

The Ramsey property for Banach spaces and Choquet simplices, and applications

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Abstract

We show that the class of finite-dimensional Banach spaces and the class of finite-dimensional Choquet simplices have the Ramsey property. As an application, we show that the group $\text{Aut}(\mathbb{G})$ of surjective linear isometries of the Gurarij space \mathbb{G} is extremely amenable, and that the canonical action $\text{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ is the universal minimal flow of the group $\text{Aut}(\mathbb{P})$ of affine homeomorphisms of the Poulsen simplex \mathbb{P} . This answers questions of Melleray–Tsankov and Conley–Törnquist.

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Résumé

Nous montrons que la classe des espaces de Banach de dimension finie et la classe des simplexes de Choquet de dimension finie ont la propriété de Ramsey. En guise d'application, nous montrons que le groupe $\text{Aut}(\mathbb{G})$ des isométries linéaires surjectives de l'espace de Gurarij \mathbb{G} est extrêmement moyennable et que l'action canonique $\text{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ est le flot minimal universel du groupe $\text{Aut}(\mathbb{P})$ des homéomorphismes affines du simplexe de Poulsen \mathbb{P} . Ceci répond aux questions de Melleray–Tsankov et Conley–Törnquist.

Pour citer cet article :

1. Introduction

Ramsey theory studies, broadly speaking, which combinatorial configurations can always be found within one of the pieces of a given finite partition (or coloring) of a certain object. A foundational result in Ramsey theory is Ramsey’s theorem from 1930, which can be stated as follows. Suppose that $r, k, m \in \mathbb{N}$, where k, m are considered as finite ordered sets with respect to the canonical orderings. Then there exists $n \in \mathbb{N}$ such that for every r -coloring (partition into r pieces) of the set $\text{Emb}(k, n)$ of order-preserving injections from k to n , there exists an order-preserving injection γ from m to n such that the set $\gamma \circ \text{Emb}(k, m) = \{\gamma \circ \rho : \rho \in \text{Emb}(k, m)\}$ is monochromatic, i.e. contained in one of the pieces of the partition. Ramsey’s theorem was used by Pestov [18] to establish a remarkable property of the group $\text{Aut}(\mathbb{Q}, <)$ of bijections of the set of rationals that preserve the canonical ordering inherited from the reals, endowed with the topology of pointwise convergence. Pestov showed that $\text{Aut}(\mathbb{Q}, <)$ is *extremely amenable*, namely any continuous action of $\text{Aut}(\mathbb{Q}, <)$ on a compact Hausdorff space has a fixed point. This was one of the first naturally occurring example of an extremely amenable group, besides the example of the unitary group $U(\mathcal{H})$ of the separable infinite-dimensional Hilbert space obtained by Gromov and Milman [7], and previous examples due to Herer and Christensen [9].

A far-reaching generalization of Pestov’s argument was considered by Kechris, Pestov, and Todorcevic in the influential paper [10]. There, the authors work in the framework of Fraïssé theory. Initially developed by Fraïssé in the 1950s, Fraïssé theory provides a correspondence between countable structures where any partial isomorphism between finitely-generated substructures can be extended to an automorphism (*ultrahomogeneous structures*), and directed classes of finitely-generated structures closed under substructures satisfying the amalgamation property (*Fraïssé classes*). Given a Fraïssé class \mathcal{C} , one can consider its canonical Fraïssé limit, which is a countable ultrahomogeneous structure. Conversely, any Fraïssé class arises as the collection of finitely-generated substructures (or *age*) of a countable ultrahomogeneous structure. The main result of [10] establishes a surprising tight correspondence between dynamical properties of the automorphism group $\text{Aut}(M)$ of a countable homogeneous structure M and combinatorial properties of the age \mathcal{C} of M . Precisely, the *KPT correspondence* asserts that $\text{Aut}(M)$ is extremely amenable if and only if \mathcal{C} satisfies the following *Ramsey property*: for every $r \in \mathbb{N}$ and $A, B \in \mathcal{C}$ there exists $C \in \mathcal{C}$ such that for every r -coloring of the space $\text{Emb}(A, C)$ of embeddings of A into C there exists $\phi \in \text{Emb}(B, C)$ such that $\phi \circ \text{Emb}(A, B)$ is monochromatic. When \mathcal{C} is the class of finite linear orders, the KPT correspondence recovers Pestov’s earlier result.

This work provided new impetus on the study of extreme amenability and topological dynamics of “large” groups and sparked new interest in structural Ramsey theory. Since then many new examples of Fraïssé classes satisfying the Ramsey property have been found, leading to many natural examples of extremely amenable groups. Recently, a body of work initiated by Ben Yaacov [2] has shown that the machinery of Fraïssé theory admits a natural generalization to the context of *metric structures*. This more general framework is applicable to “continuous” structures arising from functional analysis such as, prominently, Banach spaces. As essentially shown by Kubis and Solecki in [11], the Fraïssé limit of the class of finite-dimensional Banach spaces is the Gurarij space \mathbb{G} , initially constructed by Gurarij in 1966 [8].

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This important example motivated Melleray and Tsankov to extend the KPT correspondence to the metric setting [17], and to ask whether it can be applied to show that the group $\text{Iso}(\mathbb{G})$ of surjective linear isometries of \mathbb{G} is extremely amenable [17,16]. Our main result is that this is indeed the case.

2. The Ramsey property for Banach spaces

The notion of Ramsey property for a class of finitely-generated discrete structures admits a natural generalization in the setting of metric structures, where one replaces discrete colorings with “continuous colorings”, which are just $[0,1]$ -valued 1-Lipschitz maps [17]. In the case of Banach spaces, an easy approximation argument shows that this is in turn equivalent to Theorem 2.1. For finite-dimensional Banach spaces X, Y , we let $\text{Emb}(X, Y)$ be the space of linear isometries from X to Y endowed with the distance $d(\phi, \psi) = \sup \{\|\phi(x) - \psi(x)\| : x \in X, \|x\| \leq 1\}$. The ε -fattening of a subset A of $\text{Emb}(X, Y)$ is the set of elements of $\text{Emb}(X, Y)$ of distance at most ε from A .

Theorem 2.1 *For every $r \in \mathbb{N}$, finite-dimensional Banach spaces X, Y , and $\varepsilon > 0$, there exists a finite-dimensional Banach space Z such that for every r -coloring of $\text{Emb}(X, Z)$ there exists $\phi \in \text{Emb}(Y, Z)$ such that $\phi \circ \text{Emb}(X, Y)$ is ε -monochromatic, i.e. contained in the ε -fattening of one of the colors.*

Corollary 2.2 *The group $\text{Aut}(\mathbb{G})$ of surjective linear isometries of the Gurarij space is extremely amenable.*

We will present the proof of Theorem 2.1 in the case of real scalars, the complex case being very similar. The general statement of Theorem 2.1 follows from the particular instance when X and Y are *injective*, i.e. $X = \ell_\infty^d$ and $Y = \ell_\infty^m$ for some $d, m \in \mathbb{N}$. Indeed, assuming that Theorem 2.1 holds for such spaces, one can deduce the general form of Theorem 2.1 in two steps as follows:

- First, one proves Theorem 2.1 in the case when X and Y are *polyhedral*, that is, their unit balls have finitely many extreme points or, equivalently, they embed isometrically into a finite-dimensional injective Banach space. This can be done by using the *injective envelope* construction for a given finite-dimensional polyhedral space X . This is a finite-dimensional injective Banach space $\ell_\infty^{n_X}$ together with an isometric embedding $\phi : X \rightarrow \ell_\infty^{n_X}$ satisfying the following universal property: every isometric embedding from X into another finite-dimensional injective Banach space factors through ϕ . In this way, a coloring of $\text{Emb}(X, Y)$ naturally induces a coloring of $\text{Emb}(\ell_\infty^{n_X}, \ell_\infty^{n_Y})$.
- The result for polyhedral spaces can be used to prove the general case using: 1) the fact that a finite dimensional space is approximated by polyhedral spaces in the Banach-Mazur distance, and 2) the following approximation result established in [11]: Suppose that $\varepsilon, \delta > 0$, and X, Y are finite dimensional Banach spaces. Then there is a finite dimensional Banach space Z and an isometric embedding $J : Y \rightarrow Z$ such that for every linear map $\varphi : X \rightarrow Y$ satisfying $(1 + \varepsilon)^{-1}\|x\|_X \leq \|\varphi(x)\|_Y \leq \|x\|_X$ there exists an isometric embedding $I : X \rightarrow Z$ such that $\|J \circ \varphi - I\| \leq \varepsilon + \delta$.

It remains therefore to prove the particular instance of Theorem 2.1 when X and Y are injective. This will be obtained as an application of the Dual Ramsey Theorem of Graham and Rothschild [6]. Such a result can be stated in terms of finite linear orders and rigid surjections. Given finite linear orders $\mathcal{L} = (L, <_L)$ and $\mathcal{R} = (R, <_R)$, a *rigid surjection* from \mathcal{L} to \mathcal{R} is a surjective map $f : L \rightarrow R$ such that $\min_{<_L} f^{-1}(r_0) <_L \min_{<_L} f^{-1}(r_1)$ for every $r_0, r_1 \in R$ such that $r_0 <_R r_1$. We denote such a set by $\text{Epi}(\mathcal{L}, \mathcal{R})$. The Dual Ramsey Theorem (DRT) asserts that, for every finite orders $\mathcal{L} = (L, <_L)$ and $\mathcal{R} = (R, <_R)$, and for every $r \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for every r -coloring of $\text{Epi}(n, \mathcal{R})$ there exists $f \in \text{Epi}(n, \mathcal{L})$ such that $\text{Epi}(\mathcal{L}, \mathcal{R}) \circ f$ is monochromatic. Here the natural number n is considered as a finite linear order with respect to its canonical ordering. Using the obvious fact that two finite linear orders of the same cardinality are order-isomorphic, this statement can be easily reformulated just in terms of rigid surjections between natural numbers, or equivalently in terms of ordered partitions of finite sets.

Observe now that embeddings $\phi : \ell_\infty^d \rightarrow \ell_\infty^m$ correspond by duality to *contractive quotient mappings* $\pi : \ell_1^m \rightarrow \ell_1^d$. These are contractive linear functions that map the unit ball of ℓ_1^m onto the unit ball of ℓ_1^d . One can regard π as a $d \times m$ scalar matrix A with respect to the canonical bases $(e_i)_{i < m}$ and $(e_i)_{i < d}$ of these spaces. The fact that π is a contractive quotient mapping implies that each column of A is in the unit ball of ℓ_1^d and each row of A contains an entry of absolute value 1. Conversely, any such a matrix arises as the representative matrix of a contractive quotient mapping from ℓ_1^m to ℓ_1^d . We regard the space $Q(\ell_1^m, \ell_1^d)$ of contractive quotient mappings from ℓ_1^m to ℓ_1^d as a metric space, endowed with a metric given by $d(\pi_0, \pi_1) := \|\pi_0 - \pi_1\|_{\ell_1^m \rightarrow \ell_1^d} = \sup\{\|(\pi_0 - \pi_1)(x)\|_{\ell_1^d} : x \in \ell_1^m, \|x\|_{\ell_1^m} \leq 1\}$. These observations show that, by passing to the duals, it suffices to prove the following statement.

Lemma 2.3 *For every $r, d, m \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every r -coloring of the space $Q(\ell_1^n, \ell_1^d)$ of contractive quotient mappings from ℓ_1^n to ℓ_1^d , there exists $\pi \in Q(\ell_1^n, \ell_1^m)$ such that $Q(\ell_1^m, \ell_1^d) \circ \pi$ is ε -monochromatic.*

Here is an extremely simplified summary of our approach to the proof of Lemma 2.3. Notice that, given a finite subset D of the unit ball of ℓ_1^d such that $\{e_i\}_{i < d} \subseteq D \cup (-D)$, there is a natural assignment $f \in \text{Epi}(n, D) \mapsto \pi_f \in Q(\ell_1^n, \ell_1^d)$, obtained by defining $\pi_f(e_j) := f(j)$ for every $j < n$. Similarly one can define an assignment $f \in \text{Epi}(n, m) \mapsto \pi_f \in Q(\ell_1^n, \ell_1^m)$ as above, where D is the canonical basis of ℓ_1^m . So, an r -coloring c of $Q(\ell_1^n, \ell_1^d)$ induces an r -coloring \hat{c} of $\text{Epi}(n, D)$, given by $\hat{c}(f) := c(\pi_f)$. Given such a coloring c , apply the DRT to D , endowed with an arbitrary linear ordering, and to m , identified with the basis $(e_i)_{i < m}$, to find $n \in \mathbb{N}$ such that there exists $g \in \text{Epi}(n, m)$ such that $\text{Epi}(m, D) \circ g$ is monochromatic for \hat{c} . Then prove that, for a suitable choice of D , $Q(\ell_1^n, \ell_1^d) \circ \pi_g$ is ε -monochromatic for c , by finding for each $\psi \in Q(\ell_1^n, \ell_1^d)$ some $h \in \text{Epi}(m, D)$ such that $\|\psi \circ \pi_g - \pi_{h \circ g}\|_{\ell_1^n \rightarrow \ell_1^d} \leq \varepsilon$. While this simplified approach does not work, a suitable modification of it will yield a proof of Lemma 2.3, as follows.

Fix $r, d, m \in \mathbb{N}$ with $d \leq m$, and $\varepsilon > 0$. Let \mathcal{P} be a finite subset of the unit ball of ℓ_1^d containing the zero vector, the canonical basis of ℓ_1^d , and such that for any nonzero element v in the unit ball of ℓ_1^d there exists $v' \in \mathcal{P}$ such that $\|v - v'\|_{\ell_1^d} < \varepsilon$ and $\|v'\|_{\ell_1^d} < \|v\|_{\ell_1^d}$. Such a set can be explicitly constructed as follows. Let D be a finite $\varepsilon/2$ -subset of the unit sphere of ℓ_1^d , and let $s \in \mathbb{N}$ be such that $(1 - \varepsilon/2)^s < \varepsilon/2$. Then the set $\mathcal{P} := \bigcup_{i=0}^s (1 - \varepsilon/2)^i \cdot D$ has the required properties.

Let also \mathcal{Q} be the (finite) set of $m \times d$ matrices with entries from $\{0, 1, -1\}$ and such that every column contains exactly one nonzero entry, and every row contains at most one nonzero entry. Observe that any element of \mathcal{Q} is the representative matrix of an isometric embedding from ℓ_1^d to ℓ_1^m . In the following, we will identify an element of \mathcal{Q} with the corresponding isometric embedding.

Fix an arbitrary linear order on \mathcal{Q} , and then a linear order on \mathcal{P} with the property that $w < w'$ whenever $w, w' \in \mathcal{P}$ are such that $\|w\|_{\ell_1^d} < \|w'\|_{\ell_1^d}$. Let $\mathcal{Q} \times \mathcal{P}$ be endowed with the corresponding antilexicographic order. The key property of \mathcal{Q} is that for every $\sigma \in Q(\ell_1^m, \ell_1^d)$ there exists $M_\sigma \in \mathcal{Q}$ such that the composition $\sigma \circ M_\sigma$ is the identity of ℓ_1^d .

Consider now $n \in \mathbb{N}$ obtained from the linear orders \mathcal{P} and $\mathcal{Q} \times \mathcal{P}$ by applying the DRT. We claim that such an n satisfies the conclusions of Lemma 2.3. To simplify the presentation, we introduce some notation:

- (i) Given $f \in \text{Epi}(n, \mathcal{P})$, let $\pi_f \in Q(\ell_1^n, \ell_1^d)$ be defined as above, that is, $\pi_f(e_j) := f(j)$ for every $j < n$;
- (ii) Given $g \in \text{Epi}(n, \mathcal{Q} \times \mathcal{P})$, let $\psi_g \in Q(\ell_1^n, \ell_1^m)$ be defined by $\psi_g(e_i) := g_0(i)(g_1(i))$ for $i < n$, where $g_0 : n \rightarrow \mathcal{Q}$ and $g_1 : n \rightarrow \mathcal{P}$ are such that $g(i) = (g_0(i), g_1(i))$ for every $i < n$. (Here we applied our convention of identifying the element $g_0(i)$ of \mathcal{Q} with a linear map from ℓ_1^d to ℓ_1^m . Thus $g_0(i)(g_1(i))$ denotes the element of ℓ_1^m obtained by applying $g_0(i)$ to the element $g_1(i)$ of ℓ_1^d .)

Let c be an r -coloring of $Q(\ell_1^n, \ell_1^d)$, and let \hat{c} be the r -coloring of $\text{Epi}(n, \mathcal{P})$ defined by $\hat{c}(f) := c(\pi_f)$. By the choice of n (which we obtained by applying the DRT to \mathcal{P} and $\mathcal{Q} \times \mathcal{P}$), we can find $g \in \text{Epi}(n, \mathcal{Q} \times \mathcal{P})$ such that $\text{Epi}(\mathcal{Q} \times \mathcal{P}, \mathcal{P}) \circ g$ is monochromatic for \hat{c} . We claim that then $Q(\ell_1^m, \ell_1^d) \circ \psi_g$ is ε -monochromatic for c , which is an immediate consequence of the following claim.

Claim 1 Let $g \in \text{Epi}(n, \mathcal{Q} \times \mathcal{P})$ be as above. For every $\sigma \in Q(\ell_1^m, \ell_1^d)$ there exists $h \in \text{Epi}(\mathcal{Q} \times \mathcal{P}, \mathcal{P})$ such that $\|\pi_{h \circ g} - \sigma \circ \psi_g\|_{\ell_1^n \rightarrow \ell_1^d} \leq \varepsilon$.

In order to prove the claim, fix $\sigma \in Q(\ell_1^m, \ell_1^d)$. Let $M_\sigma \in \mathcal{Q}$ be such that $\sigma \circ M_\sigma$ is the identity on ℓ_1^d . We define $h \in \text{Epi}(\mathcal{Q} \times \mathcal{P}, \mathcal{P})$ as follows. For $(M, v) \in \mathcal{Q} \times \mathcal{P}$:

- (i) if $\sigma(Mv) = 0$, then we declare $h(M, v) := 0$,
- (ii) if $\sigma(Mv) \neq 0$ and $M = M_\sigma$, then we declare $h(M, v) := v$, and
- (iii) otherwise, we define $h(M, v)$ to be an arbitrary $w \in \mathcal{P}$ such that $\|w - \sigma(Mv)\|_1 < \varepsilon$ and $\|w\|_1 < \|\sigma(Mv)\|_1$. Such an element w of \mathcal{P} exists by definition of \mathcal{P} .

Clearly, h maps $\mathcal{Q} \times \mathcal{P}$ onto \mathcal{P} . Recall that $\mathcal{Q} \times \mathcal{P}$ is endowed with the antilexicographic ordering. Furthermore, the linear order on \mathcal{P} has the property that $w < w'$ whenever $w, w' \in \mathcal{P}$ are such that $\|w\|_{\ell_1^d} < \|w'\|_{\ell_1^d}$. We now show that h is a rigid surjection from $\mathcal{Q} \times \mathcal{P}$ to \mathcal{P} . Fix $w \in \mathcal{P}$. We compute $\min_{<_{\mathcal{Q} \times \mathcal{P}}} h^{-1}(w)$. If $w = 0$, then $\min_{<_{\mathcal{Q} \times \mathcal{P}}} h^{-1}(w) = (M_0, 0)$, where M_0 is the first element of \mathcal{Q} . Suppose now that $w \neq 0$. If $h(M, v) = w$, then either $v = w$, and $M = M_\sigma$, or else w is such that $\|w\|_{\ell_1^d} < \|\sigma(Mv)\|_{\ell_1^d}$. Since σ is a contraction and M is an isometric embedding, it follows that $\|w\|_{\ell_1^d} < \|\sigma(Mv)\|_{\ell_1^d} \leq \|Mv\|_{\ell_1^m} = \|v\|_{\ell_1^d}$. This implies that $\min_{<_{\mathcal{Q} \times \mathcal{P}}} h^{-1}(w) = (M_\sigma, w)$. These observations, together with the definition of the ordering on $\mathcal{Q} \times \mathcal{P}$ show that h is a rigid surjection. To conclude the proof, it remains to see that $\|\pi_{h \circ g} - \sigma \circ \psi_g\|_{\ell_1^n \rightarrow \ell_1^d} < \varepsilon$. Since in both the domain and the range we are using the ℓ_1 -norm, it suffices to show that for each $j < n$ one has that $\|\pi_{h \circ g}(e_j) - (\sigma \circ \psi_g)(e_j)\|_{\ell_1^d} < \varepsilon$. Suppose first that $\sigma(g_0(j)(g_1(j))) = 0$. In this case, by clause (i) in the definition of h , $\pi_{h \circ g}(e_j) = h(g_0(j), g_1(j)) = 0$. Suppose now that $\sigma(g_0(j)(g_1(j))) \neq 0$ and that $g_0(j) = M_\sigma$. In this case, $\pi_{h \circ g}(e_j) = h(g(j)) = h(M_\sigma, g_1(j)) = g_1(j)$, while $\sigma(\psi_g(e_j)) = (\sigma \circ g_0(j))(g_1(j)) = (\sigma \circ M_\sigma)(g_1(j)) = g_1(j)$. Finally, if $\sigma(g_0(j)(g_1(j))) \neq 0$ and $g_0(j) \neq M_\sigma$, then, by (iii), $\pi_{h \circ g}(e_j) = h(g(j))$ is such that $\|h(g(j)) - \sigma(g_0(j)(g_1(j)))\|_{\ell_1^d} < \varepsilon$. This concludes the proof that h satisfies the desired conclusions.

We refer the reader to [1] for a more extended and complete explanation.

3. The Ramsey property for Choquet simplices

The combinatorial argument presented above can be naturally adapted to apply to the class of *function systems* (also known as order unit spaces). Again, for simplicity, we will consider the case of real scalars. A function system is an ordered Banach space X endowed with a distinguished Archimedean *order unit* 1. This is a positive element of X such that for every $x \in X$ one has: if $nx \leq 1$ for every $n \in \mathbb{N}$, then $x \leq 0$; there exists $n \in \mathbb{N}$ such that $-n1 \leq x \leq n1$; if x is positive, then $\|x\| = \inf \{t \in \mathbb{R} : x \leq t \cdot 1\}$. A linear map between function systems is *positive* if it maps positive elements to positive elements, and *unital* if it maps the order unit to the order unit. A unital linear map is positive if and only if it is contractive. A *state* on a function system is a positive unital linear functional. There is a natural (contravariant) equivalence of categories between function systems with unital positive linear maps and compact convex sets with continuous affine maps. This correspondence assigns to a function system X its *state space* $S(X) \subset X^*$ endowed with the w^* -topology, and conversely it assigns to a compact convex set K the function system $A(K) \subset C(K)$ of continuous affine maps from K to \mathbb{R} . Under this correspondence, the space of embeddings from X to Y endowed with the topology of pointwise convergence can be identified with the space of surjective affine continuous maps from $S(Y)$ onto $S(X)$ endowed with the compact open topology. For instance, one can consider ℓ_∞^{n+1} as a function system with respect to the entrywise order and the order unit $(1, 1, \dots, 1)$. The corresponding compact convex set is the standard n -dimensional Choquet simplex Δ_n , consisting of $t \in \mathbb{R}^{n+1}$ such that $t_i \geq 0$ for $i = 0, 1, \dots, n$ and $t_0 + t_1 + \dots + t_n = 1$.

Adapting the proof of Theorem 2.1 one can show that the class of function systems *with a distinguished state* satisfies the Ramsey property. An embedding between two function systems (X, s_X) and (Y, s_Y) with a distinguished state is a unital linear isometry $\phi : X \rightarrow Y$ such that $s_Y \circ \phi = s_X$. Again, since every function system approximately embeds into (ℓ_∞^{n+1}, e_0) for some $n \in \mathbb{N}$, where $e_0 = (1, 0, 0, \dots, 0) \in \Delta_n$, it suffices to establish the Ramsey property for the class $\{(\ell_\infty^{n+1}, e_0) : n \in \mathbb{N}\}$. To this purpose, one can notice that embeddings from (ℓ_∞^{d+1}, e_0) to (ℓ_∞^{n+1}, e_0) correspond by duality to contractive quotient mappings $\pi : \ell_1^{n+1} \rightarrow \ell_1^{d+1}$ such that $\pi(e_0) = e_0$, π maps positive elements to positive elements, and $\tau_{d+1} \circ \pi = \tau_{n+1}$, where $\tau_n(x_0, \dots, x_n) = x_0 + \dots + x_n$. In other words, if $A = [a_{ij}]$ is the representative matrix for π , one has that $a_{00} = 1$, $a_{ij} \geq 0$, each column of A is in Δ_d , and each row of A contains an entry which is equal to 1. One can thus proceed as in the proof of Theorem 2.1 by considering \mathcal{P} to be a finite subset of the positive part $\text{Ball}(\ell_1^d)_+$ of the unit ball of ℓ_1^d with the property that the zero vector and the canonical basis of ℓ_1^d belong to \mathcal{P} , and such that for any nonzero element v in $\text{Ball}(\ell_1^d)_+$ there exists $v' \in \mathcal{P}$ such that $\|v - v'\|_{\ell_1^d} < \varepsilon$ and $\|v'\|_{\ell_1^d} < \|v\|_{\ell_1^d}$. Similarly one lets \mathcal{Q} be the (finite) set of $m \times d$ matrices with entries from $\{0, 1\}$ and such that every row contains at most one nonzero entry, every column contains exactly one nonzero entry, and the first row is $(1, 0, 0, \dots, 0)$. One can identify \mathcal{P} with a subset of Δ_{d+1} by mapping $v = (v_1, \dots, v_d) \in \mathcal{P}$ to the element $(1 - (v_1 + \dots + v_d), v_1, \dots, v_d) \in \Delta_d$. Via this identification, one can proceed as in the proof of Theorem 2.1. By duality, one can give the following geometric formulation of the Ramsey property for the class $\{(\ell_\infty^{n+1}, e_0) : n \in \mathbb{N}\}$. For $d, n \in \mathbb{N}$, we let $\text{Epi}_0(\Delta_n, \Delta_d)$ be the space of surjective affine continuous maps $\phi : \Delta_n \rightarrow \Delta_d$ such that $\phi(e_0) = e_0$, endowed with the metric induced by the identification with a subspace of $\text{Emb}(\ell_\infty^d, \ell_\infty^n)$.

Theorem 3.1 *For every $r, d, m \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every r -coloring of $\text{Epi}_0(\Delta_n, \Delta_d)$ there exists $\phi \in \text{Epi}_0(\Delta_n, \Delta_m)$ such that $\text{Epi}_0(\Delta_m, \Delta_d) \circ \phi$ is ε -monochromatic.*

Theorem 3.1 has implications concerning the dynamical properties of the group $\text{Aut}(\mathbb{P})$ of affine homeomorphisms of the Poulsen simplex \mathbb{P} . Initially constructed by Poulsen in 1961 [19], \mathbb{P} is the unique nontrivial metrizable Choquet simplex with dense extreme boundary as shown by Lindenstrauss–Olsen–Sternfeld in 1976 [12]. It turns out that the class of finite-dimensional function system is Fraïssé, and its limit is equal to the function system $A(\mathbb{P})$ associated with the Poulsen simplex [3]. Furthermore, the class of finite-dimensional function systems with a distinguished state is also Fraïssé, and its limit is $(A(\mathbb{P}), p)$ where p is an extreme point of \mathbb{P} [13]. Thus Theorem 3.1 implies that the stabilizer $\text{Aut}_p(\mathbb{P})$ of p in $\text{Aut}(\mathbb{P})$ is extremely amenable. The completion of the quotient $\text{Aut}(\mathbb{P})$ by $\text{Aut}_p(\mathbb{P})$, endowed with the quotient uniformity induced by the left uniformity on $\text{Aut}(\mathbb{P})$ and with the $\text{Aut}(\mathbb{P})$ -action induced by the left translation action on $\text{Aut}(\mathbb{P})$, can be equivariantly identified with \mathbb{P} endowed with the canonical action of $\text{Aut}(\mathbb{P})$. Since such an action is furthermore *minimal* as shown by Glasner [4], one can conclude that the canonical action $\text{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ is the *universal minimal* flow of $\text{Aut}(\mathbb{P})$.

Theorem 3.2 *The universal minimal flow of $\text{Aut}(\mathbb{P})$ is the canonical action $\text{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$. In other words, such an action is minimal, and it factors onto any minimal action of $\text{Aut}(\mathbb{P})$.*

This result answers a question of Conley and Törnquist from [3]. With similar methods one can establish the natural *noncommutative analogs* of the results above, stated in terms of the *noncommutative Gurarii space* NG [14] and the *noncommutative Poulsen simplex* NP [13]. These are defined similarly as \mathbb{G} and \mathbb{P} by replacing Banach spaces with *exact* operator spaces, and function systems with *exact* operator systems.

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