1) Show $\mathbf{\nabla} \times \mathbf{F} = (\frac{1}{r} \frac{\partial F_r}{\partial \theta} - \frac{\partial F_\theta}{\partial z}) \hat{\mathbf{e}}_r + (\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}) \hat{\mathbf{e}}_\theta + \frac{1}{r} \left( \frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\mathbf{e}}_z$.

Solution:

$$
\left(\frac{1}{r} \frac{\partial F_r}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right) \hat{\mathbf{e}}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) \hat{\mathbf{e}}_\theta + \frac{1}{r} \left( \frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\mathbf{e}}_z
$$

$$
= \left(\frac{1}{r} \frac{\partial F_r}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right) \left( \cos \theta \hat{\mathbf{e}}_r + \sin \theta \hat{\mathbf{e}}_z \right) + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) \left( -\sin \theta \hat{\mathbf{e}}_r + \cos \theta \hat{\mathbf{e}}_z \right) \\
+ \frac{1}{r} \left( \frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\mathbf{e}}_z
$$

Also,

$$
\mathbf{F} = F_r \hat{\mathbf{e}}_r + F_\theta \hat{\mathbf{e}}_\theta + F_z \hat{\mathbf{e}}_z = F_r \left( \cos \theta \hat{\mathbf{e}}_r + \sin \theta \hat{\mathbf{e}}_z \right) \\
+ F_\theta \left( -\sin \theta \hat{\mathbf{e}}_r + \cos \theta \hat{\mathbf{e}}_z \right) + F_z \hat{\mathbf{e}}_z
$$

$$
= (F_r \cos \theta - F_\theta \sin \theta) \hat{\mathbf{e}}_r + (F_r \sin \theta + F_\theta \cos \theta) \hat{\mathbf{e}}_z + F_z \hat{\mathbf{e}}_z
$$

So

$$
F_1 = F_r \cos \theta - F_\theta \sin \theta
$$

$$
F_2 = F_r \sin \theta + F_\theta \cos \theta
$$

$$
F_3 = F_z
$$

Using (1), (for example, multiplying first eq. by $\cos \theta$ and second eq. by $\sin \theta$ and adding the two equations), we can conclude that

$$
F_r = F_1 \cos \theta + F_2 \sin \theta
$$

$$
F_\theta = F_2 \cos \theta - F_1 \sin \theta
$$

$$
F_z = F_3
$$

cont →
Now, replacing \( F_1, F_2, \) and \( F_3 \) in (**) with these expressions in terms of \( F_1, F_2 \) and \( F_3 \), applying the chain rule when taking \( \frac{\partial}{\partial \theta} \), and grouping all \( \hat{\varepsilon}_1, \hat{\varepsilon}_2, \) and \( \hat{\varepsilon}_3 \) terms together, we find that

\[
(\*\*\* \text{ in terms of } \hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3 ) = \left( \frac{\partial F_3}{\partial x_1} - \frac{\partial F_2}{\partial x_3} \right) \hat{\varepsilon}_1 + \left( \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right) \hat{\varepsilon}_2 + \left( \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_3} \right) \hat{\varepsilon}_3,
\]

\[= \nabla \times F.\]

Remark: To write the cross product of two arbitrary vector fields \( \vec{F}, \vec{G} \), start by writing

\[\vec{F} \times \vec{G} = (F_2 G_3 - F_3 G_2) \hat{\varepsilon}_1 + (F_3 G_1 - F_1 G_3) \hat{\varepsilon}_2 + (F_1 G_2 - F_2 G_1) \hat{\varepsilon}_3. \]

Replace \( F_i \) and \( G_i \) with \( F_r, F_\theta, F_\phi, G_r, G_\theta, G_\phi \).

Using (**) on last page, and rewrite \( \hat{\varepsilon}_1, \hat{\varepsilon}_2, \) and \( \hat{\varepsilon}_3 \) in terms of \( \hat{\varepsilon}_r, \hat{\varepsilon}_\theta, \hat{\varepsilon}_\phi \):

Note:

\[
\begin{align*}
\hat{\varepsilon}_r &= \cos \Theta \hat{\varepsilon}_1 + \sin \Theta \hat{\varepsilon}_2 \\
\hat{\varepsilon}_\theta &= -\sin \Theta \hat{\varepsilon}_1 + \cos \Theta \hat{\varepsilon}_2 \\
\hat{\varepsilon}_\phi &= \hat{\varepsilon}_3
\end{align*}
\]

Using (***) , we see that

\[
\begin{align*}
\hat{\varepsilon}_1 &= \hat{\varepsilon}_r \cos \Theta - \hat{\varepsilon}_\theta \sin \Theta \\
\hat{\varepsilon}_2 &= \hat{\varepsilon}_r \sin \Theta + \hat{\varepsilon}_\theta \cos \Theta
\end{align*}
\]

After putting everything in terms of \( r, \Theta, \phi \), combine all \( \hat{\varepsilon}_r \) terms, \( \hat{\varepsilon}_\theta \) terms, and \( \hat{\varepsilon}_\phi \) terms to get cross product.
Viscous fluid flows down a pipe of circular cross section \( r = a \) under a constant pressure gradient \( P = -\frac{dp}{dz} \). Show

\[
    u_z = \frac{P}{\rho} \left( a^2 - r^2 \right) \quad \Rightarrow \quad u_r = u_\theta = 0.
\]

**Solution**: We can assume in this case that \( u_r = u_\theta = 0 \), and since the flow is steady, \( \frac{\partial u_z}{\partial t} = 0 \). Also, by assumption, \( \frac{dp}{\partial \theta} = \frac{dp}{dr} = 0 \). Plugging all of this info into (NS), we find that

\[
    \frac{\partial u_r}{\partial t} + (\vec{u} \cdot \nabla) u_r - \frac{u_r^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nabla \left( \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right)
\]

reduces to \( 0 = 0 \).

Similarly,

\[
    \frac{\partial u_\theta}{\partial t} + (\vec{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \nabla \left( \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right).
\]

The equation

\[
    \frac{\partial u_z}{\partial t} + (\vec{u} \cdot \nabla) u_z = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nabla \nabla^2 u_z
\]

reduces to

\[
    u_z \frac{\partial^2 u_z}{\partial z^2} = \frac{P}{\rho} + \nabla \nabla^2 u_z \quad (*)
\]

For this type of flow, it is reasonable to assume that \( u_z \) does not depend on \( z \) or \( \theta \) (think about this!).
so (*) reduces to
\[-\frac{P}{\mu} = \nu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_2}{\partial r} \right) \right)\]

\[\Rightarrow -\frac{P}{\mu} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_2}{\partial r} \right).\]

We solve for \( u_2 \) with the boundary condition \( u_2(r=a) = 0 \).
We have
\[-\frac{P}{\mu} \frac{r}{r} = \frac{\partial}{\partial r} \left( r \frac{\partial u_2}{\partial r} \right)\]

\[\Rightarrow -\frac{P}{2\mu} r^2 + C_1 = r \frac{\partial u_2}{\partial r} \left( \text{integrated} \right)\]

\[\Rightarrow -\frac{P}{2\mu} r + \frac{1}{r} C_1 = \frac{\partial u_2}{\partial r}\]

\[\Rightarrow u_2 = -\frac{P}{4\mu} r^2 + C_1 \ln r + C_2 \left( \text{integrated} \right)\]

Assuming \( u_2(r) < \infty \) \( \forall \ r \leq a \), we have

\[u_2(0) = -\frac{P}{4\mu} (0) + C_1 \ln (0) + C_2\]

\[\Rightarrow C_1 = 0, \text{ so}\]

\[u_2 = -\frac{P}{4\mu} r^2 + C_2. \text{ But}\]

\[u_2(a) = -\frac{P}{4\mu} a^2 + C_2 = 0 \Rightarrow C_2 = \frac{P}{4\mu} a^2,\]

So

\[u_2(r) = \frac{P}{4\mu} \left( a^2 - r^2 \right).\]
5) (Pg. 54, #2.11): Viscous fluid occupies \( 0 < z < h \) between two rigid boundaries \( z = 0 \) and \( z = h \). The lower boundary is stationary, and the upper boundary rotates with angular velocity \( \omega \) about the \( z \)-axis.

Show a solution to \((NS)\) of the form

\[ \vec{u} = u_0 (r, z) \hat{e}_\theta \] is not possible.

Solution: Assume for contradiction that such a solution is possible. Then \((NS)\) reduces to the following:

\[ \begin{align*}
\text{r-component:} & \quad - \frac{u_0^2}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} \quad (*) \\
\text{z-component:} & \quad \frac{\partial p}{\partial z} = 0.
\end{align*} \]

Since \( p \) does not depend on \( z \), \( \frac{\partial p}{\partial r} \) does not depend on \( z \). Therefore, by \((*)\), \( u_0 \) does not depend on \( z \).

This contradicts the no-slip boundary conditions.

\[ \begin{align*}
\mu \frac{\partial u_1}{\partial x_2} (x_2 + b x_1) \cdot b x_1 \\
\mu \frac{\partial u_2}{\partial x_1} (x_2 + b x_1) \cdot b x_2 \\
\mu \frac{\partial u_1}{\partial x_2} (x_1 + b x_2) \cdot b x_2 \\
\mu \frac{\partial u_2}{\partial x_2} (x_1 + b x_2) \cdot b x_1
\end{align*} \]