Tue ~~May~~ June 1, 2010

Duality

The dual of the standard problem

$$\max \sum_{j=1}^{n} c_j x_j$$

(P) primal

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i=1..m)$$

$$x_j \ge 0$$

is the problem

(D) dual

$$\min \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t.} \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad (j=1..n)$$

$$y_i \ge 0 \quad (i=1..m)$$

How did we get it?

- Transpose matrix of coeffs.
- RHS of constraints become coeff in obj. func.
- coeffs in obj. func. become RHS in constr.
- # variables becomes # of constr. & vice versa
- directions of ineqs. reversed.
- max becomes min & vice versa.
- variables are non neg.

Rules are symmetric so they give duals of both ~~max~~ standard max & min.

Ex.

$$\max 3x_1 + 2x_2 - 4x_3$$

$$\text{(P)} \quad \text{s.t.} \quad x_1 + x_2 + 3x_3 \le 6$$

$$x_1 + 2x_3 \le 2$$

$$x_1 - 2x_2 + 4x_3 \le 1$$

$$x_1, x_2, x_3 \ge 0$$

$\Longleftrightarrow$ (D)

$$\min 6y_1 + 2y_2 + y_3$$

s.t.

<u>Thm</u> (weak Duality) For every feasible soln $(x_1, \ldots, x_n)$ & every feasible $(y_1, \ldots, y_m)$ we have

$$\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i \,.$$

Pf: $$\sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i$$

$$\le \sum_{i=1}^{m} b_i y_i \,. \quad \blacksquare$$

<u>Thm</u> (The Duality Thm) If the primal (P) has an optimal solution $(x_1^*, \ldots, x_n^*)$, then the dual (D) has an optimal soln $(y_1^*, \ldots, y_m^*)$ s.t.

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Pf: ~~To do the pf,~~ We need $(y_1^*, \ldots, y_m^*)$ s.t.

Say we introduced slacks

$$(**) \qquad x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad (i = 1..m),$$

solved by simplex & eventually arrive at

$$z = z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k \,. \qquad (*)$$

each $\bar{c}_k \le 0$ & $\bar{c}_k = 0$ when x_k is basic.

z^* optimal $\Rightarrow$ $z^* = \sum_{j=1}^{n} c_j x_j^*$.

let $y_i^* = -\bar{c}_{n+i}$ $(i = 1..m)$. (Show this satisfies what we want)

Substituting $\sum c_j x_j$ for z & $(**)$ in for slacks, $(*)$ says

$$\sum_{j=1}^{n} c_j x_j = z^* + \sum_{j=1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$

which is $$\sum_{j=1}^{n} c_j x_j = \left(z^* - \sum_{i=1}^{m} y_i^* b_i \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j \,.$$

So we get

$$z^* - \sum_{i=1}^{m} b_i y_i^* = 0 \Rightarrow z^* = \sum b_i y_i^* \quad \text{(optimal)}$$

& $$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*$$

$$\bar{c}_j \le 0 \ \forall k \Rightarrow \cancel{c_j \le} \ \left.\begin{array}{l} \sum_{i=1}^{m} a_{ij} y_i^* \ge c_j \\ \& \ y_i^* \ge 0 \end{array}\right\} \text{dual feas}$$

Since dual of dual is primal again, we get

	Dual: optimal	Infeas	Unbdd
optimal	✓	X	X
Infeas	X	✓	✓
Unbdd	X	✓	X

✓ = possible
X = impossible.

Ex: Class on 5/20, did

(P)
$$\max 3x_1 + 2x_2 + 4x_3$$
$$\text{s.t. } x_1 + x_2 + 2x_3 \le 4$$
$$2x_1 + 3x_3 \le 5$$
$$2x_1 + x_2 + 3x_3 \le 7$$
$$x_1, x_2, x_3 \ge 0$$

(D)
$$\min 4y_1 + 5y_2 + 7y_3$$
$$\text{s.t. } y_1 + 2y_2 + 2y_3 \ge 3$$
$$y_1 + y_3 \ge 2$$
$$2y_1 + 3y_2 + 3y_3 \ge 4$$
$$y_1, y_2, y_3 \ge 0.$$

opt tab

			s_1	s_2	s_3	
0	1	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{3}{2}$
1	0	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$5/2$
0	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	1	$1/2$
0	0	$\frac{3}{2}$	2	$\frac{1}{2}$	0	$21/2$

So Thm says $y_1 = 2$, $y_2 = \frac{1}{2}$, $y_3 = 0$ is opt. Let's see

satisfies all constr. ✓

$$4 \cdot 2 + \frac{5}{2} = \frac{16}{2} + \frac{5}{2} = \frac{21}{2} \ ✓$$

So we have a way of reading off dual soln from optimal tableau.

But we can ~~...~~ also derive dual soln easily from ~~optimal tableau~~ opt soln $x_1^*, \ldots, x_n^*$.

<u>Thm</u> (Complementary Slackness) Let $x_1^*, \ldots, x_n^*$ be feasible soln of (P), & $y_1^*, \ldots, y_m^*$ be feas. soln of (D). These are simultaneously optimal <u>iff</u>

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \quad \text{or} \quad x_j^* = 0 \qquad j = 1 \ldots n$$

and

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \quad \text{or} \quad y_i^* = 0 \qquad i = 1 \ldots m$$

These conditions often written as $\left(c_j - \sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* = 0$.

and $\left(b_i - \sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* = 0$.

<u>Pf</u>: Recall

$$\sum_{j=1}^{n} c_j x_j^* \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \le \sum_{i=1}^{m} b_i y_i^*$$

So to have equality of LHS & RHS, ineqs must be equality.

So $x_j = 0$

<u>Ex</u>: Know $x_1 = 5/2$, $x_2 = \frac{3}{2}$, $x_3 = 0$.

$y_1 + 2y_2 + 2y_3 = 3$ $\qquad$ 1st constr tight $\Rightarrow$

$y_1 + y_3 = 2$.

$y_3 = 0$.

$\rightarrow$ $y_1 = 2$. $y_2 = \frac{1}{2}$ ✓

This can be used to verify the claim that a s/n is ~~uniq~~ optimal, by using the following

Thm: A feasible soln $x_1^* \cdots x_n^*$ of (P) is optimal iff there are $y_1^*, \dots, y_m^*$ s.t.

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \quad \text{when} \quad x_j^* > 0$$

$$y_i^* = 0 \quad \text{when} \quad \sum_{j=1}^{n} a_{ij} x_j^* < b_i$$

and $y_1^*, \dots, y_m^*$ is feasible for (D).

Consider the claim that

$(0, 2, 0, 7, 0)^T$ is optimal for $\max 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5$

$$\begin{pmatrix} 2 & -3 & 4 & 1 & 3 \\ 1 & 7 & 3 & -2 & 1 \\ 5 & 4 & -6 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \\ 22 \end{pmatrix}$$

$$-3y_1 + 7y_2 + 4y_3 = -9$$

$$y_1 - 2y_2 + 2y_3 = 4$$

$$y_2 = 0$$

$5 \cdot \frac{3}{10} = \frac{3}{2}$

$\frac{34}{5} + \frac{3}{2}$

$\frac{68}{10} + \frac{15}{10}$

$$\begin{pmatrix} 1 & 2 & 4 \\ -3 & 4 & -9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 10 & 3 \end{pmatrix}$$

$$y_3 = 3/10, \quad \Rightarrow \quad y_1 = 4 - \frac{3}{5} = \frac{17}{5}$$

$$\begin{pmatrix} 3.4 \\ 0 \\ .3 \end{pmatrix}$$

$13.6 - 1.8 = 11.8$

Does not satisfy 3rd Dual constraint.