21470 HW 4 Solutions

1) Solution: a) For each $P = (x, y)$ in the fluid domain, let $\phi(P) = \int_O^P \mathbf{u} \cdot d\mathbf{x}$, where $O$ is an arbitrary fixed point. Since $\nabla \times \mathbf{u} = 0$, we can write $\mathbf{u} = \nabla \psi$ for some scalar valued function $\psi = \psi(x, t)$. Then, by the fundamental theorem of calculus for line integrals, we have

$$\phi(P) = \int_O^P \mathbf{u} \cdot d\mathbf{x} = \int_O^P \nabla \psi \cdot d\mathbf{x} = \psi(P) - \psi(O).$$

Therefore, for each point $P$ in the fluid domain $\phi(P) = \psi(P) - \psi(O)$, so that

$$\nabla \phi = \nabla \psi = \mathbf{u}.$$

b) Define $\mathbf{w} = (-v, u)$, where $\mathbf{u} = (u, v)$. Since $\mathbf{u}$ is divergence-free, it follows that $\nabla \times \mathbf{w} = 0$. Therefore, we can write $\mathbf{w} = \nabla \eta$ for some scalar valued function $\eta = \eta(x, t)$. Setting

$$\psi(P) = \int_O^P \mathbf{w} \cdot d\mathbf{x} = \int_O^P \nabla \eta \cdot d\mathbf{x},$$

one can use an argument identical to that in part a to show that

$$\nabla \psi = \nabla \eta = \mathbf{w}.$$

2) Solution: (i) Assume $\gamma_1$ and $\gamma_2$ are two distinct paths from $O$ to $P$, and define the closed curve $\Gamma = \gamma_1 - \gamma_2$. Then

$$\int_{\gamma_1} \mathbf{u} \cdot d\mathbf{x} - \int_{\gamma_2} \mathbf{u} \cdot d\mathbf{x} = \int_{\gamma_1} \mathbf{u} \cdot d\mathbf{x} + \int_{-\gamma_2} \mathbf{u} \cdot d\mathbf{x}$$

$$= \int_{\gamma_1 - \gamma_2} \mathbf{u} \cdot d\mathbf{x} = \int_{\Gamma} \mathbf{u} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = 0,$$

where we used Stokes’ Theorem to get the second to last equality. Here $\Gamma$ is spanned by the surface $S$ and $\mathbf{n}$ denotes the outward unit normal to the surface $S$. The above series of equalities implies

$$\int_{\gamma_1} \mathbf{u} \cdot d\mathbf{x} - \int_{\gamma_2} \mathbf{u} \cdot d\mathbf{x} = 0,$$

so $\phi(P)$ is well-defined (value is independent of path chosen).

(ii) The idea is similar to part (i), but use $\mathbf{w} = (-v, u)$ in place of $\mathbf{u}$ and use Green’s Theorem in place of Stokes’ Theorem.
3) **Solution:** The idea is again similar to #1. In this case, define $F = Ru_z e_R - Ru_R e_z$. Then

$$\nabla \times F = \left( \frac{\partial}{\partial z} (Ru_z) - \frac{\partial}{\partial R} (-Ru_R) \right) e_\phi = (R \nabla \cdot u) e_\phi = 0,$$

so there exists a scalar valued function $\psi$ such that $\nabla \psi = F$. Now, for an arbitrary fixed point $A$ and for any point $P$ in the fluid region, define $\Psi(P)$ by

$$\Psi(P) = \int_A^P (-Ru_R \, dz + Ru_z \, dR) = \int_A^P F \cdot d\mathbf{x}$$

$$= \int_A^P \nabla \psi \cdot d\mathbf{x} = \psi(P) - \psi(A).$$

Therefore, $\nabla \Psi = \nabla \psi = F$. This implies (after a little more work) that $u = \nabla \times \left( \frac{\Psi}{R} e_\phi \right)$.

5) **Solution:** Assuming $-\frac{1}{2} \alpha r \omega = \nu \frac{d}{dr} \omega$, and letting $F(r) = ru_\theta$ (so that $\omega = \frac{1}{r} F'$), we have

$$-\frac{1}{2} \alpha F' = \nu \frac{d}{dr} \left( \frac{1}{r} F' \right) = \nu \left( -\frac{1}{r^2} F' + \frac{1}{r} F'' \right)$$

$$\Rightarrow \nu (F'' - \frac{1}{r} F') = -\frac{\alpha r}{2} F'.$$

Let $\eta = r \sqrt{\frac{\nu}{\nu}}$, and assume $F = f(\eta)$ for some $f$. By the chain rule, we see that $F'' = \frac{d^2}{d\eta^2} F = \frac{\alpha}{\nu} f''(\eta)$, and $F' = \sqrt{\frac{\nu}{\nu}} f'(\eta)$. Plugging this information into (1) and doing some algebra, we see that

$$\nu \left( \frac{\alpha}{\nu} f''(\eta) - \frac{1}{r} \sqrt{\frac{\alpha}{\nu}} f'(\eta) \right) = -\frac{\alpha r}{2} \sqrt{\frac{\alpha}{\nu}} f'(\eta)$$

$$\Rightarrow f''(\eta) - \left( \frac{\sqrt{\nu}}{r \sqrt{\alpha}} - \frac{1}{2} r \sqrt{\frac{\alpha}{\nu}} \right) f'(\eta) = 0$$

$$\Rightarrow f''(\eta) - \left( \frac{1}{\eta} - \frac{1}{2} \eta \right) f'(\eta) = 0.$$

Let $g = f'$. Separating variables, we get $\frac{1}{\eta} g' = \frac{1}{\eta} - \frac{1}{2} \eta$. Integrating both sides in $\eta$ gives

$$f'(\eta) = g(\eta) = C \eta e^{-\frac{1}{2} \eta^2}.$$

To obtain $f$ we integrate in $\eta$ again to get

$$f(\eta) = C e^{-\frac{\eta^2}{2}} + D.$$
Since $u_\theta$ is finite at $r = 0$, we have that $0 = ru_\theta|_{r=0} = f(0) = C + D$. So $C = -D$ and, letting $C = -\frac{\Gamma}{2\pi}$, we have
\[
ru_\theta = \frac{\Gamma}{2\pi}(1 - e^{-\frac{r^2}{4\nu}}) = \frac{\Gamma}{2\pi}(1 - e^{-\frac{\nu^2}{4\nu}}).
\]

6) Solution: By definition, $\mathbf{u} = \nabla \phi$, so $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla \cdot \mathbf{u} = 0$.

We want to show that
\[
\frac{1}{2} \rho \int_V \mathbf{u}^2 \, dV = \frac{1}{2} \rho \int_S \phi \frac{\partial \phi}{\partial n} \, dS.
\]
Write
\[
\frac{1}{2} \rho \int_V \mathbf{u}^2 \, dV = \frac{1}{2} \rho \int_V (\nabla \phi) \cdot (\nabla \phi) \, dV
\]
\[
= \frac{1}{2} \rho \int_V [\nabla \cdot (\phi \nabla \phi) - \phi \nabla \cdot (\nabla \phi)] \, dV \text{ by (A.4)}
\]
\[
= \frac{1}{2} \rho \int_V \nabla \cdot (\phi \nabla \phi) \, dV \text{ since } \nabla \cdot (\nabla \phi) = 0
\]
\[
= \frac{1}{2} \rho \int_S (\phi \nabla \phi) \cdot \mathbf{n} \, dS \text{ by the Divergence Thm}
\]
\[
= \frac{1}{2} \rho \int_S \phi \frac{\partial \phi}{\partial n} \, dS.
\]