Solutions to Review 2

1a) $y = x^3 - \frac{3}{2}x^2 - 6x$

$y' = 3x^2 - 3x - 6$

$= 3(x^2 - x - 2)$

$= 3(x-2)(x+1) = 0$ if $x = 2$ or $x = -1$

$y'' = 6x - 3 = 0$ if $x = \frac{1}{2}$

		-1		2	
$(x-2)$	$-$		$-$		$+$
$(x+1)$	$-$		$+$		$+$
y'	$+$	max	$-$	min	$+$

		$\frac{1}{2}$	
y''	$-$	inf. pt.	$+$

$y(-1) = (-1)^3 - \frac{3}{2}(-1)^2 - 6(-1)$

$= -1 - \frac{3}{2} + 6$

$= \frac{7}{2}$

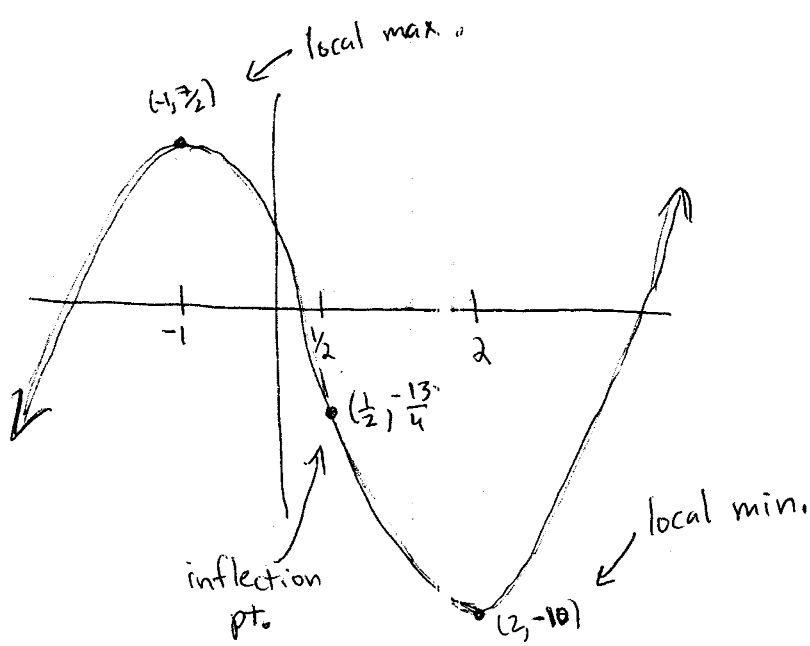
$y(2) = 2^3 - \frac{3}{2}(2)^2 - 6 \cdot 2$

$= 8 - 6 - 12 = -10$

$y(\frac{1}{2}) = (\frac{1}{2})^3 - \frac{3}{2}(\frac{1}{2})^2 - 6(\frac{1}{2})$

$= \frac{1}{8} - \frac{3}{8} - 3$

$= -\frac{13}{4}$



1b) $y = x^4 - 4x^3$

$y' = 4x^3 - 12x^2 = x^2(4x-12) = 0$ if $x=0$ or $x=3$

$y'' = 12x^2 - 24 := x(12x-24) = 0$ if $x=0$ or $x=2$

		0		3	
x^2	+		+		+
$(4x-12)$	−		−		+
y'	−		−	min	+

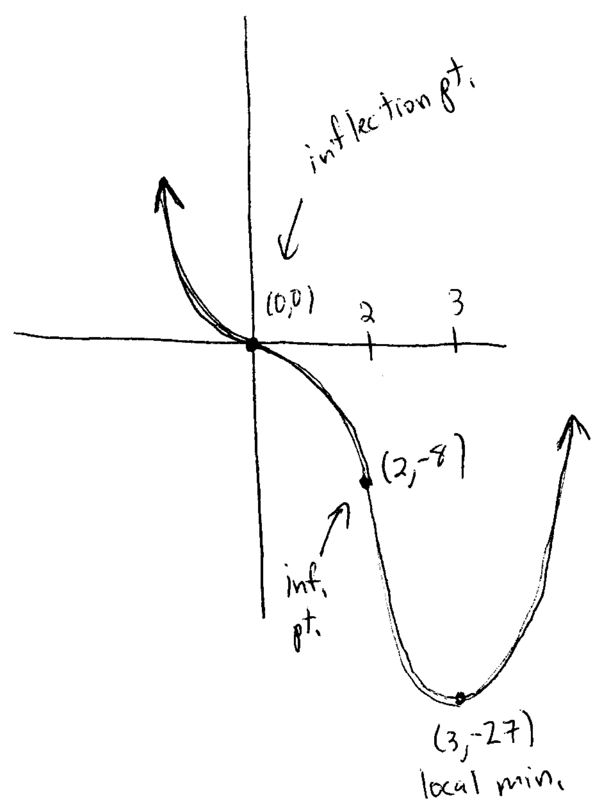
$y(0) = 0^4 - 4 \cdot 0^3 = 0$

$y(3) = 3^4 - 4(3^3) = -27$

		0		2	
x	−		+		+
$12x-24$	−		−		+
y''	+	inf.	−	inf.	+

$y(2) = 2^4 - 4 \cdot 2^3 = -8$

		0		2		3	
y'	−		−		−		+
y''	+		−		+		+
y							



2) minimize $x + y$

Subject to $xy^2 = 32$

$x \geq 0, y \geq 0$

$$x = \frac{32}{y^2}$$

$$f(y) = \frac{32}{y^2} + y$$

$$f'(y) = \frac{-64}{y^3} + 1 = 0 \quad \text{if} \quad 1 = \frac{64}{y^3}$$

$$y^3 = 64$$

$$y = 4$$

$$f''(y) = \frac{-192}{y^4}$$

$$f''(4) = \frac{-192}{4^4} < 0$$ so 4 is a local max ✓

(Note $f'' < 0$ for all $y > 0$ so 4 is abs. max)

$$x = \frac{32}{y^2} = \frac{32}{4^2} = 2$$

So max occurs if $x = 2, y = 4$

$$x + y = \boxed{6}$$

3) a) $p = 50 - .2x$

$C(x) = .1x^2 + 5x + 96$

Profit $P(x) = R(x) - C(x)$

$= x(50 - .2x) - (.1x^2 + 5x + 96)$

$= 50x - .2x^2 - .1x^2 - 5x - 96$

$= -.3x^2 + 45x - 96$

$P'(x) = -.6x + 45 = 0$ if $45 = .6x$

$75 = x$

$P''(x) = -.6 < 0$ so P concave down everywhere

$\Rightarrow 75$ is abs. max

The company should produce 75 units.

b) $p = 50 - .2x = 50 - .2(75)$

$= 50 - 15$

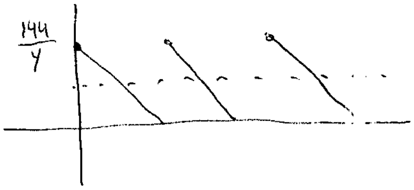
$= 35$

price will be 35

4) $x =$ batch size

$y =$ # batches made per day

$xy = 144$



$\leftarrow$ avg. inventory $= \frac{1}{2}\left(\frac{144}{y}\right) = \frac{72}{y}$

$$\text{Cost} = 8y + .25\left(\frac{72}{y}\right)$$

$$= 8y + \frac{18}{y}$$

$$C' = 8 - \frac{18}{y^2} = 0 \quad \text{if} \quad 8 = \frac{18}{y^2}$$

$$8y^2 = 18$$

$$y^2 = \frac{18}{8} = \frac{9}{4}$$

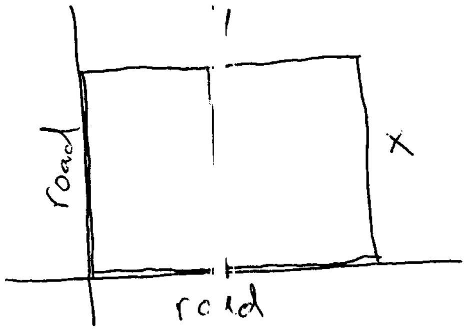
$$y = \frac{3}{2}$$

$$C'' = \frac{36}{y^3} > 0 \quad \text{for all } y > 0$$

so $y = \frac{3}{2}$ is abs. min.

They should prepare $\frac{3}{2}$ batches (?) per day.

5)



Area $= xy$

cost $= 4(2x) + 4y + 8x + 8y$

$= 8x + 8x + 12y$

$= 16x + 12y$

So the optimization problem is:

maximize $f = xy$

subject to $16x + 12y = 960$

$$12y = 960 - 16x$$

$$y = 80 - \frac{4}{3}x$$

Maximize $f = x\left(80 - \frac{4}{3}x\right)$

$$= 80x - \frac{4}{3}x^2$$

$$f' = 80 - \frac{8}{3}x = 0 \quad \text{if} \quad 80 = \frac{8}{3}x$$

$$30 = x$$

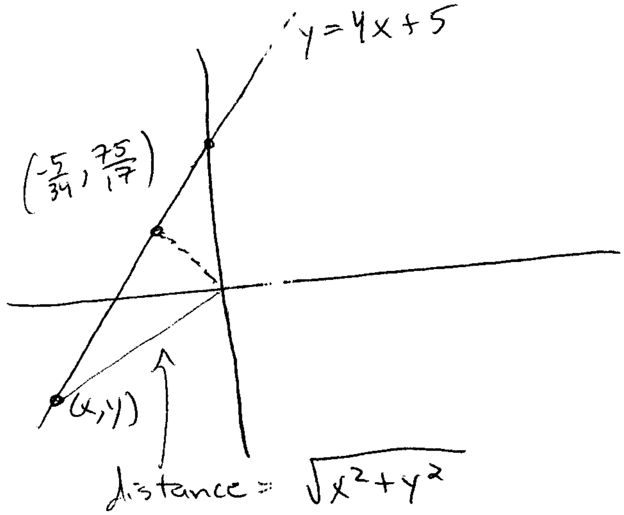
$$f'' = -\frac{8}{3} < 0 \text{ for all } x$$

so $x = 3$ is abs. max. ✓

$$y = 80 - \frac{4}{3}x = 80 - \frac{4}{3}\cdot 30 = 40$$

The farmer should make his pasture 30×40 for an area of 1200.

6)



minimize $f = \sqrt{x^2+y^2}$

subject to $y = 4x+5$

$$f = \sqrt{x^2 + (4x+5)^2}$$

$$f' = \frac{1}{2}\left(x^2+(4x+5)^2\right)^{-1/2}\left(2x + 2(4x+5)\cdot 4\right)$$

$$= \frac{1}{2}\ \frac{2x + 32x + 5}{\sqrt{x^2+(4x+5)^2}}$$

$$= \frac{34x+5}{2\sqrt{x^2+(4x+5)^2}} = 0$$

if $34x = -5$

$x = \frac{-5}{34}$

$f' < 0$ if $x < -\frac{5}{34}$ $\Rightarrow$ $\frac{-5}{34}$ is min

$f' > 0$ if $x > -\frac{5}{34}$

$$y = 4x+5 = 4\left(\frac{-5}{34}\right) + 5$$

$$= \frac{-20}{34} + \frac{170}{34} = \frac{150}{34} = \frac{75}{17}$$

The closest point is $\left(\frac{-5}{34}, \frac{75}{17}\right)$

7a) $\frac{d}{dx}\Big|_{x=2}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}\Big|_{x=2}$

$= \frac{g(2)f'(2)-f(2)g'(2)}{[g(2)]^2}$

$= \frac{(-2)(3) - 4(3)}{(-2)^2} = \frac{-6-12}{4} = \frac{-18}{4} = \boxed{\frac{-9}{2}}$

7b) $\frac{d}{dx}\Big|_{x=2}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)\Big|_{x=2}$

$= f'(2)g(2) + g'(2)f(2)$

$= 3(-2) + (3)(4)$

$= -6 + 12 = \boxed{6}$

8) $\frac{d}{dx}\Big|_{x=2} f(g(x)) = f'(g(x))g'(x)\Big|_{x=2}$

$= f'(g(2))g'(2)$

$= f'(4)\cdot 3$

$= (-3)(3) = \boxed{-9}$

9) $x^2 - y^2 - xy = 5$

Implicit differentiation:

product rule

$$2x - 2y\frac{dy}{dx} - \left(x\frac{dy}{dx} + y \cdot 1\right) = 0$$

$$2x - 2y\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$$

$$2x - y = 2y\frac{dy}{dx} + x\frac{dy}{dx}$$

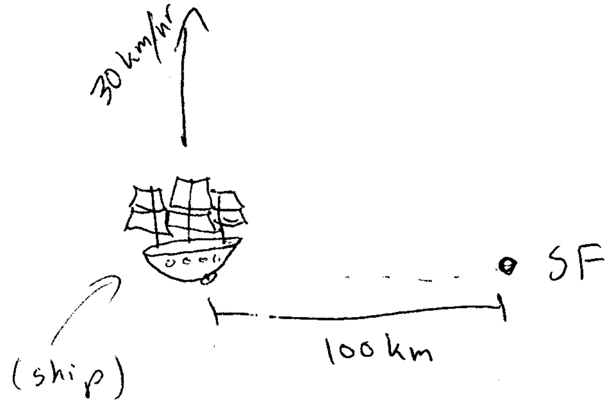
$$2x - y = \frac{dy}{dx}(2y + x)$$

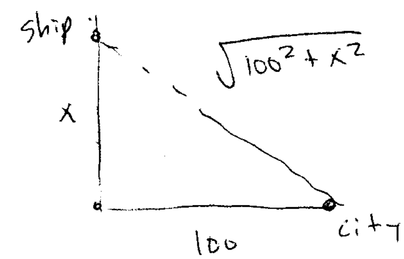
$$\frac{2x - y}{2y - x} = \frac{dy}{dx}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=3\\y=1}} = \left.\frac{2x - y}{2y - x}\right|_{\substack{x=3\\y=1}} = \frac{2 \cdot 3 - 1}{2 \cdot 1 - 3} = \frac{5}{-1} = -5$$

slope of tangent line is -5.

10)





f = distance from ship to city = $\sqrt{100^2+x^2}$

$$\frac{dx}{dt} = 30$$

$$\frac{df}{dt} = \tfrac{1}{2}(100^2+x^2)^{-1/2}(2x)\frac{dx}{dt}$$

At noon, $x=0$, $\frac{dx}{dt}=30$, so

$$\left.\frac{df}{dt}\right|_{\text{noon}} = \frac{1}{2}(100^2+0^2)^{-1/2}(2\cdot 0)\cdot 30 = \boxed{0\ \text{km}/\text{hr}}$$

(This makes sense since at noon the distance is minimized)

At 4pm, $x=120$, $\frac{dx}{dt}=30$, so

$$\left.\frac{df}{dt}\right|_{4pm} = \frac{1}{2}(100^2+120^2)^{-1/2}(2\cdot 120)\cdot 30$$

$$= \frac{1}{2}\frac{7200}{\sqrt{24400}} \boxed{\approx 23\ \text{km}/\text{h}}$$