

# ON THE AXISYMMETRIC EULER EQUATIONS WITH INITIAL VORTICITY IN BORDERLINE SPACES OF BESOV TYPE

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ABSTRACT. Borderline spaces of Besov type consist of tempered distributions satisfying the property that the partial sums of their  $B_{\infty,1}^0$ -norm diverge in a controlled way. We prove an existence and uniqueness result for the three-dimensional axisymmetric Euler equations without swirl when initial vorticity belongs to these spaces. We also prove that for this class of solutions the vanishing viscosity limit holds in the energy norm, and we give a rate of convergence.

## 1. INTRODUCTION

We consider the Navier-Stokes equations modeling incompressible viscous fluid flow in  $\mathbb{R}^3$ , given by

$$(NS) \quad \begin{cases} \partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu = -\nabla p_\nu \\ \operatorname{div} u_\nu = 0 \\ u_\nu|_{t=0} = u_\nu^0, \end{cases}$$

and the Euler equations modeling incompressible non-viscous fluid flow in  $\mathbb{R}^3$ , given by

$$(E) \quad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \operatorname{div} u = 0 \\ u|_{t=0} = u^0. \end{cases}$$

In three dimensions, breakdown of smooth solutions to the Navier-Stokes and Euler equations remains open and has proved to be one of the most difficult problems in fluid mechanics (see [8] for details). For the case of axisymmetric solutions without swirl, however, global existence and uniqueness for  $(NS)$  and  $(E)$  has been established under various assumptions on the initial data. In what follows, we say a vector field  $v$  is axisymmetric if it can be written as

$$(1.1) \quad v(t, x) = v^r(t, r, z)e_r + v^z(t, r, z)e_z,$$

where  $z = x_3$ ,  $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ ,  $(e_r, e_\theta, e_z)$  is the cylindrical basis of  $\mathbb{R}^3$ , and  $v^r$  and  $v^z$  do not depend on  $\theta$ . For fluids with axisymmetric divergence-free velocity  $u_\nu$

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solving  $(NS)$ , the vorticity  $\omega_\nu$  can be written as

$$(1.2) \quad \omega_\nu = (\partial_z u_\nu^r - \partial_r u_\nu^z) e_\theta := \omega_\nu^\theta e_\theta.$$

Moreover,  $\omega_\nu$  satisfies the equality

$$(1.3) \quad \partial_t \omega_\nu + u_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = r^{-1} \omega_\nu u_\nu^r.$$

Similarly, for axisymmetric velocity  $u$  solving  $(E)$ , the vorticity  $\omega$  satisfies

$$(1.4) \quad \omega = (\partial_z u^r - \partial_r u^z) e_\theta := \omega^\theta e_\theta,$$

and

$$(1.5) \quad \partial_t \omega + u \cdot \nabla \omega = r^{-1} \omega u^r.$$

Letting  $\alpha = \frac{\omega^\theta}{r}$ , one can use (1.3) to conclude that  $\|\alpha(t)\|_{L^p} \leq \|\alpha^0\|_{L^p}$  for all  $p \in [1, \infty]$ . When  $\nu = 0$ , (1.5) implies that the  $L^p$ -norms of  $\alpha$  are conserved.

In [12], Ukhovskii and Yudovich proved that when initial velocity is axisymmetric and satisfies  $u^0 \in L^2(\mathbb{R}^3)$ ,  $\omega(u^0) = \omega^0 \in L^2 \cap L^\infty(\mathbb{R}^3)$ , and  $\frac{\omega^0}{r} \in L^2 \cap L^\infty(\mathbb{R}^3)$ , there exists a unique global in time solution to  $(NS)$  and  $(E)$  with  $u \in L^\infty([0, T]; L^2 \cap L^p(\mathbb{R}^2))$  for some  $p \in (3, 6]$  and with  $\omega \in L^1([0, T]; L^2(\mathbb{R}^3))$  (in fact, for  $(NS)$  the assumption that  $\omega^0$  is bounded is unnecessary to prove existence). These conditions on the initial data follow when, for example,  $u^0$  belongs to  $H^s(\mathbb{R}^3)$  for  $s > \frac{7}{2}$ . Shirota and Yanagisawa proceeded to show in [9] that there exists a unique solution  $u$  to  $(NS)$  in  $H^s$  for  $s > \frac{5}{2}$  with  $u^0$  axisymmetric and belonging to  $H^s(\mathbb{R}^3)$ . Finally, in a recent paper of Abidi (see [1]), the author demonstrated existence and uniqueness of a solution to  $(NS)$  in  $H^{\frac{1}{2}}(\mathbb{R}^3)$  with axisymmetric initial velocity in  $H^{\frac{1}{2}}(\mathbb{R}^3)$ . For our purposes, the most relevant result appears in a recent paper of Abidi, Hmidi, and Keraani regarding the Euler equations. In [2], the authors prove that if  $u^0$  is an axisymmetric vector field in the critical Besov space  $B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3)$  (see Definition 1) with  $p \in [1, \infty]$ , and if  $\frac{\omega^0}{r}$  belongs to the Lorentz space  $L^{3,1}(\mathbb{R}^3)$  (see Definition 3), then there exists a unique solution  $u$  to  $(E)$  in  $C(\mathbb{R}^+; B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))$ .

In this paper, we study the three-dimensional Euler equations with axisymmetric initial velocity under the assumption that  $\omega^0$  does not necessarily belong to  $B_{p,1}^{\frac{3}{p}}(\mathbb{R}^3)$  for any  $p \in [1, \infty]$ . We do require, however, that the partial sums of the  $B_{\infty,1}^0$ -norm of  $\omega^0$  diverge in a controlled way. Specifically, we assume  $\omega^0$  belongs to the space  $B_\Gamma(\mathbb{R}^3)$ . This space was introduced by Vishik in [13]. Roughly speaking, the space  $B_\Gamma(\mathbb{R}^3)$  consists of tempered distributions which satisfy the property that the partial sums of the  $B_{\infty,1}^0$ -norm diverge no faster than  $\Gamma(N)$  (see Definition 2 in Section 2 for more information about  $B_\Gamma$  spaces). In [13], Vishik proved that the solution to  $(E)$  is unique as long as vorticity remains in the space  $L^\infty([0, T]; B_\Gamma(\mathbb{R}^d))$  for  $d \geq 2$  with  $\Gamma(N)$  behaving roughly like  $N \log N$ . He also proved that for  $d = 2$ , a solution

in his uniqueness class exists locally in time for  $\omega^0$  in  $B_\Gamma(\mathbb{R}^2)$  with  $\Gamma(N) = \log N$ , and the solution exists globally in time for  $\omega^0 \in B_\Gamma$  with  $\Gamma(N) = \log^{\frac{1}{2}}(N)$ .

In [5], with Kelliher we studied the vanishing viscosity limit in the plane for Vishik's existence and uniqueness class. The goal of this paper is to extend the results of [5] to the three-dimensional case with axisymmetric initial velocity. Specifically, in this paper we apply the techniques of [5] to prove two results regarding the three-dimensional fluid equations with axisymmetric initial velocity. In the first result we prove that when  $\Gamma(N) = \log^\kappa(N)$  for  $\kappa \in [0, 1)$ , there exists a unique solution  $u$  to (E) in  $L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^3))$  with  $u^0 \in H^1(\mathbb{R}^3)$ ,  $\omega^0 \in B_\Gamma(\mathbb{R}^3)$  and  $\frac{\omega^0}{r} \in L^{3,1}(\mathbb{R}^3)$ . With  $\Gamma(N) = \log N$ , we prove an analogous existence and uniqueness statement for short time. In the second result, we show that the vanishing viscosity limit holds in  $L^\infty([0, T]; L^2(\mathbb{R}^3))$  for these types of solutions to (E). The proof of the vanishing viscosity result closely follows the argument used in [5] and relies on new estimates from [2] for solutions with axisymmetric initial data.

The paper is organized as follows. In Section 2, we define the Littlewood-Paley operators and various function spaces. We also state a few useful lemmas. Finally, we state the main theorem of the paper. In Section 3 we prove the existence and uniqueness portion of the main theorem. In Section 4, we show that the vanishing viscosity limit from the main theorem holds.

## 2. BACKGROUND AND MAIN RESULT

We first define the Littlewood-Paley operators (for a more thorough discussion of these operators, we refer the reader to [11]). We let  $\varphi \in S(\mathbb{R}^d)$  satisfy  $\text{supp } \varphi \subset \{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ , and for every  $j \in \mathbb{Z}$  we let  $\varphi_j(\xi) = \varphi(2^{-j}\xi)$  (so  $\check{\varphi}_j(x) = 2^{jn}\check{\varphi}(2^jx)$ ). We define  $\psi_n \in S(\mathbb{R}^d)$  by the equality

$$\psi_n(\xi) = 1 - \sum_{j \geq n} \varphi_j(\xi)$$

for all  $\xi \in \mathbb{R}^d$ , and for  $f \in S'(\mathbb{R}^d)$  we define the operator  $S_n$  by

$$S_n f = \check{\psi}_n * f.$$

In the following sections we will make frequent use of the Littlewood-Paley operators. For  $f \in S'(\mathbb{R}^d)$  and  $j \in \mathbb{Z}$ , we define the Littlewood-Paley operators  $\Delta_j$  by

$$\Delta_j f = \begin{cases} 0, & j < -1, \\ \check{\psi}_0 * f, & j = -1, \\ \check{\varphi}_j * f, & j > -1. \end{cases}$$

We now define the Besov spaces.

**Definition 1.** Let  $s \in \mathbb{R}$ ,  $(p, q) \in [1, \infty] \times [1, \infty)$ . We define the inhomogeneous Besov space  $B_{p,q}^s(\mathbb{R}^d)$  to be the space of tempered distributions  $f$  on  $\mathbb{R}^d$  such that

$$\|f\|_{B_{p,q}^s} := \|S_0 f\|_{L^p} + \left( \sum_{j=0}^{\infty} 2^{jq_s} \|\Delta_j f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty.$$

When  $q = \infty$ , we write

$$\|f\|_{B_{p,\infty}^s} := \|S_0 f\|_{L^p} + \sup_{j \geq 0} 2^{jq_s} \|\Delta_j f\|_{L^p}.$$

We define the  $B_\Gamma$  spaces as in [13].

**Definition 2.** We define  $B_\Gamma$  to be the set of all  $f$  in  $S'(\mathbb{R}^d)$  satisfying

$$\sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} = O(\Gamma(N))$$

with the norm

$$\|f\|_\Gamma = \sup_{N \geq -1} \frac{1}{\Gamma(N)} \sum_{j=-1}^N \|\Delta_j f\|_{L^\infty}.$$

Finally, we must define the Lorentz spaces.

**Definition 3.** The nonincreasing rearrangement of a measurable function  $f$  is given by

$$f^*(t) := \inf \{s, m(\{x : |f(x)| > s\}) \leq t\}.$$

For  $p, q \in [1, \infty]$ , we define the Lorentz space  $L^{p,q}(\mathbb{R}^d)$  to be the set of tempered distributions  $f$  satisfying  $\|f\|_{L^{p,q}} < \infty$ , where

$$\|f\|_{L^{p,q}} = \begin{cases} \left( \int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}, & q \in [1, \infty) \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & q = \infty. \end{cases}$$

We remark that Lorentz spaces relate to  $L^p$  spaces by the equality  $L^{p,p} = L^p$ . We refer the reader to [10] and [3] for other properties of Lorentz spaces.

Before we state the main theorem, we give a few lemmas which we use throughout the paper. We begin with Bernstein's Lemma. We refer the reader to [4] for a proof of the lemma.

**Lemma 1.** (*Bernstein's Lemma*) Let  $r_1$  and  $r_2$  satisfy  $0 < r_1 < r_2 < \infty$ , and let  $p$  and  $q$  satisfy  $1 \leq p \leq q \leq \infty$ . There exists a positive constant  $C$  such that for every integer  $k$ , if  $u$  belongs to  $L^p(\mathbb{R}^d)$ , and  $\text{supp } \hat{u} \subset B(0, r_1 \lambda)$ , then

$$(2.1) \quad \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^k \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}.$$

Furthermore, if  $\text{supp } \hat{u} \subset C(0, r_1\lambda, r_2\lambda)$ , then

$$(2.2) \quad C^{-k}\lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^k \lambda^k \|u\|_{L^p}.$$

We also make frequent use of the following lemma, which is proved in [5].

**Lemma 2.** *Let  $u$  be a divergence-free vector field in  $L^2_{loc}(\mathbb{R}^d)$  with vorticity  $\omega$ . Then there exists an absolute constant  $C$  such that for all  $q \geq 0$ ,*

$$\|\Delta_q \nabla v\|_{L^\infty} \leq C \|\Delta_q \omega\|_{L^\infty}.$$

We now state a key lemma regarding the vorticity equation in three dimensions. This lemma allows us to control the growth of the  $L^p$ -norm of  $\omega(t)$  with time using the  $L^{3,1}$ -norm of  $\frac{\omega^0}{r}$ . We refer the reader to Proposition 2 and Corollary 1 of [6] for a proof of the lemma. (In fact, the author proves a more general lemma for Lorentz spaces  $L^{p,q}$ , but the lemma for  $L^p$  spaces follows by the equality  $L^{p,p} = L^p$ .)

**Lemma 3.** *Let  $u$  be a divergence-free vector field with coefficients in the space  $L^1([0, T]; B^\infty_{\infty, \infty}(\mathbb{R}^3))$ , and let  $\omega$  satisfy the vorticity equation*

$$\partial_t \omega + u \cdot \nabla \omega = r^{-1} \omega u^r.$$

*There exists a constant  $C$  such that for all  $p \in [1, \infty]$  and  $t \in [0, T]$ ,*

$$(2.3) \quad \|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} \exp(C_1 t \|r^{-1} \omega^0\|_{L^{3,1}}).$$

*Remark 2.4.* An inequality similar to that in (2.3) holds for the Navier-Stokes vorticity. Indeed, by (1.3) we have that for all  $p \in [1, \infty]$ ,

$$(2.5) \quad \|\omega_\nu(t)\|_{L^p} \leq \|\omega_\nu^0\|_{L^p} + \int_0^t \|r^{-1} u_\nu(s)\|_{L^\infty} \|\omega_\nu(s)\|_{L^p} ds.$$

Using the Biot-Savart law, we can bound  $\|r^{-1} u_\nu(s)\|_{L^\infty}$  above by  $\|r^{-1} \omega_\nu^\theta(s)\|_{L^{3,1}}$  (see for example [6] and [2]). Moreover, we have that  $\|r^{-1} \omega_\nu^\theta(s)\|_{L^{3,1}} \leq \|r^{-1} \omega_\nu^\theta(0)\|_{L^{3,1}}$ . This follows from the inequality  $\|r^{-1} \omega_\nu^\theta(s)\|_{L^p} \leq \|r^{-1} \omega_\nu^\theta(0)\|_{L^p}$  for all  $p \in [1, \infty]$  and a standard interpolation argument. Finally, an application of Gronwall's Lemma yields an estimate analogous to (2.3).

We are now in a position to state the main theorem of the paper.

**Theorem 1.** *Let  $u^0$  be an axisymmetric vector field belonging to  $L^2(\mathbb{R}^3)$  and let  $\omega^0$  belong to  $B_\Gamma \cap L^2(\mathbb{R}^3)$  with  $\Gamma(N) = \log^\kappa(N)$  for  $0 \leq \kappa \leq 1$ . Also assume  $r^{-1} \omega^0$  is in the Lorentz space  $L^{3,1}(\mathbb{R}^3)$ . For  $\kappa = 1$ , there exists  $T_0 > 0$  and a unique solution  $(u, \nabla p)$  to (E) with  $u$  in the space  $L^\infty([0, T_0]; H^1(\mathbb{R}^3))$  and with  $u|_{t=0} = u^0$ . If  $\kappa \in [0, 1)$ , then for every  $T > 0$  there exists a unique solution  $(u, \nabla p)$  to (E) with  $u$  in  $L^\infty([0, T]; H^1(\mathbb{R}^3))$  and with  $u|_{t=0} = u^0$ .*

*Moreover, if  $u_\nu$  is the unique solution to (NS) from [1] in the space  $H^{\frac{1}{2}}(\mathbb{R}^3)$  with the same initial data  $u^0$ , then the following estimate holds for fixed  $T > 0$ :*

$$(2.6) \quad \|u_\nu - u\|_{L^\infty([0, T]; L^2(\mathbb{R}^3))} \leq \sqrt{\nu e^{C_1 T}} \exp\left(e^{C_1 T \Gamma(-\frac{1}{2} \log(\nu e^{C_1 T}))} e^{C_1 T}\right).$$

We remark that the right hand side of (2.6) converges to 0 as  $\nu$  approaches 0 for short time when  $\Gamma(N) = \log(N)$  and for any finite time when  $\Gamma(N) = \log^\kappa(N)$  with  $\kappa \in [0, 1)$ . Therefore, the vanishing viscosity limit holds for the entire existence and uniqueness class established in the first part of Theorem 1.

The proof of Theorem 1 is contained in Sections 3 and 4.

### 3. EXISTENCE AND UNIQUENESS

Let  $u^0$  satisfy the conditions of Theorem 1, and let  $u_p$  be the unique solution to (E) in  $C(\mathbb{R}^+; B_{\infty,1}^1(\mathbb{R}^3))$  with initial data  $S_p u^0$  (for a proof that  $S_p u^0$  is axisymmetric, see [2], Proposition 3.1). We will show the sequence  $\{u_p\}$  is Cauchy in  $L^\infty([0, T]; L^2(\mathbb{R}^3))$  for  $\Gamma(N)$  growing sufficiently slowly with  $N$ . We fix positive integers  $m$  and  $n$ , with  $m > n$ , and we let  $u_m$  and  $u_n$  be solutions to (E) with initial data  $S_m u^0$  and  $S_n u^0$ , respectively. An energy argument identical to that in [7] yields the estimate

$$(3.1) \quad \|u_n - u_m\|_{L^\infty([0,T]; L^2)} \leq C \|u_m^0 - u_n^0\|_{L^2} e^{\int_0^T \|\nabla u_n\|_{L^\infty}}.$$

Since  $\omega^0$  belongs to  $L^2(\mathbb{R}^3)$ , Bernstein's inequality and boundedness of Calderon-Zygmund operators on  $L^2$  imply that  $\|u_m^0 - u_n^0\|_{L^2} \leq C 2^{-n} \|\omega^0\|_{L^2}$ . We can conclude from (3.1) that

$$\|u_n - u_m\|_{L^\infty([0,T]; L^2)} \leq C 2^{-n} e^{\int_0^T \|\nabla u_n\|_{L^\infty}}.$$

It remains to estimate the growth of the quantity  $\|\nabla u_n\|_{L^\infty}$  with  $n$ . To do this, we use estimates proved in [2]. Specifically, in the proof of Proposition 4.4 of [2], the authors show that if the axisymmetric initial velocity belongs to  $B_{\infty,1}^1(\mathbb{R}^3)$  and if  $\frac{\omega^0}{r}$  is in  $L^{3,1}(\mathbb{R}^3)$ , then for fixed  $\hat{N} \in \mathbb{N}$ ,

$$\|\omega(t)\|_{B_{\infty,1}^0} \leq C \|\omega^0\|_{B_{\infty,1}^0} (2^{-\hat{N}} e^{CU(t)} + \hat{N} e^{Ct} \|\frac{\omega^0}{r}\|_{L^{3,1}}),$$

where  $U(t) = \int_0^t \|u\|_{B_{\infty,1}^1}$  and  $C$  is an absolute constant. Applying this estimate to  $\omega_n$  and using the inequality  $\|\frac{\omega_n^0}{r}\|_{L^{3,1}} \leq \|\frac{\omega^0}{r}\|_{L^{3,1}}$  (see, for example, Lemma 1 of [6]), it follows that

$$(3.2) \quad \begin{aligned} \|\omega_n(t)\|_{B_{\infty,1}^0} &\leq C \|\omega_n^0\|_{B_{\infty,1}^0} (2^{-\hat{N}} e^{CU_n(t)} + \hat{N} e^{Ct} \|\frac{\omega_n^0}{r}\|_{L^{3,1}}) \\ &\leq C \|\omega_n^0\|_{B_{\infty,1}^0} (2^{-\hat{N}} e^{CU_n(t)} + \hat{N} e^{Ct} \|\frac{\omega^0}{r}\|_{L^{3,1}}), \end{aligned}$$

with  $U_n(t)$  now equal to  $\int_0^t \|u_n\|_{B_{\infty,1}^1}$ . As in [2], we choose  $\hat{N} = CU_n(t) + 1$ . Using Bernstein's Lemma, energy conservation, Lemma 2, and the definition of  $B_{\infty,1}^0$ , we can conclude that  $\|u_n\|_{B_{\infty,1}^1} \leq C \|u_n\|_{L^2} + \sum_{j \geq 0} \|\Delta_j \omega_n\|_{L^\infty} \leq C \|u^0\|_{L^2} + \|\omega_n\|_{B_{\infty,1}^0}$ .

This estimate, combined with our choice of  $\hat{N}$  and (3.2), gives

$$(3.3) \quad \begin{aligned} \|\omega_n(t)\|_{B_{\infty,1}^0} &\leq C\|\omega_n^0\|_{B_{\infty,1}^0} e^{Ct\|\frac{\omega^0}{r}\|_{L^{3,1}}} (U_n(t) + 1) \\ &\leq Ce^{C_1 t} \|\omega_n^0\|_{B_{\infty,1}^0} \left( \int_0^t \|\omega_n(s)\|_{B_{\infty,1}^0} ds + t\|u^0\|_{L^2} + 1 \right), \end{aligned}$$

where the constant  $C_1$  depends on  $\|\frac{\omega^0}{r}\|_{L^{3,1}}$ . An application of Gronwall's inequality yields

$$(3.4) \quad \|\omega_n(t)\|_{B_{\infty,1}^0} \leq Ce^{C_1 t} \|\omega_n^0\|_{B_{\infty,1}^0} (t+1) e^{Ct\|\omega_n^0\|_{B_{\infty,1}^0}} e^{C_1 t}.$$

Again by Bernstein's Lemma, energy conservation, Lemma 2, and the definition of  $B_{\infty,1}^0$ , it follows that  $\|\nabla u_n\|_{L^\infty} \leq \|u^0\|_{L^2} + \|\omega_n\|_{B_{\infty,1}^0}$ . Using this estimate, (3.4), and the estimate  $\|\omega_n^0\|_{B_{\infty,1}^0} \leq \|\omega^0\|_{B_\Gamma(n)}$ , we write

$$(3.5) \quad \begin{aligned} \|u_n - u_m\|_{L^\infty([0,T];L^2)} &\leq C2^{-n} e^{\int_0^T \|\nabla u_n\|_{L^\infty}} \\ &\leq C2^{-n} \exp\left(\int_0^T Ce^{C_1 t} \Gamma(n)(t+1) e^{Ct\Gamma(n)e^{C_1 t}}\right) \\ &\leq C2^{-n} \exp\left(e^{CT\Gamma(n)e^{C_1 T}} - 1\right) \end{aligned}$$

for sufficiently large  $n$ , where we integrated in time to get the last inequality. We conclude that the sequence  $\{u_p\}$  is Cauchy in  $L^\infty([0, T]; L^2(\mathbb{R}^3))$  for sufficiently small  $T$  when  $\Gamma(n) = \log(n)$  and for every  $T$  when  $\Gamma(n) = \log^\kappa(n)$  with  $\kappa \in [0, 1)$ . From this we conclude that  $\{u_p\}$  converges to a vector field  $u$  in  $L^\infty([0, T]; L^2(\mathbb{R}^3))$ .

A straightforward argument shows that since  $\{u_p\}$  converges to  $u$  in the space  $L^\infty([0, T]; L^2(\mathbb{R}^3))$ , we can pass to the limit in (E). To see that  $u$  belongs to  $L^\infty([0, T]; H^1(\mathbb{R}^3))$ , we observe that by Lemma 3, we have the estimate

$$(3.6) \quad \|\omega_n(t)\|_{L^2} \leq \|\omega^0\|_{L^2} e^{Ct\|r^{-1}\omega^0\|_{L^{3,1}}}$$

for all  $n$ , where we also used the inequality  $\|\frac{\omega_n^0}{r}\|_{L^{3,1}} \leq \|\frac{\omega^0}{r}\|_{L^{3,1}}$ . A weak compactness argument shows that  $\|\omega(t)\|_{L^2} \leq \|\omega^0\|_{L^2} e^{Ct\|r^{-1}\omega^0\|_{L^{3,1}}}$ , giving  $u \in L^\infty([0, T]; H^1(\mathbb{R}^3))$ . Finally, the membership of  $u$  to  $L^\infty([0, T]; L^2(\mathbb{R}^3))$  allows us to uniquely determine  $\nabla p$  from  $u$ . We have thus constructed a weak solution  $(u, \nabla p)$  to (E) satisfying the assumptions of Theorem 1.

To show uniqueness of  $u$  in  $L^\infty([0, T]; H^1(\mathbb{R}^3))$ , we first observe that by the existence proof,  $\|u_p - u\|_{L^\infty([0,T];L^2(\mathbb{R}^2))}$  approaches 0 as  $p$  approaches infinity. Since the sequence  $u_p$  is uniquely determined by the initial data  $u_0$ , two solutions to (E) with the same initial data and initial vorticity in  $B_\Gamma$  will have the same approximating sequence and will therefore be equal on  $[0, T]$ .

## 4. THE VANISHING VISCOSITY LIMIT

The proof that the vanishing viscosity limit holds in  $L^\infty([0, T], L^2(\mathbb{R}^3))$  is similar to that in [5]. Given our solution  $(u, \nabla p)$  to  $(E)$  from Theorem 1 with initial data  $u^0$ , we again construct the sequence of solutions  $\{u_n\}$  to  $(E)$  with initial data  $S_n u^0$  (this notation is admittedly somewhat confusing as we used  $\{u_p\}$  to denote a sequence of velocities in Section 3), and we write

$$(4.1) \quad \|u_\nu - u\|_{L^\infty([0, T]; L^2)} \leq \|u_\nu - u_n\|_{L^\infty([0, T]; L^2)} + \|u_n - u\|_{L^\infty([0, T]; L^2)}.$$

To estimate  $\|u_n - u\|_{L^\infty([0, T]; L^2)}$ , we apply the same argument as that used to show existence in Section 3, but with  $u$  in place of  $u_m$ , to conclude that

$$(4.2) \quad \|u_n - u\|_{L^\infty([0, T]; L^2)} \leq C 2^{-n} e^{\int_0^T \|\nabla u_n\|_{L^\infty}}.$$

To estimate  $\|u_\nu - u_n\|_{L^\infty([0, T]; L^2)}$ , we again apply the energy argument from [7], this time to the solutions  $u_\nu$  and  $u_n$  to  $(NS)$  and  $(E)$ , respectively. Before applying Gronwall's Lemma, the energy argument yields the inequality

$$(4.3) \quad \begin{aligned} \|(u_\nu - u_n)(t)\|_{L^2}^2 &\leq \|u_n^0 - u^0\|_{L^2}^2 + C\nu \int_0^t \|\nabla u_\nu(s)\|_{L^2} \|(\nabla u_\nu - \nabla u_n)(s)\|_{L^2} ds \\ &\quad + \int_0^t \int_{\mathbb{R}^3} |(u_\nu - u_n)(s)|^2 |\nabla u_n(s)|^2 dx ds. \end{aligned}$$

Note that we now have an extra term on the right hand side, resulting from the fact that  $\nu > 0$ . To handle this term, we bound  $\|\nabla u_\nu\|_{L^2}$  and  $\|\nabla u_n\|_{L^2}$  with  $\|\omega_\nu\|_{L^2}$  and  $\|\omega_n\|_{L^2}$ , respectively, and we apply Remark 2.4 and (3.6). We also use the membership of  $\omega^0$  to  $L^2(\mathbb{R}^3)$  and Bernstein's Lemma to again write  $\|u_n^0 - u^0\|_{L^2}^2 \leq C 2^{-2n}$ . These bounds combined with an application of Gronwall's Lemma to (4.3) yield

$$(4.4) \quad \|u_\nu - u_n\|_{L^\infty([0, T]; L^2)} \leq C \left( 2^{-n} + \sqrt{\nu e^{C_1 T}} \right) e^{\int_0^T \|\nabla u_n\|_{L^\infty}},$$

where  $C_1$  depends on  $\|r^{-1}\omega^0\|_{L^{3,1}}$ . Combining (4.1), (4.2), and (4.4) gives

$$\|u_\nu - u\|_{L^\infty([0, T]; L^2)} \leq C \left( 2^{-n} + \sqrt{\nu e^{C_1 T}} \right) e^{\int_0^T \|\nabla u_n\|_{L^\infty}}.$$

To complete the proof, we apply the bound  $e^{\int_0^T \|\nabla u_n\|_{L^\infty}} \leq \exp\left(e^{CT\Gamma(n)e^{C_1 T}} - 1\right)$  from (3.5) in Section 3, and we let  $n = -\frac{1}{2} \log(\nu T)$ . This completes the proof of Theorem 1.

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