xi) = \chi(2^{-3}\xi)(1-\chi(2\xi))\frac{\xi_i\xi_j}{|\xi|^2},$$

$\check{h}_q(x) = 2^{2q}\check{h}(2^q x)$, and thus by a change of variables, $\|\check{h}_q\|_{L^1} = \|\check{h}\|_{L^1} = C$.

Then using Young's inequality,

$$\begin{aligned}\|\Delta_q(\check{h}_q(\xi)*\omega)\|_{L^\infty} &= \|\check{h}_q(\xi)*\Delta_q\omega\|_{L^\infty} \le \|\check{h}_q\|_{L_1}\|\Delta_q\omega\|_{L^\infty}\\ &\le C\|\Delta_q\omega\|_{L^\infty},\end{aligned}$$

which completes the proof. □

Lemma 2.3.9. *Assume ω is a C^∞ solution to (V) with $\omega^0 \in L^p(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ for some $p \in [1,2)$, and with v determined from ω using the Biot-Savart law. Then v belongs to $L^\infty_{loc}(\mathbb{R}^+; L^\infty(\mathbb{R}^2))$, and $||v(t)||_{L^\infty} \le C||\omega^0||_{L^p\cap L^\infty}$.*

Proof. The proof of Lemma 2.3.9 closely follows arguments in [21], Theorem 3.1. We break v into low and high frequencies and write

$$\begin{aligned} ||v(t)||_{L^\infty} &\leq ||\Delta_{-1}v(t)||_{L^\infty} + \sum_{j\geq 0} ||\Delta_j v(t)||_{L^\infty} \\ &\leq ||\Delta_{-1}v(t)||_{L^\infty} + \sum_{j\geq 0} 2^{-j}||\Delta_j \nabla v(t)||_{L^\infty} \\ &\leq ||\Delta_{-1}v(t)||_{L^\infty} + C||\omega^0||_{L^\infty}, \end{aligned} \tag{2.3.25}$$

where we used Lemma 2.3.8, Bernstein's inequality, and conservation of vorticity along flow lines for the high frequency terms. To bound the low frequency term, we use the Biot-Savart law. Let χ be a smooth bump function with support in the unit ball. Write

$$\begin{aligned} ||\Delta_{-1}v(t)||_{L^\infty} &\leq ||\chi\partial_i E_2 \Delta_{-1}\omega(t)||_{L^\infty} + ||(1-\chi)\partial_i E_2 \Delta_{-1}\omega(t)||_{L^\infty} \\ &\leq ||\chi\partial_i E_2||_{L^1}||\Delta_{-1}\omega(t)||_{L^\infty} + ||(1-\chi)\partial_i E_2||_{L^{q_0}}||\Delta_{-1}\omega(t)||_{L^{p_0}} \\ &\leq C||\Delta_{-1}\omega(t)||_{L^{p_0}} \leq C||\omega^0||_{L^{p_0}}, \end{aligned} \tag{2.3.26}$$

where $\frac{1}{p_0} + \frac{1}{q_0} = 1$. Again, we used conservation of the vorticity along flow lines. This completes the proof. □

The following lemma and proof can be found in [21], Theorem 3.1.

Lemma 2.3.10. *Assume v is a solution to (E) and $\omega(v^0) = \omega^0$ belongs to $L^p(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for some $p < \infty$. Then v belongs to $L^1_{loc}(\mathbb{R}^+; LL(\mathbb{R}^2))$.*

Proof. We begin by showing v is bounded on $\mathbb{R}^2$. Write

$$\begin{aligned} ||v(t)||_{L^\infty} &\le ||\Delta_{-1}v(t)||_{L^\infty} + \sum_{j\ge 0} ||\Delta_j v(t)||_{L^\infty} \\ &\le ||\Delta_{-1}v(t)||_{L^\infty} + \sum_{j\ge 0} 2^{-j}||\Delta_j \nabla v(t)||_{L^\infty} \\ &\le ||\Delta_{-1}v(t)||_{L^\infty} + C||\omega(t)||_{L^\infty} \\ &\le C||\Delta_{-1}v(t)||_{L^p} + C||\omega^0||_{L^\infty}, \end{aligned}$$

where we used Bernstein's inequality, Lemma 2.3.8, and conservation of the L^∞-norm of ω over time by Remark 2.3.1. To bound $||\Delta_{-1}v(t)||_{L^p}$, we use the proof of Lemma 6.2 in [20]. We first bound $||\Delta_{-1}v(t)||_{L^p}$ by $||v(t)||_{L^p}$. Then, because the Weyl projection operator is bounded from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$, we can write

$$\begin{aligned} ||v(t)||_{L^p} - ||v^0||_{L^p} &\le \int_0^t ||-v\cdot\nabla v(\tau) - \nabla p(\tau)||_{L^p} d\tau \\ &\le C\int_0^t ||v\cdot\nabla v(\tau)||_{L^p} d\tau \\ &\le C\int_0^t ||v(\tau)||_{L^\infty}||\nabla v(\tau)||_{L^p} d\tau \\ &\le C||\omega^0||_{L^p}\int_0^t ||v(\tau)||_{L^\infty} d\tau \\ &\le C||\omega^0||_{L^p}\int_0^t (||\Delta_{-1}v(t)||_{L^\infty} + C||\omega^0||_{L^p}) d\tau \\ &\le Ct||\omega^0||_{L^p} + C_1||\omega^0||_{L^p}\int_0^t ||v(t)||_{L^p} d\tau, \end{aligned} \tag{2.3.27}$$

where we used the boundedness of Calderon-Zygmund operators on $L^p(\mathbb{R}^2)$, as well as (2.3.30), and conservation of the L^p-norm of the vorticity. An application of Gronwall's inequality gives

$$||v(t)||_{L^p} \le Ce^{C_1 t||\omega^0||_{L^p}}. \tag{2.3.28}$$

We conclude that v belongs to $L^\infty_{loc}(\mathbb{R}^+; L^\infty(\mathbb{R}^2))$. To complete the proof, we write

$$
\begin{aligned}
|v(t,x) - v(t,y)| &\leq \sum_{j=-1}^{\infty} |\Delta_j v(t,x) - \Delta_j v(t,y)| \\
&\leq ||\Delta_{-1}\nabla v(t)||_{L^\infty}|x-y| + \sum_{j=0}^{N-1} ||\Delta_j \nabla v(t)||_{L^\infty}|x-y| + 2\sum_{j=N}^{\infty} ||\Delta_j v(t)||_{L^\infty} \\
&\leq C||\Delta_{-1}\nabla v(t)||_{L^p}|x-y| + CN||\Delta_j \omega(t)||_{L^\infty}|x-y| + 2\sum_{j=N}^{\infty} 2^{-j}||\Delta_j \nabla v(t)||_{L^\infty} \\
&\leq C||\omega(t)||_{L^p}|x-y| + CN||\omega(t)||_{L^\infty}|x-y| + C2^{-N}||\omega(t)||_{L^\infty}
\end{aligned}
\tag{2.3.29}
$$

where we used Bernstein's inequality on the first and third terms, and we used Lemma 2.3.8 on the second and third terms. We also used boundedness of Calderon-Zygmund operators from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ on the first term. Letting $N = -\log_2|x-y|$, we get

$$
\begin{aligned}
|v(t,x) - v(t,y)| &\leq C(||\omega(t)||_{L^p} + N||\omega(t)||_{L^\infty})|x-y| + C2^{-N}||\omega(t)||_{L^\infty} \\
&\leq C||\omega(t)||_{L^p}(1 - \log_2|x-y|)|x-y| + C|x-y|||\omega(t)||_{L^\infty} \\
&\leq C||\omega(t)||_{L^p \cap L^\infty}(1 - \log_2|x-y|)|x-y|.
\end{aligned}
$$

Since the L^p-norm and L^∞-norm of vorticity are conserved over time, $||\omega(t)||_{L^p\cap L^\infty} = ||\omega^0||_{L^p\cap L^\infty}$, and the lemma is proved. □

Lemma 2.3.11. *Assume v is a solution to (E) with $\omega^0 \in L^p(\mathbb{R}^2)$ for some $p \in (2,\infty)$. Then v belongs to $L^\infty_{loc}(\mathbb{R}^+; L^\infty(\mathbb{R}^2))$, and there exist two positive constants C and C_1 such that $||v(t)||_{L^\infty} \leq C||\omega^0||_{L^p} e^{C_1 t||\omega^0||_{L^p}}$.*

Proof. The proof of Lemma 2.3.11 is almost identical to the proof that v belongs to $L^\infty_{loc}(\mathbb{R}^+; L^\infty(\mathbb{R}^2))$ in Lemma 2.3.10. Here we are assuming ω^0 belongs to $L^p(\mathbb{R}^2)$ for fixed $p \in (2, \infty)$, but ω^0 is not necessarily bounded. We therefore treat low frequencies of $||v(t)||_{L^\infty}$ as we did in the proof of Lemma 2.3.11. For high frequencies we modify the proof slightly. We write

$$\begin{aligned} ||v(t)||_{L^\infty} &\leq ||\Delta_{-1} v(t)||_{L^\infty} + \sum_{j \geq 0} ||\Delta_j v(t)||_{L^\infty} \\ &\leq ||\Delta_{-1} v(t)||_{L^p} + \sum_{j \geq 0} 2^{j(\frac{2}{p}-1)} ||\Delta_j \nabla v(t)||_{L^p} \\ &\leq C e^{C_1 t ||\omega^0||_{L^p}} + C ||\omega^0||_{L^p}, \end{aligned} \tag{2.3.30}$$

for $p \in (2, \infty)$. This completes the proof. □

Chapter 3

An initial value problem for the Euler equations in the plane

3.1 Introduction and history

In this chapter, we study regularity of solutions to the Euler equations in the plane with initial velocity in the critical or subcritical Sobolev spaces $W^{s+1,p}(\mathbb{R}^2)$, where $sp = 2$ or $sp < 2$, respectively. For the supercritical case, when $sp > 2$, Kato and Ponce prove persistence of regularity in [11].

For the critical case, much less is known. In [21] Vishik proves global well-posedness of the two-dimensional Euler equations in the critical Besov space $B^{s+1}_{p,1}(\mathbb{R}^2)$. To prove local existence, he observes that $B^{s+1}_{p,1}(\mathbb{R}^2)$ continuously imbeds into $B^1_{\infty,1}(\mathbb{R}^2)$. This implies that ∇v belongs to $B^0_{\infty,1}(\mathbb{R}^2)$ and therefore to $L^\infty(\mathbb{R}^2)$. He then uses paradifferential calculus to prove an estimate on the $B^s_{p,1}$-norm of the vorticity:

$$||\omega(t)||_{B^s_{p,1}} \leq ||\omega^0||_{B^s_{p,1}} \exp\left(C\int_0^t ||\nabla v(\tau)||_{L^\infty} d\tau\right). \tag{3.1.1}$$

To give global existence, he proves the following logarithmic estimate on the L^∞-norm of ∇v:

$$||\nabla v(t)||_{L^\infty} \leq C(1 + \log(||g(t)||_{lip}||g^{-1}(t)||_{lip})||\omega^0||_{B^0_{\infty,1}}. \tag{3.1.2}$$

In [4], Chae proves global well-posedness for the two-dimensional Euler equations in the critical Triebel-Lizorkin spaces $F^3_{1,q}(\mathbb{R}^2)$, for $q \in [1, \infty]$. Like Vishik, he uses paradifferential calculus to prove the estimate

$$||\omega(t)||_{F^2_{1,q}} \leq ||\omega^0||_{F^2_{1,q}} \exp\left(C \int_0^t ||\nabla v(\tau)||_{L^\infty} d\tau\right). \tag{3.1.3}$$

In order to obtain a logarithmic estimate like that in (3.1.2), Chae utilizes the following embedding theorem (see, for example, [9], Theorem 2.1):

Theorem 3.1.1. *Let $s_0 > s_1$, $p_0, p_1 \in (0, \infty)$, and $q, r \in (0, \infty]$. If $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$, then*

$$F^{s_0}_{p_0,q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1,p_0}(\mathbb{R}^n) \hookrightarrow B^0_{\infty,p_0}(\mathbb{R}^n). \tag{3.1.4}$$

In fact, the first embedding is the interesting part of the theorem. The second embedding follows from a straightforward application of Bernstein's inequality. It follows from Theorem 3.1.1 that $F^3_{1,q}(\mathbb{R}^2)$ embeds into $B^0_{\infty,1}(\mathbb{R}^2)$.

Chae applies the embedding $B^0_{\infty,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and boundedness of Calderon-Zygmund operators from $B^0_{\infty,1}(\mathbb{R}^2)$ into $B^0_{\infty,1}(\mathbb{R}^2)$ to (3.1.3) and concludes that

$$||\nabla v(\tau)||_{L^\infty} \leq ||\nabla v(\tau)||_{B^0_{\infty,1}} \leq ||\omega(\tau)||_{B^0_{\infty,1}}. \tag{3.1.5}$$

He then applies (3.1.2) and Theorem 3.1.1 to conclude that

$$||\nabla v(\tau)||_{L^\infty} \leq C(1 + \log(||g(t)||_{lip}||g^{-1}(t)||_{lip})||\omega^0||_{F^2_{1,q}}, \tag{3.1.6}$$

and using this inequality he proves global well-posedness when initial velocity belongs to $F^3_{1,q}(\mathbb{R}^2)$.

Remark 3.1.1. The space $F^3_{1,q}(\mathbb{R}^2)$ is equivalent to the space $W^{3,\tilde{1}}(\mathbb{R}^2)$, where the $\tilde{1}$ denotes the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ in place of the usual Lebesgue space $L^1(\mathbb{R}^2)$. Therefore, Chae's result gives global well-posedness for the Euler equations in this space.

Despite several results in the critical Besov and Triebel-Lizorkin spaces in the plane, many interesting problems remain open for the critical case. In particular, regularity of solutions to the Euler equations with initial velocity in $W^{s+1,p}(\mathbb{R}^2)$, for $sp = 2$, is not well understood. The main motivation for this chapter is a theorem of Bahouri and Chemin found in [1] which studies this problem. The authors show that a lower bound for the Sobolev exponent of $\omega(t)$ is determined by the log-Lipschitz norm of $v(t)$. They define

$$V(t) = \sup_{|x-y|\leq 1} \frac{|v(t,x) - v(t,y)|}{|x-y|(1-\log|x-y|)},$$

and they prove the following:

Theorem 3.1.2. *Let v be a solution to (E) such that $\omega(v^0) = \omega^0 \in L^\infty(\mathbb{R}^2) \cap W^{s,p}(\mathbb{R}^2)$, for $sp \leq 2$ and $s \in (0,1]$. Fix $s' < s$, and define $\sigma(s',t) = s' \exp(-\int_0^t V(\tau)d\tau)$. Then $\omega(t) \in W^{\sigma(s',t),p}(\mathbb{R}^2)$ for all $t \in \mathbb{R}$.*

To prove the theorem, the authors show loss of regularity in the Triebel-Lizorkin space $F^s_{p,\infty}(\mathbb{R}^2)$. They use the following definition of $F^s_{p,\infty}(\mathbb{R}^2)$:

Definition 3.1.1. Let $s \in (0,1)$ and $p \in (1,\infty]$. The space $F^s_{p,\infty}(\mathbb{R}^2)$ is the set of all tempered distributions u in $L^p(\mathbb{R}^2)$ such that there exists a function

U in $L^p(\mathbb{R}^2)$ satisfying, for all $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\frac{|u(x)-u(y)|}{|x-y|^s} \leq U(x) + U(y). \tag{3.1.7}$$

The $F^s_{p,\infty}$-norm is given by

$$||u||_{F^s_{p,\infty}} = ||u||_{L^p} + \inf\{||U||_{L^p} : U \text{ satisfies } (3.1.7)\}.$$

For a proof that this definition is equivalent to Definition 2.2.4, we refer the reader to [1], Proposition 3.2.

The authors proceed by considering what happens to the Triebel-Lizorkin exponent of ω^0 under composition with the measure-preserving map $g(t)$. Using the fact that $g(t) - Id$ is Holder continuous with Holder exponent $\alpha(t) = \exp(-\int_0^t V(\tau)d\tau)$, they write

$$\frac{|\omega^0(g_{t^{-1}}(x)) - \omega^0(g_{t^{-1}}(y))|}{|x-y|^{s\alpha(t)}} = \frac{|\omega^0(g_{t^{-1}}(x)) - \omega^0(g_{t^{-1}}(y))|}{|g_{t^{-1}}(x) - g_{t^{-1}}(y)|^s} \frac{|g_{t^{-1}}(x) - g_{t^{-1}}(y)|^s}{|x-y|^{s\alpha(t)}}$$

$$\leq C(t)^s (U(g_{t^{-1}}(x)) + U(g_{t^{-1}}(y))), \tag{3.1.8}$$

where $C(t)$ depends on the Holder norm of $g(t) - Id$. Since the flow is measure-preserving, it follows from 3.1.8 and the definition of the $F^s_{p,\infty}$-norm that

$$||\omega^0 \circ g_{t^{-1}}||_{F^{s\alpha(t)}_{p,\infty}} \leq (C(t)^s + 1)||\omega^0||_{F^s_{p,\infty}}.$$

The theorem follows from the embeddings $W^{s,p}(\mathbb{R}^2) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^2) \hookrightarrow W^{s',p}(\mathbb{R}^2)$ for all $s' < s$.

We claim that if we also assume ω^0 is uniformly continuous, then we can improve the lower bound for loss of regularity to an arbitrarily small amount. We will show that, given $\epsilon > 0$ arbitrarily small and $T > 0$ fixed, if $\omega^0 \in W^{s,p}(\mathbb{R}^2) \cap UC(\mathbb{R}^2)$ with $sp \leq 2$, $p \in (1, \infty)$, and $s \in (0, 2)$, then $\omega(t)$ belongs to $W^{s-\epsilon,p}(\mathbb{R}^2) \cap UC(\mathbb{R}^2)$ for all $t \in [0, T]$. As in [1], we study the vorticity equation (V) corresponding to the Euler equations.

To prove our result, we show that $\omega(t)$ belongs to the Besov space $B^{s-\epsilon}_{p,\infty}$ (see Definition 2.2.3) for all t in a finite time interval $[0, T]$. Our approach is to localize the frequency of the terms of (V), which results in a new equation with a commutator term on the right-hand side:

$$\begin{aligned} &\partial_t \Delta_q \omega + v \cdot \nabla \Delta_q \omega = [v \cdot \nabla, \Delta_q]\omega, \\ &\Delta_q \omega|_{t=0} = \Delta_q \omega^0. \end{aligned} \tag{3.1.9}$$

We then prove the necessary estimate on the L^p-norm of the commutator on the right-hand side of (3.1.9), and apply a Gronwall argument to show that $\omega(t)$ is in $B^{s-\epsilon}_{p,\infty}(\mathbb{R}^2)$.

The main novelty of this proof is that our methods allow us to draw conclusions for $\omega^0 \in B^s_{p,\infty}(\mathbb{R}^2) \cap UC(\mathbb{R}^2)$ for *all* $s \in (0, 2)$. If we restrict our attention to the case $s \in (0, 1]$, we can also prove our result by modifying the proof found in [1] (which we have described above) using the following theorem:

Theorem 3.1.3. *Let v be a solution to (E) such that $\omega(v^0) = \omega^0$ belongs to $UC(\mathbb{R}^2)$. Let $g(t)$ be the measure-preserving homeomorphism in $\mathbb{R}^2$ satisfying*

$\partial_t g(t,x) = v(t, g(t,x))$. Given $\delta > 0$ and $T > 0$, it follows that $||g(t)^{-1} - Id||_{C^{1-\delta}}$ belongs to $L^\infty([0,T])$.

We devote the next section to a proof of Theorem 3.1.3. We prove the main result in Section 3.4.

3.2 A proof of Theorem 3.1.4

An argument using Osgood's Lemma (see, for example, chapter 5 of [6]) implies that the Holder exponent of the flow $g(t) - Id$ is determined by the log-Lipschitz norm of v. Precisely,

$$\sup_{|x-y|\leq 1} |g(t)(x) - g(t)(y)| \leq C|x-y|^{\alpha(t)}, \tag{3.2.1}$$

where $\alpha(t) = \exp(-\int_0^t V(\tau)d\tau)$, and $V(t)$ is defined as in section 3.1. One can characterize log-Lipschitz functions using the following inequality (see [1]):

$$C^{-1}||f||_{LL} \leq ||S_0 f||_{L^\infty} + \sup_{q\geq 1} \frac{||\nabla S_q f||_{L^\infty}}{q+1} \leq C||f||_{LL} \tag{3.2.2}$$

for a constant $C > 0$. When computing the Holder exponent of $g(t) - Id$, (3.2.1) and (3.2.2) motivate us to study the behavior of the quantity $||\nabla S_q v(t)||_{L^\infty}$. In this section, we assume ω^0 belongs to $UC(\mathbb{R}^2)$, and we show that, given $\epsilon > 0$ and $T > 0$, $||\nabla S_q v(t)||_{L^\infty} \leq \epsilon(q+1)$ for sufficiently large q and for all $t \in [0,T]$. We then conclude that $g(t)$ is locally Holder continuous with Holder exponent arbitrarily close to 1. We begin with the following lemma:

Lemma 3.2.1. *Let u be a uniformly continuous function on $\mathbb{R}^2$. Given $\epsilon > 0$, there exists an $N > 0$ such that $||\Delta_j u||_{L^\infty} < \epsilon$ for all $j > N$.*

Proof. Since u is bounded and uniformly continuous, it can be approximated uniformly by the sequence $(S_N u)_{N=1}^{\infty}$. Therefore, for N sufficiently large and for all $j > N$ we have

$$\begin{aligned} ||\Delta_j u||_{L^\infty} &\leq ||\Delta_j (S_N u - u)||_{L^\infty} + ||\Delta_j S_N u||_{L^\infty} \\ &\leq C||S_N u - u||_{L^\infty} < \epsilon. \end{aligned}$$

This completes the proof. □

Observe that by Remark 2.3.1 in Section 2.3.2, the solution to (V) satisfies $\omega(t,x) = \omega^0(g(t)^{-1}(x))$. Therefore, if we assume that ω^0 is uniformly continuous on $\mathbb{R}^2$, then $\omega(t)$ is uniformly continuous on $\mathbb{R}^2$ for all t in $\mathbb{R}$. We now apply Lemma 3.2.1 to $\omega(t)$ and conclude that, for fixed t, given $\epsilon > 0$, there exists N_t such that $\sup_{j>N_t} ||\Delta_j \omega(t)||_{L^\infty} < \epsilon$. In what follows, we need N_t to be time independent. We therefore prove the following lemma:

Lemma 3.2.2. *Let v be a solution to (E) such that $\omega(v^0) = \omega^0$ belongs to $UC(\mathbb{R}^2)$. Given $\epsilon > 0$ and $T > 0$, there exists an $N = N(T,\epsilon)$ such that $\sup_{j>N} ||\Delta_j \omega(t)||_{L^\infty} < \epsilon$ for all $t \in [0,T]$.*

Proof. Observe that, for x_1 and x_2 in $\mathbb{R}^2$ satisfying $g(t_1,x_1) = g(t_2,x_2) = x$, we have,

$$\begin{aligned}|x_1 - x_2| &\leq |t_1 - t_2| \sup_{\tau \in [t_1, t_2]} ||v(\tau)||_{L^\infty} \\ &\leq C|t_1 - t_2| ||\omega^0||_{L^p} e^{C_1 t_2 ||\omega^0||_{L^p}} \\ &\leq C|t_1 - t_2| ||\omega^0||_{L^p} e^{C_1 T ||\omega^0||_{L^p}},\end{aligned}$$

where we used Lemma 2.3.11. Therefore, by uniform continuity of ω^0, given $\epsilon > 0$ there exists $\delta = \delta(T) > 0$ such that for $|t_1 - t_2| < \delta$ and for all $x \in \mathbb{R}^2$,

$$|\omega(t_1, x) - \omega(t_2, x)| = |\omega^0(x_1) - \omega^0(x_2)| < \frac{\epsilon}{2}.$$

We now break $[0, T]$ into intervals of length less than or equal to δ: $[0, t_1]$, $[t_1, t_2]$, ..., $[t_{M-1}, T]$. We observe that, for each $t \in [0, T]$, there exists an i, $1 \leq i \leq M$, such that $|t - t_i| < \delta$, and therefore $||\Delta_j \omega(t)||_{L^\infty} \leq \frac{\epsilon}{2} + ||\Delta_j \omega(t_i)||_{L^\infty}$. Furthermore, given $\omega(t_i)$, there exists an N_{t_i} such that $\sup_{j > N_{t_i}} ||\Delta_j \omega(t_i)||_{L^\infty} < \frac{\epsilon}{2}$. Let $N = \max\{N_{t_1}, ...N_{t_M}\}$. Then, for all $t \in [0, T]$,

$$\sup_{j > N} ||\Delta_j \omega(t)||_{L^\infty} \leq \frac{\epsilon}{2} + \sup_{j > N} ||\Delta_j \omega(t_i)||_{L^\infty} \leq \epsilon.$$

This completes the proof. □

We use Lemma 2.3.8 to bound $||\Delta_j \nabla v||_{L^\infty}$ by $C||\Delta_j \omega||_{L^\infty}$ if $j \geq 0$, and we bound $||\Delta_{-1} \nabla v(t)||_{L^\infty}$ with $C||\omega^0||_{L^p}$ for $p \in (1, \infty)$ using Bernstein's inequality. From Lemma 3.2.2, we conclude that, for N sufficiently large, and for all $t \in [0, T]$,

$$||S_N \nabla v(t)||_{L^\infty} \leq \epsilon(N + 1). \tag{3.2.3}$$

Remark 3.2.1. One can also prove (3.2.3) using Calderon-Zygmund theory. As in Lemma 3.2.1, since ω is bounded and uniformly continuous, it can be approximated uniformly by the sequence of functions $(S_N\omega)_{N=1}^{\infty}$. Because ω is L^p integrable for some $p \in (1, \infty)$, the functions in this sequence vanish at infinity; therefore, ω vanishes at infinity. By the theory of Calderon-Zygmund operators, since ω is continuous, in $L^p(\mathbb{R}^2)$ for some $p \in (1, \infty)$, and vanishes at infinity, ∇v belongs to the space $vmo(\mathbb{R}^2)$ (see, for example, [16]). This implies by Definition 2.2.8 that there exists a sequence of compactly supported C^∞ functions $(u_N)_{N=1}^{\infty}$ converging to ∇v in $bmo(\mathbb{R}^2)$. But $bmo(\mathbb{R}^2)$ imbeds into the Zygmund space $C_*^0(\mathbb{R}^2)$ by Proposition 2.2.1; thus the sequence converges to ∇v in $C_*^0(\mathbb{R}^2)$, and we can write

$$||\Delta_j \nabla v||_{L^\infty} \leq ||\Delta_j(\nabla v - u_n)||_{L^\infty} + ||\Delta_j u_n||_{L^\infty}. \tag{3.2.4}$$

We choose n so that the first term on the right hand side of (3.2.4) is small for all j. Given this n, we choose j large enough to make the second term on the right hand side of (3.2.4) as small as we would like. This gives Lemma 3.2.1, and, following the argument above, implies (3.2.3).

We now use (3.2.3) to compute the Holder exponent for the flow corresponding to the velocity of a fluid with initial vorticity in $UC(\mathbb{R}^2)$, which will complete the proof of Theorem 3.1.3. We show that, given $\epsilon > 0$ and $T > 0$, $g(t)^{-1} - Id$ belongs to $C^{\sigma(t)}(\mathbb{R}^2)$ for all $t \in [0, T]$, where $\sigma(t) = e^{-Ct\epsilon}$, and C is an absolute constant.

Fix $\epsilon > 0$. Write $v = v_{1,N} + v_{2,N}$, where $v_{1,N} = S_{N-1}v$, and $v_{2,N} = (Id - S_{N-1})v$. By (3.2.3), it follows that $|v_{1,N}(t,x) - v_{1,N}(t,y)| \le ||\nabla S_{N-1}v(t)||_{L^\infty}|x - y| \le CN\epsilon|x-y|$ for large enough N. Similarly, for large N we can conclude that $|v_{2,N}(t,x) - v_{2,N}(t,y)| \le C\sum_{j=N-1}^{\infty} 2^{-j}||\Delta_j \nabla v(t)||_{L^\infty} \le C2^{-N}\epsilon$. Letting $N = -\log_2|x-y|$, we have that for $|x-y|$ sufficiently small,

$$\begin{aligned} |v(t,x) - v(t,y)| &\le |(v_{1,N}(t,x) + v_{2,N}(t,x)) - (v_{1,N}(t,y) + v_{2,N}(t,y))| \\ &\le C(\epsilon(-\log_2|x-y|)|x-y| + \epsilon|x-y|) \qquad (3.2.5) \\ &\le C\epsilon(1 - \log_2|x-y|)|x-y|. \end{aligned}$$

We now use (3.2.5) and Osgood's Lemma (see Lemma 2.3.4) to compute properties of the flow. We write

$$|v(t, g(t,x)) - v(t, g(t,y))| \le C\epsilon(1 - \log_2|g(t,x) - g(t,y)|)|g(t,x) - g(t,y)|$$

whenever $|g(t,x) - g(t,y)| < \delta$. From (2.3.6), we see that $|g(t,x) - g(t,y)|$ is bounded above by

$$|x-y| + \int_0^t C\epsilon(1 - \log_2|g(\tau,x) - g(\tau,y)|)|g(\tau,x) - g(\tau,y)|d\tau.$$

By Osgood's Lemma, we conclude that

$$-\log(1 - \log|g(t,x) - g(t,y)|) + \log(1 - \log|x-y|) \le Ct\epsilon.$$

Taking the exponential twice, we get

$$\frac{|g(t,x) - g(t,y)|}{|x-y|^{e^{-Ct\epsilon}}} \le e^{1-e^{-Ct\epsilon}} \le e$$

whenever $|x-y| < \tilde{\delta}$, where $\tilde{\delta} = e^{1-e^{Ct\epsilon}}$. This gives

$$\frac{|(g(t,x) - x) - (g(t,y) - y)|}{|x-y|^{e^{-Ct\epsilon}}} \le e + 1.$$

In the case $|x-y| \geq \tilde{\delta}$, we have

$$\frac{|(g(t,x)-x)-(g(t,y)-y)|}{|x-y|^{e^{-Ct\epsilon}}} \leq 2\tilde{\delta}^{-e^{-Ct\epsilon}}||g(t)-Id||_{L^\infty}.$$

To see that $||g(t)^{-1}-Id||_{C^{e^{-Ct\epsilon}}} \in L^\infty_{loc}(\mathbb{R}^+)$, we observe that

$$\begin{aligned}\sup_{x,y\in\mathbb{R}^2} \frac{|(g(t,x)-x)-(g(t,y)-y)|}{|x-y|^{e^{-Ct\epsilon}}} &\leq (e+1) + 2\tilde{\delta}^{-e^{-Ct\epsilon}}||\int_0^t v(\tau, g(\tau,\cdot))d\tau||_{L^\infty} \\ &\leq (e+1) + 2\tilde{\delta}^{-1}T(||\omega^0||_{L^p} + ||\omega^0||_{L^\infty})\end{aligned}$$

for all $t \in [0,T]$. This completes the proof of Theorem 3.1.3.

3.3 Paradifferential estimates for the vorticity equation

In this section, we consider an initial value problem for the vorticity equation corresponding to the two-dimensional Euler equations. Recall that in dimension two, the vorticity equation is given by

$$(V) \quad \begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \omega|_{t=0} = \omega^0. \end{cases}$$

Note that, if ω satisfies (V), then $\Delta_q \omega$ satisfies the following equation:

$$(V^*) \quad \begin{cases} \partial_t \Delta_q \omega + v \cdot \nabla \Delta_q \omega = [v \cdot \nabla, \Delta_q]\omega, \\ \Delta_q \omega|_{t=0} = \Delta_q \omega^0. \end{cases}$$

We want to prove the following estimate:

Proposition 3.3.1. *Let $p \in (1,\infty)$ and $\sigma > 0$ be fixed. Then there exist two positive constants $C_1(\sigma)$ and C_2 such that*

$$||[v\cdot\nabla, \Delta_q]\omega||_{L^p} \leq C_1(\sigma)(C_2 + ||S_{q-1}\nabla v||_{L^\infty})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.$$

Remark 3.3.1. To simplify notation, we suppress the time variable in the statement and proof of Proposition 3.3.1 (time is fixed throughout the proof). However, we will reintroduce the time variable when we use Proposition 3.3.1 to prove the main theorem in Section 3.4. Therefore, it is important to emphasize that the constants C_1 and C_2 will depend on time in Section 3.4. In the proof of Proposition 3.3.1, we find that C_1 depends on σ, and in Section 3.4 we let σ vary with time. We also find in the proof of Proposition 3.3.1 that $C_2 = C||\omega^0||_{L^p \cap L^\infty} e^{Ct||\omega^0||_{L^p}}$, where C is an absolute constant, and $p \in (2, \infty)$. This dependence arises from bounding $||v(t)||_{L^\infty}$ as in Lemma 2.3.11. (If in fact ω^0 belongs to $L^p(\mathbb{R}^2)$ for some $p < 2$, then it is possible to bound $||v(t)||_{L^\infty}$ by a constant which depends only on initial vorticity and does not depend on time.)

Proof. We consider the cases $q \geq 4$ and $q < 4$ separately. We first assume $q \geq 4$ and use Bony's decomposition to write

$$[v \cdot \nabla, \Delta_q]\omega = \sum_{j=1}^{2} [T_{v^j} \partial_j, \Delta_q]\omega + [T_{\partial_j \cdot} v^j, \Delta_q]\omega + [\partial_j R(v^j, \cdot), \Delta_q]\omega.$$

We address each piece of the sum separately. We start with $[T_{v^j} \partial_j, \Delta_q]\omega$. Write

$$[T_{v^j} \partial_j, \Delta_q]\omega = \sum_{q'=q-4}^{q+4} [S_{q'-1}(v^j), \Delta_q]\Delta_{q'} \partial_j \omega.$$

Letting $u = \Delta_{q'} \partial_j \omega$, letting $h = \check{\phi}$ (recall we are assuming $q \geq 4$ here), and keeping in mind that $|q'-q|\leq 4$, we have

$$
\begin{aligned}
&||[S_{q'-1}(v^j),\Delta_q]u||_{L^p}\\
&\quad = ||\int_{\mathbb{R}^2} h(y)(S_{q'-1}(v^j)(x-2^{-q}y)-S_{q'-1}(v^j)(x))u(x-2^{-q}y)dy||_{L^p}\\
&\quad \leq C||S_{q'-1}\nabla v||_{L^\infty}2^{-q}||u||_{L^p}\\
&\quad \leq C2^{4\sigma}||S_{q'-1}\nabla v||_{L^\infty}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}},
\end{aligned}
$$

where we used the fact that $h\in S$ and therefore $zh(z)$ is integrable, as well as Bernstein's inequality. We now sum over q' to get

$$
\begin{aligned}
||[T_{v^j}\partial_j,\Delta_q]\omega||_{L^p} &\leq C2^{4\sigma}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}\sum_{q'=q-4}^{q+4}||S_{q'-1}\nabla v||_{L^\infty}\\
&\leq C2^{4\sigma}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}(||S_{q-1}\nabla v||_{L^\infty}+||\nabla v||_{C^0_*}).
\end{aligned}
\tag{3.3.1}
$$

We now consider $[T_{\partial_j\cdot}v^j,\Delta_q]\omega$. To bound $||T_{\partial_j\Delta_q\omega}v^j||_{L^p}$, we use Bernstein's inequality and our assumption that $q\geq 4$, as well as properties of our partition of unity, to write

$$
\begin{aligned}
||T_{\partial_j\Delta_q\omega}v^j||_{L^p} &\leq \sum_{q'=q}^{\infty}C2^q2^{-q'}||S_{q'-1}\Delta_q\omega||_{L^p}||\Delta_{q'}\nabla v||_{L^\infty}\\
&\leq C||\Delta_q\omega||_{L^p}\sup_{q'\geq q}||\Delta_{q'}\nabla v||_{L^\infty}\\
&\leq C||\nabla v||_{C^0_*}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.
\end{aligned}
\tag{3.3.2}
$$

Furthermore, since the Fourier support of $S_{q'-1}\partial_j\omega\Delta_{q'}v^j$ is contained in an annulus with inner and outer radius $C2^{q'}$ and $\tilde{C}2^{q'}$ respectively, we can write

$$\begin{aligned} ||\Delta_q(T_{\partial_j\omega}v^j)||_{L^p} &\leq \sum_{q'=q-4}^{q+4} ||S_{q'-1}\partial_j\omega||_{L^\infty}||\Delta_{q'}v||_{L^p} \\ &\leq \sum_{q'=q-4}^{q+4} C2^{q'}2^{-q'}||S_{q'-1}\omega||_{L^\infty}||\Delta_{q'}\nabla v||_{L^p} \\ &\leq \sum_{q'=q-4}^{q+4} C||S_{q'-1}\nabla v||_{L^\infty}||\Delta_{q'}\omega||_{L^p} \\ &\leq C2^{4\sigma}(||S_{q-1}\nabla v||_{L^\infty} + ||\nabla v||_{C^0_*})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \end{aligned} \tag{3.3.3}$$

Once again, we used Bernstein's inequality in the second inequality. It is in this inequality that our assumption that $q \geq 4$ is necessary. For the third inequality, we used Lemma 2.3.8. Combining (3.3.2) and (3.3.3), we see that

$$||[T_{\partial_j\cdot}v^j, \Delta_q]\omega||_{L^p} \leq C2^{4\sigma}(||S_{q-1}\nabla v^j||_{L^\infty} + ||\nabla v||_{C^0_*})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}} \tag{3.3.4}$$

for $q \geq 4$.

We now study the remainder term, $[\partial_j R(v^j, \cdot), \Delta_q]\omega$. We need the following lemma:

Lemma 3.3.2. *If $s + \sigma > 0$, then $||R(a,b)||_{B^{s+\sigma}_{p,\infty}} \leq C(s,\sigma)||a||_{B^s_{\infty,\infty}}||b|||_{B^\sigma_{p,\infty}}$, where $C(s,\sigma) = C2^{M\sigma}(\frac{1}{1-(\frac{1}{2})^{s+\sigma}}+2^{N(s+\sigma)}+...+2^{1(s+\sigma)})$ for fixed positive integers M and N.*

Proof. For a proof of this lemma, see [22]. □

To handle the remainder term, we will consider low and high frequencies of v^j separately. We begin with $[\partial_j R((Id - \Delta_{-1})v^j, \cdot), \Delta_q]\omega$. Using Lemma

3.3.2 with $s = 1$, Bernstein's inequality, and the fact that the Fourier transform of $(Id - \Delta_{-1})\nabla v$ vanishes in a neighborhood of the origin, we write

$$\begin{aligned} ||\Delta_q \partial_j R((Id - \Delta_{-1})v^j, \omega)||_{L^p} &\leq 2^{-q\sigma}||R((Id - \Delta_{-1})v^j, \omega)||_{B^{\sigma+1}_{p,\infty}} \\ &\leq C(\sigma)2^{-q\sigma}||(Id - \Delta_{-1})v||_{B^1_{\infty,\infty}}||\omega||_{B^\sigma_{p,\infty}} \\ &\leq C(\sigma)2^{-q\sigma}||\nabla v||_{C^0_*}||\omega||_{B^\sigma_{p,\infty}}. \end{aligned} \tag{3.3.5}$$

Here $C(\sigma) = C(1, \sigma)$ from Lemma 3.3.2. To bound $||\partial_j R((Id - \Delta_{-1})v^j, \Delta_q \omega)||_{L^p}$, note that

$$\begin{aligned} ||\partial_j R((Id - \Delta_{-1})v^j, \Delta_q \omega)||_{L^p} &\leq \sum_{\substack{q', q'' \\ |q' - q''| \leq 1 \\ |q' - q| \leq 1}} 2^q ||\Delta_{q''}(Id - \Delta_{-1})v||_{L^\infty}||\Delta_{q'}\Delta_q \omega||_{L^p} \\ &\leq C||\nabla v||_{C^0_*}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \end{aligned} \tag{3.3.6}$$

In the first inequality above, we used the fact that the support of the Fourier transform of $\Delta_{q''}(Id - \Delta_{-1})v\Delta_{q'}\Delta_q \omega$ is contained in a ball with radius $C2^q$, along with Bernstein's inequality, to get the factor 2^q. In the second inequality, we used the inequality $||(Id - \Delta_{-1})v||_{B^1_{\infty,\infty}} \leq C||\nabla v||_{C^0_*}$. We now combine (3.3.6) with (3.3.5) to conclude that

$$||[\partial_j R((Id - \Delta_{-1})v^j, \cdot), \Delta_q]\omega||_{L^p} \leq C(\sigma)||\nabla v||_{C^0_*}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \tag{3.3.7}$$

We now estimate $||[\partial_j R(\Delta_{-1}v^j, \cdot), \Delta_q]\omega||_{L^p}$. Using the definition of the remainder operator, as well as the properties of our partition of unity, we write

$$\begin{aligned} [\partial_j R(\Delta_{-1}v^j, \cdot), \Delta_q]\omega &= \partial_j R(\Delta_{-1}v^j, \Delta_q \omega) - \Delta_q(\partial_j R(\Delta_{-1}v^j, \omega)) \\ &= \partial_j (\sum_{\substack{i,k \\ |i-k| \leq 1}} \Delta_k \Delta_{-1}v^j \Delta_i \Delta_q \omega) - \Delta_q \partial_j (\sum_{\substack{i,k \\ |i-k| \leq 1}} \Delta_k \Delta_{-1}v^j \Delta_i \omega). \end{aligned}$$

We begin by estimating $\Delta_q\partial_j(\sum_{\substack{i,k\\|i-k|\leq 1}} \Delta_k\Delta_{-1}v^j\Delta_i\omega)$. We first reintroduce the sum over j, allowing us to use the fact that div $v = 0$ to move ∂_j inside the parentheses and differentiate ω. This, along with properties of our partition of unity, gives

$$\begin{aligned}
&\sum_{j=1}^{2} ||\Delta_q\partial_j(\sum_{\substack{i,k\\|i-k|\leq 1}} \Delta_k\Delta_{-1}v^j\Delta_i\omega)||_{L^p} \\
&\qquad\qquad \leq C\sum_{j=1}^{2}\sum_{l=-1}^{1}\sum_{k=-1}^{0} ||\Delta_q(\Delta_k\Delta_{-1}v^j\Delta_{k-l}\partial_j\omega)||_{L^p}.
\end{aligned} \tag{3.3.8}$$

Note that in the second line of (3.3.8), the Fourier support of $\Delta_k\Delta_{-1}v^j\Delta_{k-l}\partial_j\omega$ is contained in a ball with radius $C2^k$. Therefore, the sum in the second line is 0 if $q \geq k + M$, for a constant M. Furthermore, $k \leq 0$. Therefore, we are only considering $q \leq M$. We then write

$$\begin{aligned}
&\sum_{j=1}^{2}\sum_{l=-1}^{1}\sum_{k=-1}^{0} ||\Delta_q(\Delta_k\Delta_{-1}v^j\Delta_{k-l}\partial_j\omega)||_{L^p} \\
&\leq C2^{M\sigma}||v||_{L^\infty}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.
\end{aligned} \tag{3.3.9}$$

To bound $||\partial_j(\sum_{\substack{i,k\\|i-k|\leq 1}} \Delta_k\Delta_{-1}v^j\Delta_i\Delta_q\omega)||_{L^p}$, we use the fact that the Fourier transform of $\Delta_{-1}v^j$ is supported in the neighborhood of the origin, and we recognize that div $v = 0$ allows us to move ∂_j inside the parentheses. Therefore,

$$\begin{aligned}
&\sum_{j=1}^{2} ||\partial_j(\sum_{\substack{i,k\\|i-k|\leq 1}} \Delta_k\Delta_{-1}v^j\Delta_i\Delta_q\omega)||_{L^p} \\
&\leq C\sum_{j=1}^{2}\sum_{l=-1}^{1}\sum_{k=-1}^{0} ||\Delta_k\Delta_{-1}v||_{L^\infty}2^{k-l}||\Delta_{k-l}\Delta_q\omega||_{L^p} \\
&\leq C||v||_{L^\infty}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}},
\end{aligned} \tag{3.3.10}$$

where we used Bernstein's inequality and Holder's inequality to get the first inequality in (3.3.10). We now combine (3.3.8) through (3.3.10) to conclude that

$$||[\partial_j R(\Delta_{-1}v^j, \cdot), \Delta_q]\omega||_{L^p} \leq C2^{M\sigma}||v||_{L^\infty}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \tag{3.3.11}$$

Combining (3.3.1), (3.3.4), (3.3.7), and (3.3.11), we conclude that for $q \geq 4$,

$$\begin{aligned}||[v\cdot\nabla, \Delta_q]\omega||_{L_p} \leq C(2^{M\sigma} + C(\sigma))(||S_{q-1}\nabla v^j||_{L^\infty} \\ + ||\nabla v||_{C^0_*} + ||v||_{L^\infty})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.\end{aligned}$$

To complete the proof for the case $q \geq 4$, we use Lemma 2.3.11 to bound $||\nabla v||_{C^0_*} + ||v||_{L^\infty}$ by C_2, where C_2 is defined as in Remark 3.3.1. This completes the proof for the case $q \geq 4$.

For the case $q < 4$, write:

$$[v\cdot\nabla, \Delta_q]\omega = v\cdot\nabla\Delta_q\omega - \Delta_q(v\cdot\nabla\omega). \tag{3.3.12}$$

Keeping in mind that $q \leq 3$, it is easy to see that

$$||v\cdot\nabla\Delta_q\omega||_{L^p} \leq C_2 2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}, \tag{3.3.13}$$

where again we used Lemma 2.3.11 to bound $||v||_{L^\infty}$. We now write the second term of (3.3.12) as

$$\Delta_q(v\cdot\nabla\omega) = \sum_{j=1}^{2}\Delta_q(T_{v^j}\partial_j\omega + T_{\partial_j\omega}v^j + R(v^j, \partial_j\omega)). \tag{3.3.14}$$

We successfully bounded the L^p-norm of the remainder term of (3.3.14) in (3.3.5), (3.3.8), and (3.3.9) of the proof for the $q \geq 4$ case (note that (3.3.5), (3.3.8), and (3.3.9) hold for all q). Therefore, we are only concerned with $\sum_{j=1}^{2} \Delta_q(T_{v^j}\partial_j\omega + T_{\partial_j\omega}v^j)$. Using the fact that $S_{q'-1}v^j\Delta_{q'}\partial_j\omega$ has Fourier support in an annulus with inner radius $C2^{q'}$ and outer radius $\tilde{C}2^{q'}$, and, once again, keeping in mind that $q \leq 3$, we have

$$\begin{aligned} ||\Delta_q(T_{v^j}\partial_j\omega)||_{L^p} &\leq \sum_{q'=q-4}^{q+4} C||S_{q'-1}v||_{L^\infty}2^{q'}||\Delta_{q'}\omega||_{L^p} \\ &\leq C_2 2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \end{aligned} \tag{3.3.15}$$

Furthermore, since $q \leq 3$, we write

$$\begin{aligned} ||\Delta_q(T_{\partial_j\omega}v^j)||_{L^p} &\leq \sum_{q'=q-4}^{q+4} ||S_{q'-1}\partial_j\omega\Delta_{q'}v^j||_{L^p} \\ &\leq C\sum_{q'=q-4}^{q+4}\sum_{k=-1}^{q'-2} 2^{-k\sigma}||\omega||_{B^\sigma_{p,\infty}}||v||_{L^\infty} \\ &\leq C_2 2^{3\sigma}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \end{aligned}$$

This completes the proof of the case $q < 4$, and therefore completes the proof of the estimate for all q. We conclude that, for all q,

$$||[v\cdot\nabla, \Delta_q]\omega||_{L_p} \leq C_1(\sigma)(||S_{q-1}\nabla v^j||_{L^\infty} + C_2)2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.$$

□

If we define $h(t,x) = g(t)^{-1}(x) - x$, then h satisfies the following:

$$\begin{aligned} &\partial_t h + v\cdot\nabla h + v = 0, \\ &h|_{t=0} = 0. \end{aligned} \tag{3.3.16}$$

Since h satisfies (3.3.16), it also satisfies

$$\begin{aligned} &\partial_t\Delta_q h + v\cdot\nabla\Delta_q h = -\Delta_q v + [v\cdot\nabla, \Delta_q]h, \\ &\Delta_q h|_{t=0} = 0. \end{aligned} \tag{3.3.17}$$

This motivates us to prove a similar commutator estimate with h in place of ω. We prove that the following estimate holds:

Proposition 3.3.3. *Let $p \in (1, \infty)$, $\delta > 0$, and $\sigma > 0$ be fixed. There exist two positive constants $C_1(\sigma)$ and C_2 such that*

$$||[v \cdot \nabla, \Delta_q]h||_{L_p} \leq C_1(\sigma)(C_2 + ||S_{q-1}\nabla v||_{L^\infty})2^{-q\sigma}||h||_{B^\sigma_{p,\infty}}$$
$$+ \sum_{q'=q-4}^{q+4} C(\delta)2^{q'\delta}||\Delta_{q'}v||_{L^p}||h||_{C^{1-\delta}}.$$

Proof. The proof of Proposition 3.3.3 is identical to the proof of Proposition 3.3.1 with h in place of ω for every term except $\Delta_q(T_{\partial_j h}v^j)$. Therefore, we restrict our attention to this term. This portion of the proof will result in the second piece on the right-hand side in Proposition 3.3.3.

Note that, in the proof of Proposition 3.3.1, we use the assumption that $q \geq 4$ only when bounding $||\Delta_q(T_{\partial_j \omega}v^j)||_{L^p}$. For all other terms, $q \geq 0$ suffices. This observation, combined with the fact that we will only need to assume $q \geq 0$ to bound $||\Delta_q(T_{\partial_j h}v^j)||_{L^p}$, leads us to consider the cases $q = -1$ and $q \geq 0$ separately.

We first assume $q \geq 0$, and we write

$$\Delta_q(T_{\partial_j h}v^j) = \Delta_q(\sum_{q'=1}^{\infty} S_{q'-1}\partial_j h \Delta_{q'}v^j). \tag{3.3.18}$$

Using the fact that $S_{q'-1}\partial_j h\Delta_{q'}v^j$ has Fourier support in an annulus with inner radius $C2^{q'}$ and outer radius $\tilde{C}2^{q'}$, we apply Bernstein's inequality and Holder's inequality to get

$$\begin{aligned}||\Delta_q(T_{\partial_j h} v^j)||_{L^p} &\leq \sum_{q'=q-4}^{q+4} \sum_{k=-1}^{q'-2} 2^{k\delta} 2^{k(1-\delta)} ||\Delta_k h||_{L^\infty} ||\Delta_{q'} v||_{L^p} \\ &\leq \sum_{q'=q-4}^{q+4} C(\delta) 2^{q'\delta} ||h||_{C^{1-\delta}} ||\Delta_{q'} v||_{L^p}.\end{aligned} \tag{3.3.19}$$

We now consider the case $q = -1$. As in the proof of Proposition 3.3.1 when assuming $q < 4$, we begin by writing:

$$[v \cdot \nabla, \Delta_q] h = v \cdot \nabla \Delta_q h - \Delta_q (v \cdot \nabla h). \tag{3.3.20}$$

For the first term of (3.3.20), we use the fact that $q = -1$ to get

$$||v \cdot \nabla \Delta_q h||_{L^p} \leq C(||\omega^0||_{L_0^p} + ||\omega^0||_{L^\infty}) 2^{-q\sigma} ||h||_{B^\sigma_{p,\infty}}, \tag{3.3.21}$$

for $p \in (1, 2)$. The second term of (3.3.20) can be written as

$$\Delta_q(v \cdot \nabla h) = \sum_{j=1}^{2} \Delta_q (T_{v^j} \partial_j h + T_{\partial_j h} v^j + R(v^j, \partial_j h)). \tag{3.3.22}$$

The proof of the bound on the L^p-norm of the remainder term in (3.3.22) is identical to the proofs of (3.3.5), (3.3.8), and (3.3.9), (which hold for all q), with h in place of ω. Furthermore, we handled the L^p-norm of $\Delta_q(T_{\partial_j h} v^j)$ (see (3.3.19), which also holds for all q). Therefore, in (3.3.22), we are only concerned with $\Delta_q(T_{v^j} \partial_j h)$. For this term, we refer the reader to the proof for ω with $q < 4$, given in (3.3.15). This completes the proof of the case $q = -1$, and therefore completes the proof of the estimate for all q. □

3.4 Statement and proof of the main results

In this section we study an initial value problem for ideal incompressible fluids with uniformly continuous vorticity in a Besov space.

3.4.1 Besov regularity of the vorticity

We prove the following theorem:

Theorem 3.4.1. *Let $v^0 \in B^{s+1}_{p,\infty}(\mathbb{R}^2)$, div $v^0 = 0$, and let $\omega(v^0) = \omega^0 \in UC(\mathbb{R}^2)$, where $sp \leq 2$, $s \in (0,2)$, and $p \in (1,\infty)$. Let $\epsilon > 0$. There exists a unique solution to (E) such that $||v(t)||_{B^{s+1-\epsilon}_{p,\infty}}$ belongs to $L^\infty_{loc}(\mathbb{R}^+)$.*

Remark 3.4.1. Uniqueness in Theorem 3.4.1 follows from [23].

Proof. Our approach is as follows: we fix $\epsilon' > 0$ and $T > 0$, and we define $\sigma(t) = s\exp\{-\frac{C}{s}\epsilon' t\}$, where C is an absolute constant. We then show that $||v(t)||_{B^{\sigma(t)+1}_{p,\infty}} \in L^\infty([0,T])$. Letting $\epsilon = s - s\exp\{-\frac{C}{s}\epsilon' T\}$, we make ϵ as small as we would like by our choice of ϵ'.

We first prove the theorem on a sufficiently small time interval $[t_0, t]$. We then use a bootstrapping argument to show that the theorem holds on any finite time interval $[0,T]$. From (V^*) and Proposition 3.3.1, it follows that

$$\begin{aligned}||\Delta_q\omega(t)||_{L^p} &\leq ||\Delta_q\omega(t_0)||_{L^p} \\ &+ \int_{t_0}^t C_1(\sigma(\tau))(Ce^{C\tau} + ||S_{q-1}\nabla v(\tau)||_{L^\infty})2^{-q\sigma(\tau)}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}}d\tau.\end{aligned}$$

We used Remark 3.3.1 to replace $C_2(t)$ with Ce^{Ct} in the above inequality, where C depends only on the initial vorticity. Moreover, we see from the proof of Proposition 3.3.1 that $C_1(\sigma(t))$ can be bounded by an absolute constant for all $\sigma(t) \in (0,2)$. Therefore, for the remainder of the proof, we drop the dependence of C_1 on $\sigma(t)$. We multiply both sides of the equation by $2^{q\sigma(t)}$ and take the supremum over q to get

$$\begin{aligned}||\omega(t)||_{B^{\sigma(t)}_{p,\infty}} &\leq ||\omega(t_0)||_{B^{\sigma(t_0)}_{p,\infty}} \\ &+ \sup_q \{\int_{t_0}^t C_1(e^{C\tau} + ||S_{q-1}\nabla v(\tau)||_{L^\infty})2^{q\sigma(t)-q\sigma(\tau)}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau\}.\end{aligned}$$

We now show that the supremum over q on the right-hand side is finite. We claim that the loss of regularity in the Besov exponent, resulting in the term $2^{q(\sigma(t)-\sigma(\tau))}$, is enough to combat the growth of $||S_{q-1}\nabla v(\tau)||_{L^\infty}$.

When taking the supremum over q of the time integral, we consider two cases separately: the supremum over $q \leq N$, and the supremum over $q > N$. We then use (3.2.3) to handle the supremum over $q > N$. We write

$$||\omega(t)||_{B^{\sigma(t)}_{p,\infty}} \leq ||\omega(t_0)||_{B^{\sigma(t_0)}_{p,\infty}} + I_1 + I_2,$$

where

$$I_1 = \sup_{q\leq N}\{\int_{t_0}^t C_1(e^{C\tau} + ||S_{q-1}\nabla v(\tau)||_{L^\infty})2^{q\sigma(t)-q\sigma(\tau)}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau\},$$

and

$$I_2 = \sup_{q> N}\{\int_{t_0}^t C_1(e^{C\tau} + ||S_{q-1}\nabla v(\tau)||_{L^\infty})2^{q\sigma(t)-q\sigma(\tau)}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau\}.$$

To bound I_2, we apply (3.2.3), and we conclude that

$$I_2 \leq \int_{t_0}^t Ce^{C\tau}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau + \sup_{q>N}\{\int_{t_0}^t C_1\epsilon q 2^{q\sigma(t)-q\sigma(\tau)}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau\}. \quad (3.4.1)$$

To handle the second integral in (3.4.1), we integrate by parts. Letting $\sigma(t) = s\exp(-\frac{2C_1}{s}\epsilon t)$ (this is the definition of $\sigma(t)$ with $C = 2C_1$), we let $u = e^{\frac{2C_1}{s}\epsilon\tau}$ and $dv = C_1\epsilon q e^{-\frac{2C_1}{s}\epsilon\tau}2^{q\sigma(t)-q\sigma(\tau)}d\tau$. Then, substituting u and dv into the second integral in (3.4.1), and recognizing that du and v are positive for all $\tau \in [t_0, t]$, we write

$$\begin{aligned}\sup_{q>N}\{\sup_{\tau\in[t_0,t]}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}}\int_{t_0}^{t} udv\} &\leq \sup_{q>N}\{\sup_{\tau\in[t_0,t]}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}}(uv|_{t_0}^{t})\} \\ &= \sup_{\tau\in[t_0,t]}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}}e^{\frac{2C_1}{s}\epsilon t}\frac{1}{2\ln 2}.\end{aligned} \tag{3.4.2}$$

We now bound the first time integral on the right-hand side of (3.4.1) by

$$Ce^{Ct}\sup_{\tau\in[t_0,t]}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}}(t-t_0). \tag{3.4.3}$$

Combining (3.4.3) with (3.4.2), we conclude that

$$I_2 \leq \sup_{\tau\in[t_0,t]}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}}\{e^{\frac{C}{s}\epsilon t}\frac{1}{2\ln 2} + Ce^{Ct}(t-t_0)\}. \tag{3.4.4}$$

To bound I_1, we first observe that $||S_{q-1}\nabla v(\tau)||_{L^\infty} \leq q||\nabla v(\tau)||_{C^0_*}$. Then, bounding $||\nabla v(\tau)||_{C^0_*}$ by $C(||\omega^0||_{L^\infty} + ||\omega^0||_{L^{p_0}})$, for $p_0 \in (1,\infty)$, and recognizing that $2^{q(\sigma(t)-\sigma(\tau))} \leq 1$ for all q, we conclude that

$$I_1 \leq CNe^{Ct}(t-t_0)\sup_{\tau\in[t_0,t]}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}}. \tag{3.4.5}$$

We now combine our estimates for I_1 and I_2 given in (3.4.4) and (3.4.5), which gives

$$\sup_{\tau\in[t_0,t]}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} \leq ||\omega(t_0)||_{B^{\sigma(t_0)}_{p,\infty}} + C^*\sup_{\tau\in[t_0,t]}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}},$$

where we let

$$C^* = e^{\frac{C}{s}\epsilon t}\frac{1}{2\ln 2} + CNe^{Ct}(t-t_0). \tag{3.4.6}$$

To complete the proof, we must make the constant $C^* < 1$. Fix $t > 0$. Given this t, choose $\epsilon > 0$ small enough to ensure that $\frac{e^{\frac{C}{s}\epsilon t}}{2\ln 2} < 1$. Depending on our choice of t and ϵ, $N = N(t, \epsilon)$ may be very large. Given this N, make $t - t_0$ small enough so that $C^* < 1$. Note that, under these assumptions, $C^* < 1$ when we are working on an interval of length less than or equal to $t - t_0$, as long as the right endpoint of the interval is less than or equal to t. We therefore break $[0, t]$ into a finite number $M = M(t, \epsilon)$ of intervals of length $t - t_0$, and we apply a bootstrapping argument. This gives

$$\sup_{\tau \in [0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} \leq C^M ||\omega^0||_{B^s_{p,\infty}},$$

where $C = \frac{1}{1-C^*}$, and C^M depends on t and ϵ. More precisely, larger initial choice of t and smaller choice of ϵ result in larger M and thus larger C^M.

This completes the proof for regularity of vorticity. To show that this implies regularity of the velocity, we need the following estimate:

Lemma 3.4.2. *Let $v^0 \in B^{s+1}_{p,\infty}(\mathbb{R}^2)$. Then there exists two positive constants C_0 and C_1 such that*

$$||v(t)||_{B^{\sigma(t)+1}_{p,\infty}} \leq C_0 e^{C_1 t} + ||\omega(t)||_{B^{\sigma(t)}_{p,\infty}}.$$

Proof. We follow the proof of Lemma 6.2 in [20]. We begin by writing

$$\begin{aligned} ||v(t)||_{B^{\sigma(t)+1}_{p,\infty}} &\leq ||\Delta_{-1} v(t)||_{B^{\sigma(t)+1}_{p,\infty}} + ||(Id - \Delta_{-1})\Delta_j v(t)||_{B^{\sigma(t)+1}_{p,\infty}} \\ &\leq ||\Delta_{-1} v(t)||_{L^p} + ||\omega(t)||_{B^{\sigma(t)}_{p,\infty}}, \end{aligned} \tag{3.4.7}$$

where we used Lemma 2.3.8 in Section 2.3.3 on the high frequency term. We bound $||\Delta_{-1} v(t)||_{L^p}$ as we did in Lemma 2.3.10. This gives the lemma. □

This completes the proof of Theorem 3.4.1. □

3.4.2 Besov regularity of the flow

In this section, we prove the following theorem:

Theorem 3.4.3. *Let v^0 and ω^0 satisfy the assumptions of Theorem 3.4.1. Let $g(t,x)$ be the measure-preserving homeomorphism satisfying $\partial_t g(t,x) = v(t, g(t,x))$. Define $h(t,x) = g(t)^{-1}(x) - x$. Then, for fixed $\delta > 0$, $||h(t)||_{B^{s+1-\delta}_{p,\infty}}$ belongs to $L^\infty_{loc}(\mathbb{R}^+)$.*

Proof. We show that $h(t) \in B^{\sigma'(t)}_{p,\infty}(\mathbb{R}^2)$, where $\sigma'(t) = \sigma(t) + 1 - \delta$, and δ is the Holder exponent of $h(t)$ (see Theorem 3.1.3). The proof of Theorem 3.4.3 is similar to that for Theorem 3.4.1. However, we must deal with the extra term which shows up in the commutator estimate given in Proposition 3.3.3.

We begin by applying Proposition 3.3.3 to (3.3.17), where, once again, we drop the dependence of C_1 on σ', and we introduce the dependence of C_2 on time, as given in Remark 3.3.1. This gives

$$||\Delta_q h(t)||_{L^p} \leq ||\Delta_q h(t_0)||_{L^p} + \int_{t_0}^t C_1(e^{C\tau} + ||S_{q-1}\nabla v(\tau)||_{L^\infty})2^{-q\sigma'(\tau)}||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} d\tau$$
$$+ \int_{t_0}^t \{ \sum_{q'=q-4}^{q+4} (||h(\tau)||_{C^{1-\delta}} ||\Delta_{q'} v(\tau)||_{L^p} C(\delta) 2^{q'\delta}) + ||\Delta_q v(\tau)||_{L^p} \} d\tau.$$

We now multiply both sides of the inequality by $2^{q\sigma'(t)}$ and take the supremum over q to bound $||h(t)||_{B^{\sigma'(t)}_{p,\infty}}$ above by

$$||h(t_0)||_{B^{\sigma'(t_0)}_{p,\infty}} + \sup_q \{ \int_{t_0}^{t} C_1 (e^{C\tau} + ||S_{q-1}\nabla v(\tau)||_{L^\infty}) 2^{q(\sigma'(t)-\sigma'(\tau))} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} d\tau \}$$
$$+ \int_{t_0}^{t} \{ C(\delta) ||h(\tau)||_{C^{1-\delta}} ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} + ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} \} d\tau.$$

Here we used the fact that $\sigma'(t) = \sigma(t) + 1 - \delta$, with $\delta > 0$. The constant $C(\delta)$ now depends on $\sigma(\tau)$, but it is uniformly bounded for all $\sigma(\tau) \in (0, 2)$.

We replace $\sigma'(t) - \sigma'(\tau)$ with $\sigma(t) - \sigma(\tau)$ in the first time integral, and we get

$$||h(t)||_{B^{\sigma'(t)}_{p,\infty}} \leq ||h(t_0)||_{B^{\sigma'(t_0)}_{p,\infty}} + J_1 + J_2,$$

where

$$J_1 = \sup_q \{ \int_{t_0}^{t} C_1 (e^{C\tau} + ||S_{q-1}\nabla v(\tau)||_{L^\infty}) 2^{q(\sigma(t)-\sigma(\tau))} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} d\tau \},$$

and

$$J_2 = \int_{t_0}^{t} \{ C(\delta) ||h(\tau)||_{C^{1-\delta}} ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} + ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} \} d\tau.$$

The argument for dealing with J_1 is identical to the argument we used to handle I_1 and I_2 when proving the first part of Theorem 3.4.1. Following this approach, we conclude that

$$\sup_{\tau \in [t_0,t]} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} \leq ||h(t_0)||_{B^{\sigma'(t_0)}_{p,\infty}} + C^* \sup_{\tau \in [t_0,t]} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} + J_2,$$

where C^* is given by (3.4.6). Arguing as we did with ω, we make $C^* < 1$ on a sufficiently short time interval and use a bootstrapping argument, as well as the fact that $h(0,x)=0$, to conclude that

$$\sup_{\tau\in[0,t]} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} \leq C\int_0^t \{C(\delta)||h(\tau)||_{C^{1-\delta}}||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} + ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}}\}d\tau. \tag{3.4.8}$$

We now observe that the right hand side of (3.4.8) is finite by Theorem 3.1.3 and by the first part of Theorem 3.4.1. This completes the proof of the theorem. □

Chapter 4

Vanishing viscosity in the plane

4.1 Introduction and history

The problem of vanishing viscosity addresses whether or not a solution of the Navier-Stokes equations with initial data v^0 converges in some norm to a solution of the Euler equations with the same initial data as viscosity tends to 0. This area of research is closely tied to uniqueness of solutions to the Euler equations, primarily because the methods used to prove uniqueness can be applied to show vanishing viscosity. Two of the most important uniqueness results in the plane are due to Yudovich. He proves in [23] the uniqueness of a solution v to the Euler equations with $v^0 \in L^2(\mathbb{R}^2)$ and $\omega^0 \in L^p(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for some $p < \infty$. He extends this uniqueness class in [24] to solutions with unbounded vorticity, but such that the L^p-norms of the vorticity grow sufficiently slowly (for example, like $\log p$).

For Yudovich's uniqueness class with bounded vorticity, Chemin proves in [5] that the vanishing viscosity limit holds in the energy norm, and he establishes a rate of convergence:

Theorem 4.1.1. *Let $v^0 \in L^2(\mathbb{R}^2)$. Assume v_ν is the unique solution to (NS) with initial data v^0, and v is the solution to (E) with initial data v^0. Also* *assume that* $\omega(v^0) = \omega^0$ *belongs to* $L^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. *Then* v_ν *converges to* v *in* $L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2))$ *as* ν *tends to zero. Specifically,*

$$||v_\nu - v||_{L^\infty([0,T];L^2)} \leq C(4\nu T)^{\frac{1}{2}\exp(-C||\omega^0||_{L^2\cap L^\infty}T)}.$$

In [8], Hmidi and Keraani show that the vanishing viscosity limit holds in the $B^0_{\infty,1}$-norm when v^0 belongs to $B^1_{\infty,1}(\mathbb{R}^2)$. (Global well-posedness for the Euler equations under these assumptions was proved by Vishik in [20].) They prove the following theorem:

Theorem 4.1.2. *Assume* v^0 *belongs to* $B^1_{\infty,1}(\mathbb{R}^2)$, *and div* $v = 0$. *Let* v_ν *be the solution to (NS) with initial data* v^0, *and let* v *be the solution to (E) with initial data* v^0. *Then the following inequality holds:*

$$\|v_\nu - v\|_{L^\infty([0,T];B^0_{\infty,1}(\mathbb{R}^2))} \leq c(\nu T)^{1/2}(1+\nu T)^{1/2}e^{e^{cT}},$$

where the constants depend on the $B^0_{\infty,1}$*-norm of* ω^0.

4.2 Statement of the main results

We consider a uniqueness class in which initial vorticity is in a larger space than $B^0_{\infty,1}(\mathbb{R}^2)$. This uniqueness class was established by Vishik in [21]. As in [21], let $\Gamma : \mathbb{R} \to [1, \infty)$ be a locally Lipschitz continuous monotonically nondecreasing function that satisfies conditions (i)-(iii) p. 771 of [21]. Condition (i) is that $\Gamma = 1$ on the interval $(-\infty, -1]$ and $\lim_{\beta\to\infty}\Gamma(\beta) = \infty$. For the other (minor technical) conditions see [21]. Define the space

$$B_\Gamma = \{f \in \mathcal{S}'(\mathbb{R}^2) : \sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty} = O(\Gamma(N))\}$$

with the norm

$$\|f\|_\Gamma = \sup_{N \geq -1} \frac{1}{\Gamma(N)} \sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty}.$$

The following fundamental result for initial vorticities in B_Γ is from Theorems 7.1 and 8.1 of [21]:

Theorem 4.2.1. *Define $\Gamma_1 : \mathbb{R} \to [1, \infty)$ by*

$$\Gamma_1(\beta) = \begin{cases} 1, & \beta < -1, \\ (\beta+2)\Gamma(\beta), & \beta \geq -1 \end{cases}$$

and add the assumption (on Γ) that Γ_1 is convex. Finally, assume that Γ satisfies

$$(\beta + 2)\Gamma'(\beta) \leq C \tag{4.2.1}$$

for almost all $\beta \in [-1, \infty)$. Given initial vorticity ω^0 in $B_\Gamma \cap L^{p_0} \cap L^{p_1}$ with $1 < p_0 < 2 < p_1 \leq \infty$ there exists a short-time solution to (E) unique in the class of vorticities lying in $L^\infty([0,T]; L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0,T]; B_{\Gamma_1})$. With the added assumption that

$$\Gamma'(\beta)\Gamma_1(\beta) \leq C \tag{4.2.2}$$

for almost all $\beta \geq -1$, there exists a solution to (E) unique in the class of vorticities lying in $L^\infty_{loc}([0,\infty); L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0,\infty); B_{\Gamma_1})$. Here, C_{w_*} is the space of weak-continuous functions (see [21] for details).*

Observe that the vorticity degrades immediately in that (as far as is known) it belongs to a larger space at all positive times than it does at time zero.

Remark 4.2.1. In Theorem 4.2.2, Corollary 4.2.3, and Corollary 4.2.4 below, for the case where $\lim_{n\to\infty}\Gamma(n)=\infty$, the symbol C represents an unspecified *absolute* constant (that is, independent of the initial data). For the case where $\Gamma(n)$ is bounded in n, the constant C depends on both the L^2-norm and the $B^0_{\infty,1}$-norm of initial vorticity. This dependence arises in Equation (4.3.6) below.

We now state our main result.

Theorem 4.2.2. *Let $\Gamma:\mathbb{R}\to[0,\infty)$ (without making any of the assumptions on Γ of [21]) and assume that v^0 is in $L^2(\mathbb{R}^2)$ with $\omega^0=\omega(u^0)$ in $B_\Gamma(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$. Then there exists a unique solution v_ν to (NS) and a (not necessarily unique) solution v to (E), both lying in $L^\infty([0,\infty);H^1(\mathbb{R}^2))$. For any such v,*

$$\|v_\nu-v\|_{L^\infty([0,T];L^2(\mathbb{R}^2))}\le C(\nu T)^{1/2}\|\omega^0\|_{L^2}\exp\left(e^{C\alpha T\Gamma(-\log(\nu T)/2)}\right)\qquad(4.2.3)$$

for all $T>0$, where $\alpha=\|\omega^0\|_{B_\Gamma}$.

Proof. The existence of a global-in-time solution to (E) with vorticity in $L^\infty([0,\infty);L^p(\mathbb{R}^2))$ for ω^0 in $L^p(\mathbb{R}^2)$, $p>1$, is due to Yudovich in [23] (see, for instance, Theorem 4.1 p. 126 of [14]). The existence and uniqueness of solutions to (NS) lying in $L^\infty([0,T];L^2(\mathbb{R}^2))\cap L^2([0,T];L^2(\mathbb{R}^2))$ for v^0 in $L^2(\mathbb{R}^2)$

is classical (see, for instance, Theorems III.3.1 and III.3.2 of [17]). Because our solutions to (NS) are in the whole plane, all L^p-norms of the vorticity are non-increasing, so, in fact, v_ν lies in $L^\infty([0,\infty); H^1(\mathbb{R}^2))$.

The proof of Equation (4.2.3) is contained in the sections that follow.

□

It is possible to loosen the finite energy requirement in Theorem 4.2.2 that v^0 lie in $L^2(\mathbb{R}^2)$, allowing it to lie, for instance, in the space E_m of [6].

Without restrictions on Γ it is of course possible that the right-hand side of Equation (4.2.3) will not go to zero with ν. In order to establish the vanishing viscosity limit, $\Gamma(n)$ cannot grow any faster than $C \log n$. We have the following immediate corollary of Theorem 4.2.2:

Corollary 4.2.3. *When $\Gamma(n) = O(\log n)$, $v_\nu \to v$ in $L^\infty([0,T]; L^2(\mathbb{R}^2))$ for $T < (C\alpha)^{-1}$, with*

$$\begin{aligned} &\|v_\nu - v\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \\ &\qquad \le C\|\omega^0\|_{L^2}(\nu T)^{1/2} \exp\left(\left(-\frac{1}{2}\log(\nu T)\right)^{C\alpha T}\right). \end{aligned} \tag{4.2.4}$$

In Corollary 4.2.4, we extend the class of solutions for which both existence and uniqueness of solutions to (E) can be demonstrated globally in time. Note that we obtain uniqueness in Corollary 4.2.4 in spite of lacking knowledge of whether the solution to (E) remains in the class $L^\infty([0,T]; B_{\Gamma_1}(\mathbb{R}^2))$ for arbitrarily large T, this being (almost) the class for which Vishik demonstrates uniqueness in [21] (see the comment on p. 771 of [21]).

Corollary 4.2.4. *When* $\Gamma(n) = O(\log^\kappa n)$ *with* $0 \le \kappa < 1$*, the solution* v *to (E) is unique in* $L^\infty([0,\infty); H^1(\mathbb{R}^2))$*. Also,* $v_\nu \to v$ *in* $L^\infty_{loc}([0,\infty); L^2(\mathbb{R}^2))$*, and for all* $T > 0$*,*

$$\|v_\nu - v\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \le C(\nu T)^{1/2}\|\omega^0\|_{L^2} \exp\left(e^{C\alpha T \log^\kappa(-\log(\nu T)/2)}\right). \quad (4.2.5)$$

Proof. The rate in Equation (4.2.5) follows immediately from Theorem 4.2.2. By Equation (4.2.5), any solution v to (E) lying in $L^\infty([0,\infty); H^1(\mathbb{R}^2))$ is the strong limit in $L^\infty_{loc}([0,\infty); L^2(\mathbb{R}^2))$ of the solutions v_ν to (NS); since strong limits are unique, we conclude that the solution v is unique. □

In Corollary 4.2.4, one can show that a solution to (E) in $L^\infty([0,\infty); H^1(\mathbb{R}^2))$ is unique without using the vanishing viscosity limit. Indeed, given a solution v to (E) with initial data v^0, we construct in the proof of Theorem 4.2.2 a sequence of $C^\infty(\mathbb{R}^2)$ solutions v_n to (E) with initial data $S_n v^0$. We then show that $\omega^0 \in B_\Gamma(\mathbb{R}^2)$ implies $||v_n - v||_{L^\infty([0,T];L^2(\mathbb{R}^2))}$ goes to 0 as n approaches infinity (see Equation (4.3.3) and Equation (4.3.7) in the sections that follow), where Γ satisfies the conditions in Corollary 4.2.4. Since the sequence v_n is uniquely determined by the initial data v^0, two solutions to (E) with the same initial data and initial vorticity in $B_\Gamma(\mathbb{R}^2)$ will have the same approximating sequence and will therefore be equal on $[0,T]$.

The restriction Equation (4.2.1) on Γ ensures that $\Gamma(N)$ grows no faster than $C \log N$ for large N. Therefore, Corollaries 4.2.3 and 4.2.4 establish a rate of convergence for the entire short time existence and uniqueness class in

[21]. Similarly, the assumption Equation (4.2.2) on Γ ensures that $\Gamma(N)$ grows no faster than $C\log^{\frac{1}{2}} N$ for large N. Therefore, Corollary 4.2.4 establishes a rate of convergence for the entire global existence and uniqueness class in [21] as well.

4.2.1 A comparison of rates of convergence

Note that when $\kappa = 0$, our result reduces to the case where $\omega^0 \in B^0_{\infty,1}(\mathbb{R}^2)$, and our rate of convergence becomes

$$\|v_\nu - v\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \le C(\nu T)^{1/2}\|\omega^0\|_{L^2} \exp\left(e^{C\alpha T}\right). \tag{4.2.6}$$

Since $B^0_{\infty,1}(\mathbb{R}^2) \subseteq L^\infty(\mathbb{R}^2)$, we would expect this rate to be faster than that established by Chemin in 4.1.1, which it is. Chemin's rate is substantially better than that of 4.2.4 and of 4.2.6 with $0 < \kappa < 1$; however, the two spaces $B_{O(\log^\kappa n)}(\mathbb{R}^2)$ and $L^\infty(\mathbb{R}^2)$ are not comparable for $0 < \kappa \le 1$, since the vorticity can be unbounded for the first space while $\Gamma(n) = O(n)$ for the second.

Finally, the rate established by Hmidi and Keraani in Theorem 4.1.2 is the same as that in Equation (4.2.5) up to dependence of constants on time and initial data, although they show convergence in a different norm.

4.3 Proof of the main theorem

We now begin the proof of Theorem 4.2.2. Let

$$v_n = \text{the solution to } (E) \text{ with initial velocity } v_n^0,$$

where v_n^0, $n = 1, 2, \dots$, is a divergence-free initial velocity smoothed to lie in $C^\infty(\mathbb{R}^2)$ and such that $v_n^0 \to v^0$ in $L^2(\mathbb{R}^2)$ as $n \to \infty$. Letting

$$X = L^\infty([0, T]; L^2(\mathbb{R}^2))$$

we have, for any solution v to (E) in $L^\infty([0, \infty); H^1(\mathbb{R}^2))$,

$$\|v_\nu - v\|_X \le \|v_\nu - v_n\|_X + \|v - v_n\|_X .$$

To bound the first term on the right hand side, we use an energy estimate which can be found in [13]:

$$\begin{aligned} \|v_\nu(t) - v_n(t)\|_{L^2}^2 \le & C\nu t \|\omega^0\|_{L^2} \|\omega(v_n^0)\|_{L^2} + \|v^0 - v_n^0\|_{L^2}^2 \\ & + 2 \int_0^t \int_{\mathbb{R}^2} |v_\nu(s,x) - v_n(s,x)|^2 \, |\nabla v_n(s,x)| \; dx \, ds. \end{aligned} \tag{4.3.1}$$

As long as we insure that the initial velocity is smoothed in such a way that

$$\|\omega(v_n^0)\|_{L^2} \le C \|\omega^0\|_{L^2} \tag{4.3.2}$$

we can conclude from Gronwall's inequality and (4.3.1) that

$$\|v_\nu(t) - v_n(t)\|_{L^2}^2 \le \left(C\nu t \|\omega^0\|_{L^2}^2 + \|v^0 - v_n^0\|_{L^2}^2 \right) e^{2 \int_0^t \|\nabla v_n\|_{L^\infty}}$$

so

$$\|v_\nu - v_n\|_X \le \left((C\nu T)^{1/2} \|\omega^0\|_{L^2} + \|v^0 - v_n^0\|_{L^2} \right) e^{\int_0^T \|\nabla v_n\|_{L^\infty}},$$

using $(A^2 + B^2)^{1/2} \le A + B$ for $A, B \ge 0$.

The energy argument for bounding $\|v - v_n\|_X$ is identical except that the term involving ν is absent and of course we have v in place of v_ν. (In this energy argument, although the norm of $v(t)$ in $H^1(\mathbb{R}^2)$ does not appear, the membership of $v(t)$ in $H^1(\mathbb{R}^2)$ for almost all t is required to insure the vanishing of one of the two nonlinear terms, so we *are* using the membership of v in $L^\infty([0,\infty); H^1(\mathbb{R}^2))$.)

We thus have

$$\|v - v_n\|_X \le \|v^0 - v_n^0\|_{L^2} e^{\int_0^T \|\nabla v_n\|_{L^\infty}} \tag{4.3.3}$$

and so

$$\begin{aligned} \|v_\nu - v\|_X \le (C\nu T)^{1/2} \|\omega^0\|_{L^2} e^{\int_0^T \|\nabla v_n\|_{L^\infty}} \\ + 2\|v^0 - v_n^0\|_{L^2} e^{\int_0^T \|\nabla v_n\|_{L^\infty}}. \end{aligned} \tag{4.3.4}$$

Now suppose we can show that for some sequence $(v_n^0)_{n=1}^\infty$ of approximations to v^0 satisfying Equation (4.3.2),

$$\|v^0 - v_n^0\|_{L^2} e^{\int_0^T \|\nabla v_n\|_{L^\infty}} \to 0 \text{ as } \nu \to 0. \tag{4.3.5}$$

Then letting $n = f(\nu)$ with $f(\nu) \to \infty$ as $\nu \to 0$, the second term in Equation (4.3.4) will vanish with the viscosity. By choosing f to increase to infinity sufficiently slowly, we can always make the first term in Equation (4.3.4) vanish with the viscosity as well. Thus, to establish the vanishing viscosity limit, we need only show that Equation (4.3.5) holds; to determine a bound on the rate of convergence, however, we must choose the function f explicitly.

What we have done is in effect decouple the vanishing viscosity limit from the Navier-Stokes equations and from the viscosity itself. Also, we have yet to use the information we gain from ω^0 lying in B_Γ; this information is encoded in the approximate solution v_n and will be exploited in the next section.

4.3.1 Convergence in Equation (4.3.5)

To smooth the initial velocity let

$$v_n^0 = S_n v^0.$$

Then $\omega_n^0 = S_n \omega^0$ and Equation (4.3.2) is satisfied. Also,

$$\begin{aligned}
\|v^0 - v_n^0\|_{L^2} = \|(\mathrm{Id} - S_n) v^0\|_{L^2} &= \left\| \sum_{q=n+1}^{\infty} \Delta_q v^0 \right\|_{L^2} \le \sum_{q=n+1}^{\infty} \|\Delta_q v^0\|_{L^2} \\
&\le C \sum_{q=n+1}^{\infty} 2^{-q} \|\Delta_q \nabla v^0\|_{L^2} \\
&\le C \left(\sum_{q=n+1}^{\infty} 2^{-2q} \right)^{1/2} \left(\sum_{q=n+1}^{\infty} \|\Delta_q \nabla v^0\|_{L^2}^2 \right)^{1/2} \\
&\le C 2^{-n} \left(\sum_{q=n+1}^{\infty} \|\Delta_q \omega^0\|_{L^2}^2 \right)^{1/2} \le C \|\omega^0\|_{L^2} 2^{-n},
\end{aligned}$$

where we used Minkowski's inequality, Bernstein's inequality, and the Cauchy-Schwarz inequality. From Lemma 4.3.1, below,

$$\begin{aligned}
\|\nabla v_n(t)\|_{L^\infty} &\le C \left(\|\omega_n^0\|_{L^2} + \|\omega_n^0\|_{B^0_{\infty,1}} \right) e^{Ct \|\omega_n^0\|_{B^0_{\infty,1}}} \\
&\le C \left(\|\omega^0\|_{L^2} + \alpha \Gamma(n) \right) e^{Ct\alpha\Gamma(n)} \le C \alpha \Gamma(n) e^{C\alpha t \Gamma(n)},
\end{aligned} \tag{4.3.6}$$

where $\alpha = \|\omega^0\|_{B_\Gamma}$. When $\lim_{n \to \infty} \Gamma(n) = \infty$, Equation (4.3.6) holds for an absolute constant C for all sufficiently large n; it holds for all n for a constant that depends upon the initial vorticity. (See Remark (4.2.1)). This applies as well to the inequalities that follow. Also, in Equation (4.3.6) we used

$$\|\omega_n^0\|_{B^0_{\infty,1}} = \sum_{q\geq -1} \|\Delta_q \omega_n^0\|_{L^\infty} \leq \sum_{q=-1}^{n+1} \|\Delta_q \omega^0\|_{L^\infty} \leq \alpha\Gamma(n).$$

Thus,

$$\int_0^T \|\nabla v_n(t)\|_{L^\infty} \leq \frac{C\alpha\Gamma(n)}{C\alpha\Gamma(n)} \left(e^{C\alpha T\Gamma(n)} - 1\right) \leq e^{C\alpha T\Gamma(n)}$$

and

$$\|v^0 - v_n^0\|_{L^2} e^{\int_0^T \|\nabla v_n\|_{L^\infty}} \leq C\|\omega^0\|_{L^2} 2^{-n} \exp\left(e^{C\alpha T\Gamma(n)}\right). \tag{4.3.7}$$

To bound the rate of convergence of v_ν to v, we must decide how to choose n as a function of ν in Equation (4.3.4). Using Equation (4.3.7), we have

$$\|v_\nu - v\|_X \leq C\|\omega^0\|_{L^2} \left((\nu T)^{1/2} + 2^{-n}\right) \exp\left(e^{C\alpha T\Gamma(n)}\right).$$

Viewing this as a sum of two rates, when $n = -(1/2)\log(\nu T)$ the two rates are equal. If n increases more rapidly as $\nu \to 0$ then the first term decreases more slowly as $\nu \to 0$; if n increases more slowly as $\nu \to 0$ then the second term decreases more slowly as $\nu \to 0$. Since the slower decreasing of the two terms limits the convergence rate, we conclude that letting $n = -(1/2)\log(\nu T)$ optimizes the convergence rate, giving the bound in Theorem 4.2.2 and completing its proof.

Lemma 4.3.1. *Let v be a C^∞-solution to (E) with initial velocity v^0, where ω^0 is in $L^{p_0}(\mathbb{R}^2) \cap B^0_{\infty,1}(\mathbb{R}^2)$, with p_0 in $(1, \infty)$. Then*

$$\|\nabla v(t)\|_{L^\infty} \leq C\left(\|\omega^0\|_{L^{p_0}} + \|\omega^0\|_{B^0_{\infty,1}}\right) e^{Ct\|\omega^0\|_{B^0_{\infty,1}}}.$$

Proof. We have,

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\leq \|\Delta_{-1}\nabla v(t)\|_{L^\infty} + \sum_{q\geq 0}\|\Delta_q \nabla v(t)\|_{L^\infty}\\ &\leq C\,\|\Delta_{-1}\omega(t)\|_{L^{p_0}} + C\sum_{q\geq 0}\|\Delta_q\omega(t)\|_{L^\infty}\\ &\leq C\|\omega^0\|_{L^{p_0}} + C\,\|\omega(t)\|_{B^0_{\infty,1}}.\end{aligned}$$

Here we used Bernstein's inequality with the Calderon-Zygmund inequality for the first term and Lemma 2.3.8 for the sum.

From Theorem 4.2 of [20],

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C(1+\log(\|g(t)\|_{lip}\|g^{-1}(t)\|_{lip})\|\omega^0\|_{B^0_{\infty,1}},$$

where g is the flow associated to v; that is,

$$g(t,x) = x + \int_0^t v(s, g(s,x))\,ds.$$

It follows from Gronwall's inequality that

$$\|g(t)\|_{lip},\ \|g^{-1}(t)\|_{lip} \leq \exp\int_0^t \|\nabla v(s)\|_{L^\infty}\,ds.$$

Combining the three inequalities above gives

$$\|\nabla v(t)\|_{L^\infty} \leq C\|\omega^0\|_{L^{p_0}} + C\left(1+2\int_0^t \|\nabla v(s)\|_{L^\infty}\,ds\right)\|\omega^0\|_{B^0_{\infty,1}},$$

and the proof is completed by another application of Gronwall's inequality. □

In the proof of Lemma 4.3.1 we used the existence of a flow associated with a smooth solution to (E), which allowed us to apply Theorem 4.2 of

[20]. This is where our approach differs markedly from that of Vishik's in [21], where required properties of the flow are inferred from the membership of the vorticity in the spaces $L^{p_0}(\mathbb{R}^2) \cap B_\Gamma(\mathbb{R}^2)$ and $L^{p_0}(\mathbb{R}^2) \cap L^{p_1}(\mathbb{R}^2)$ and where the constraints on the values of p_0 and p_1 of Theorem 4.2.1 are required. Vishik also requires that $p_0 < 2$ so that the velocity can be recovered uniquely from the vorticity using the Biot-Savart law, since he uses the vorticity formulation of a weak solution to (E) in [21]. By contrast, in Theorem 4.2.2 we in effect require that $p_0 = p_1 = 2$, so that we can make the basic energy argument in Section 4.3.

It is also possible to prove Lemma 4.3.1 using an argument like that in [8].

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Vita

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