MOD SUM NUMBER OF COMPLETE BIPARTITE GRAPHS

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by

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APPROVAL

This is to certify that the Graduate Committee of

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met on the
1st day of April, 1999.

The committee read and examined his thesis, supervised his defense of it in an oral examination, and decided to recommend that his study be submitted to the Graduate Council, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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ABSTRACT

MOD SUM NUMBER OF COMPLETE BIPARTITE GRAPHS

by

Christopher D. Wallace

A graph $G = (V, E)$ is a mod sum graph if there exists a positive integer $Z$ and a labelling of vertices with distinct elements of $\{1, 2, ..., Z - 1\}$ such that $\{u, v\} \in E$ if and only if $u \neq v$ and $u + v \ (mod \ Z) \in V$. First we discuss conditions which $K_{n_1, n_2, ..., n_m}$ must satisfy to be a mod sum graph and then we determine the minimum number of isolated vertices such that $K_{n, m} \geq n \geq 3$ is a mod sum graph except when $2n \leq m < 3n - 3$ and $m$ is odd.
DEDICATION

To my wife Olivia, my parents Billie and Bill, and my sister Sheena. Their support, encouragement, and love made this thesis possible.
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CHAPTER 1
INTRODUCTION

Definitions

To begin, we first must define a simple undirected graph. A simple undirected graph $G = (V, E)$ consists of a set $V$ which is a nonempty, finite set of elements called vertices, and a set $E$ which is a set of unordered pairs $\{u, v\}$ called edges, where $u, v$ are distinct elements of $V$. $V$ is called the vertex set of $G$, and $E$ is called the edge set of $G$. A simple graph can have at most one edge between any two vertices. From now on we will use the word graph to mean a simple undirected graph.

We now define some classes of graphs. The complete graph, $K_n$, has $n$ vertices and every pair of vertices share an edge. The complete $m$-partite graph, $K_{n_1, n_2, ..., n_m}$, has partite sets $V_1, V_2, ..., V_m$ with $|V_i| = n_i$ having the property that if $u \in V_i$, $v \in V_j$, $i \neq j$ then $\{u, v\} \in E$ and $\{u, v\} \notin E$ when $i = j$. One class of complete $m$-partite graphs is $H_{n,m}$ which is a $m$-partite graph which each partite set containing $n$ vertices. A cycle containing $n$ vertices is denoted $C_n$. Also a wheel $W_n$ contains a cycle of length $n$ and an additional vertex which is adjacent to every vertex on the cycle. A tree $T_n$ is a acyclic connected graph containing $n$ vertices, while a forest $F_n$ is a acyclic graph containing $n$ vertices.

The concept of sum graph was introduced by Harary at the Nineteenth Southeastern Conference on Combinatorics at Baton Rouge in 1988. A graph, $G = (V, E)$, is a sum graph if there exist a set $\mathcal{F} \subset \mathbb{Z}^+$ and a bijection $f : \mathcal{F} \to V$ such that if $x, y$ are
distinct elements of $\mathcal{F}$, then $x + y \in \mathcal{F}$ if and only if $\{f(x), f(y)\} \in E$. In this paper, we will refer to vertices by their corresponding label in $\mathcal{F}$. The first obvious class of graphs which are not sum graphs are connected graphs. Then not all graphs are sum graphs and so it is a natural extension to define sum number of a graph. The sum number of graph $G$, denoted $\sigma(G)$, is the smallest number of isolated vertices that can be added to $G$ to yield a sum graph.

**Survey on Sum Graphs**

Most papers thus far on sum graphs have involved finding the sum number for a particular class of graphs. These include

- $\sigma(K_n) = 2n - 3$ for $n \geq 4$ [1]
- $\sigma(K_{m,n}) = \frac{3m + n - 3}{2}$ for $n \geq m \geq 2$ [6]
- $\sigma(H_{2,n}) = 4n - 5$ for $n \geq 2$ [10]
- $\sigma(C_n') = 2$ for $n \neq 4$ [5]
- $\sigma(W_n) = \frac{n}{2} + 2$ for even $n \geq 4$, $\sigma(W_n) = n$ for odd $n \geq 5$ [7]
- $\sigma(F_n) = 1$ for all $n \geq 2$ [3]

**Mod Sum Graph Definitions and Survey**

Boland, Laskar, Turner and Domke introduced in [2] a generalization of sum graph. A graph is a mod sum graph if there exist a positive integer $Z$ and a labelling of the
vertices with distinct elements of \(\{1, 2, ..., Z - 1\}\) such that \(u, v\) \(\in E\) if and only if \(u \neq v\) and \(u + v \mod Z \in F\). We similarly define the mod sum number of a graph \(\rho(G)\) as the smallest number of isolated vertices that must be added to \(G\) such that the resulting graph is a mod sum graph (MSG). So far the classifications on graphs are

- \(\rho(K_n) = n\) for \(n \geq 4\) \([8]\)
- \(\rho(K_{2,n}) = 0\) and \(\rho(K_{1,n}) = 0\) for \(n \geq 2\) \([2]\)
- \(\rho(K_{p+1,q}) = 0\) for \(q \geq 1\) and \(p \geq r_q + r_{q-1} - 1\) where \(r_1 = 1\), \(r_2 = 2\), and \(r_j = r_{j-2} + r_{j-1} + 1\) for \(j \geq 3\) \([2]\)
- \(\rho(K_{n,m}) > 0\) for \(n \geq 3\) \([9]\)
- \(\rho(H_{2,n}) = 0\) for \(n \geq 2\) \([2]\)
- \(\rho(H_{m,n}) > 0\) for \(m > n \geq 2\) \([8]\)
- \(\rho(C_n) = 0\) for \(n \neq 4\) \([2]\)
- \(\rho(W_n) > 0\) for \(n \geq 5\) \([9]\)
- \(\rho(T_n) = 0\) for all \(n \geq 3\) \([2]\)

**Overview**

In this paper we determine conditions for when the complete m-partite graph is not a mod sum graph. Also we show \(K_{n,m}\) is a mod sum graph when \(m\) is even and \(m \geq 2n\) or when \(m\) is odd and \(m \geq 3n - 3\) and \(\rho(K_{n,m}) = n\) when \(3 \leq n \leq m < 2n\).
In Chapter 2 we prove these results and we present conjectures and conclude with Chapter 3.
CHAPTER 2
MAIN RESULTS

Notation

For \( K_{n_1,n_2,...,n_m} \) the complete m-partite graph, there exist a set of labels \( \mathcal{F} \), some nonnegative integer \( r \), and positive integer \( Z \), such that \( K_{n_1,n_2,...,n_m} \cup K_r \) is a mod sum graph (MSG) modulo \( Z \). Denote \( \mathcal{I} K_{n_1,n_2,...,n_m} = K_{n_1,n_2,...,n_m} \cup K_r \). Let \( V(K_{n_1,n_2,...,n_m}) = V, V(K_r) = R, E(K_{n_1,n_2,...,n_m}) = E \). Denote partite sets \( V_i \) for \( 1 \leq i \leq m \). We refer to vertices by their label, so \( V(\mathcal{I} K_{n_1,n_2,...,n_m}) = \mathcal{F} \). For easier notation, if \( G = (V, E) \) is a MSG, then \( v \in V \) means \( v \pmod{Z} \in V \), and \( \{u, v\} \in E \) means \( u + v \pmod{Z} \in \mathcal{F} \).

Define \( x \equiv y \) if and only if \( x = y \pmod{Z} \).

Complete Partite Graphs

We use the following observation in Lemma 2.6 and Lemma 2.7.

Observation 2.1 Let \( Z \in \mathbb{Z}^+ \) and let \( 0 < a_i < Z \) for \( 1 \leq i \leq n \) be integers. Let \( d = \gcd(a_1, a_2, ..., a_n) \). Then there exist nonnegative integers \( L_i \) for \( 1 \leq i \leq n \) such that \( \sum_{i=1}^{n} L_i a_i \equiv d \).

Discussion. Since \( d = \gcd(a_1, a_2, ..., a_n) \), there exist \( K_i \in \mathbb{Z} \) such that \( d = \sum_{i=1}^{n} K_i a_i \). Let \( f: \mathbb{Z} \rightarrow \{0, 1, ..., Z - 1\} \) by \( f(x) = x \pmod{Z} \). Then \( 0 \leq f(x) < Z \), \( f(xy) \equiv f(x)f(y) \), and \( f(x + y) \equiv f(x) + f(y) \). Since \( 0 < a_i < Z \), for \( 1 \leq i \leq n \),
then \( f(a_i) = a_i \). Also \( 0 < d < a_i \) for all \( a_i \). So \( f(d) = d \). Therefore \( d = f(d) \equiv f(\sum_{i=1}^{n} K_i a_i) \equiv \sum_{i=1}^{n} f(K_i) f(a_i) \equiv \sum_{i=1}^{n} f(K_i) a_i \), where \( f(K_i) \) are nonnegative integers for \( 1 \leq i \leq n \). \( \square \)

The following lemma is needed for Corollary 2.3.

**Lemma 2.2** For \( K_{n_1,n_2,...,n_m} \), with \( a_1,a_2 \in V_1, b \in V - V_1, \) and \( |V_1| \geq 3 \), if \( a_1 + b \in V - V_1 \), then \( a_2 + b \not\equiv a_1 \).

**Proof.** If \( a_2 = a_1 \), then \( a_2 + b = a_1 + b \in V - V_1 \), and the result follows. Suppose for the sake of contradiction, \( a_2 + b \equiv a_1 \). Since \( |V_1| \geq 3 \), there exist \( a_3 \in V_1 \) such that \( a_3 \not\equiv \{a_1,a_2\} \). Now \( \{a_3,b + a_1\} \in E \), since \( a_3 \in V_1 \) and \( b + a_1 \in V - V_1 \). Thus \( a_3 + b + a_1 \in \mathcal{F} \). Notice \( a_3 + b \in \mathcal{F} \) and \( a_3 + b \not\equiv a_2 + b \equiv a_1 \). So \( \{a_3+b,a_1\} \in E \) which implies \( a_3 + b \in V - V_1 \). Hence \( \{a_3+b,a_2\} \in E \) and so \( a_3 + b + a_2 \in \mathcal{F} \). By assumption \( a_3 \not\equiv a_1 \equiv b + a_2 \), therefore \( \{a_3,b + a_2\} \in E \), a contradiction. \( \square \)

The following corollary is cited in Lemma 2.4, Lemma 2.5, and Lemma 2.6.

**Corollary 2.3** Let \( V_1 \) be a partite set of \( K_{n_1,n_2,...,n_m} \) with \( |V_1| \geq 3 \). If \( a_1 \in V_1, b \in V - V_1 \), then either for all \( a_i \in V, a_i + b \not\in V - V_1 \), or for all \( a_i \in V, a_i + b \in V - V_1 \).

**Proof.** If for all \( a_i \in V_1, a_i + b \not\in V - V_1 \), the result follows. Assume there exist a \( a_i \in V \) such that \( a_i + b \in V - V_1 \). Pick \( a_j \in V_1 \). Then \( \{a_j,a_i + b\} \in E \) since \( a_j \in V_1 \) and \( a_i + b \in V - V_1 \). So \( a_j + a_i + b \in \mathcal{F} \). Also \( a_j + b \in \mathcal{F} \), since \( a_j \in V \) and \( b \in V - V_1 \). By Lemma 2.2, \( a_j + b \not\equiv a_i \). So \( \{a_i,a_j + b\} \in E \). This implies \( a_j + b \in V - V_1 \). \( \square \)

The next lemma is used in Lemma 2.5.

**Lemma 2.4** Let \( V_1 \) and \( V_2 \) be partite sets of \( K_{n_1,n_2,...,n_m} \), with \( |V_1| = n_1,|V_2| = n_2 \) and \( n_2,n_1 \geq 3 \). Suppose \( a \in V_1, b \in V_2 \). If \( a + b \in V_1 \), then for all \( b \in V_2, a + b \in V_1 \).
**Proof.** Denote $V_1 = \{a_1, a_2, ..., a_n\}$, $V_2 = \{b_1, b_2, ..., b_m\}$, and suppose $a_1 + b_1 \in V_1$. By Corollary 2.3, since $a_1 + b_1 \in V - V_2$, then for every $b_i \in V_2$, $a_1 + b_i \in V - V_2$. Likewise since $a_1 + b_1 \notin V - V_1$, for every $a_j \in V_1$, $a_j + b_1 \notin V - V_1$. If for every $b_i$, $a_1 + b_i \in V_1$, the result follows. Suppose there exist a $b_2$ such that $a_1 + b_2 \notin V_1$. Now $b_2 + a_1 \in V - V_2$. So $b_2 + a_1 \in V - V_2 - V_1 \subset V - V_1$. Thus $b_2 + a_i \in V - V_1$, for all $a_i \in V_1$ by Corollary 2.3. Pick $a_i \in V_1$. Since $b_1 + a_i \in V_1$ and $b_2 + a_i \in V - V_1$, then $\{b_1 + a_1, b_2 + a_i\} \in E$. Hence $b_1 + a_1 + b_2 + a_i \in F$. Now we showed $b_1 + a_i \notin V - V_1$, then $b_1 + a_i \in V_1$ or $b_1 + a_i \in R$. Well $b_2 + a_i \in V - V_1$, so $b_1 + a_i \notin b_2 + a_1$. Hence $\{b_1 + a_1, b_2 + a_i\} \in E$ which implies $b_1 + a_i \in V_1$. Therefore $b_1 + a_i \in V_1$ for all $a_i \in V_1$.

So $a_1, b_1 + a_1, b_1 + a_1, ..., nb_1 + a_1, ... \in V_1$. Hence there exist a smallest $J \in \mathbb{Z}^+$ such that $b_1(J + 1) \equiv 0 \pmod{kb_1 + a_1} \in V_1$ for $0 \leq k \leq J$. Since $b_2 + a_i \in V - V_1$ for all $a_i \in V_1$, then $b_2 + kb_1 + a_1 \in V - V_1$ for $0 \leq k \leq J$. Then $\{b_2 + kb_1 + a_1, a_1\} \in E$ which implies $b_2 + kb_1 + 2a_1 \in F$ for $0 \leq k \leq J$. Hence for $0 \leq k < J$, $b_2 + kb_1 + 2a_1 \neq 0$, so $b_2 + (k + 1)b_1 + 2a_1 \neq b_1$ for $1 \leq k \leq J$. Also $b_2 + 2a_1 \neq b_1$, since $b_2 + Jb_1 + 2a_1 \neq 0$. Notice $b_2 + kb_1 + 2a_1 + b_1 \in F$ for $0 \leq k \leq J$. So $\{b_2 + kb_1 + 2a_1, b_1\} \in E$ implies $\{b_2 + kb_1 + 2a_1, b_2\} \in E$ implies $2b_2 + kb_1 + 2a_1 \in F$ for $0 \leq k \leq J$.

Suppose for $l > 0$, $lb_2 + kb_1 + 2a_1 \in F$ for $0 \leq k \leq J$. So by the previous argument for $0 \leq k \leq J$, $lb_2 + kb_1 + 2a_1 \neq b_1$ and $lb_2 + (k + 1)b_1 + 2a_1 \in F$. Hence $\{lb_2 + kb_1 + 2a_1, b_1\} \in E$ implies $\{lb_2 + kb_1 + 2a_1, b_2\} \in E$ implies $(l+1)b_2 + kb_1 + 2a_1 \in F$ for $0 \leq k \leq J$. Whence $lb_2 + b_1 + 2a_1 \in F$ for all $l \geq 0$. So there must exist $K > N$ such that $Kb_2 + b_1 + 2a_1 \equiv Nb_2 + b_1 + 2a_1$. So $(K - N)b_2 \equiv 0$. Therefore $(K - N)b_2 + b_1 + 2a_1 \equiv b_1 + 2a_1 \in F$. Now $b_1 + a_1 \neq a_1$ since $b_1 \neq 0$. So $\{b_1 + a_1, a_1\} \in E$, a contradiction. □
The following result is an extension to a result by Hartfield and Smyth in [6], the lemma is referenced in Lemma 2.6, Theorem 2.10, and Theorem 2.11.

**Lemma 2.5** Let $V_1$ and $V_2$ be partite sets of $K_{n_1,n_2,...,n_m}$, with $|V_1| = n_1, |V_2| = n_2$ and $n_2 \geq n_1 \geq 3$. Suppose $b_1 \in V_2$. Then either for all $a_i \in V_1, a_i + b_1 \in R$, or for all $a_i \in V_1, a_i + b_1 \in V - V_1$.

**Proof.** Suppose $a_1 \in V_1, b_1 \in V_2$, and $a_1 + b_1 \in V - V_1$. So by Corollary 2.3 for all $a_i \in V_1, a_i + b_1 \in V - V_1$.

Now suppose $a_1 + b_1 \in V_1$. By Lemma 2.4 for every $b_i \in V_2, a_1 + b_i \in V_1$. So $a_1, a_1 + b_i \in V_1$ for $1 \leq i \leq n_2$, but $a_1 \neq a_1 + b_i$. Also $a_1 + b_i \neq a_1 + b_j$ for $i \neq j$.

Thus $a_1, a_1 + b_i$ for $1 \leq i \leq n_2$ are $n_2 + 1$ distinct elements in $V_1$, a contradiction since $n_2 \geq |V_1|$.

Finally, observe that if $a_1 + b_1 \in R$, then by the conclusion of the previous paragraph, there exists no vertex $a_i \in V_1$ such that $a_i + b_1 \in V$. This completes the proof. □

The next lemma is needed in Lemma 2.7.

**Lemma 2.6** Suppose $K_{n_1,n_2,...,n_m}$ is a MSG, i.e. $V = F$, with $V_1$ and $V_2$ partite sets and $|V_2| = n_2 \geq |V_1| = n_1 \geq 3$. If $V_1 = \{a_1, a_2, ..., a_{n_1}\}$ with $gcd(a_1, a_2, ..., a_{n_1}) = d$, then there exist $L \in \mathbb{Z}^+$ such that $(L + 1)d \equiv Z$ and for all $b \in V_2, i \in \mathbb{Z}, b + id \in V - V_1$.

**Proof.** Denote $V_2 = \{b_1, b_2, ..., b_{n_2}\}$. By assumption $|R| = 0$, so by Lemma 2.5, $b_i + a_j \in V - V_1$ for $1 \leq i \leq n_2, 1 \leq j \leq n_2$. Pick $b \in V_2$. 
Assume there exist nonnegative integers $C_i$ for $1 \leq i \leq n_1$ such that $b + \sum_{i=1}^{n_1} C'_i a_i \in V - V_1$ for $0 \leq C'_i \leq C_i$. For sake of contradiction suppose there exist $k$ and $0 \leq K_i \leq C_i$ for $i \neq k$, such that $b + \sum_{i=1, i \neq k}^{n_1} K_i a_i + (C_k + 1) a_k \in V_1$. Now by assumption $b + \sum_{i=1, i \neq k}^{n_1} K_i a_i + (C_k) a_k \in V - V_1$. Also since $b + a_k \notin V_1$ either $C_k > 0$ or there exist $K_i > 0$ for some $1 \leq i \leq n_1, i \neq k$, denote $K_c$.

Case 1: $(C_k > 0)$ Since $b + \sum_{i=1, i \neq k}^{n_1} K_i a_i + C_k a_k \in V - V_1$ and $b + \sum_{i=1, i \neq k}^{n_1} K_i a_i + C_k a_k + a_k \in V_1$, by Corollary 2.3, $b + \sum_{i=1, i \neq k}^{n_1} K_i a_i + (C_k - 1) a_k + a_l \equiv 0$, but since $C_k > 0$ by our assumption $b + \sum_{i=1, i \neq k}^{n_1} K_i a_i + (C_k - 1) a_k \in V - V_1$. Thus $b + \sum_{i=1, i \neq k}^{n_1} K_i a_i + (C_k - 1) a_k + a_l \in \mathcal{F}$, a contradiction.

Case 2: $(C_k = K_k = 0)$ So $K_c > 0$ for some $c \neq k$. So by Corollary 2.3, $b + \sum_{i=1, i \neq c}^{n_1} K_i a_i + C_c a_c + a_j \in V_1$ for $1 \leq j \leq n_1$, and also notice they are $n_1$ distinct elements. Since $|V_1| = n_1$ and $a_c \in V_1$, there exist $l$ such $a_c \equiv b + \sum_{i=1, i \neq c}^{n_1} K_i a_i + C_c a_c + a_l \equiv 0$. Therefore $b + \sum_{i=1, i \neq c}^{n_1} K_i a_i + (C_c - 1) a_c + a_l \equiv 0$, a contradiction by the previous argument. Hence $b + \sum_{i=1}^{n_1} C_i a_i \in V - V_1$ for all $C_i \geq 0$.

Now since $d = \text{gcd}(a_1, \ldots, a_{n_1})$, by Observation 2.1, there exist nonnegative integers $K_i$ for $1 \leq i \leq n_1$ such that $d \equiv \sum_{i=1}^{n_1} K_i a_i$. Then it follows that $b + \sum_{i=1}^{n_1} K_i a_i \equiv b + d \in V - V_1$. Since $b + \sum_{i=1}^{n_1} C_i a_i \in V - V_1$ for all $C_i \geq 0$, then for all $l \geq 0$, $b + l \sum_{i=1}^{n_1} K_i a_i \equiv b + ld \in V - V_1$. So there must exist $i,j$ where $i \neq j$ such that $b + id \equiv b + jd$, since otherwise $|V - V_1| = \infty$. From whence we conclude that there exist $L \in \mathbb{Z}_+$ such that $(L + 1)d \equiv \mathcal{Z}$, and so for any $i \in \mathbb{Z}$, $b + id \equiv b + ld$ for some nonnegative integer $l$, thus $b + id \ (\text{mod} \ \mathcal{Z}) \in V - V_1$. \(\Box\)
The following lemma is the major result in this paper. With it, Lemma 2.8 shows $K_{n,m}$ with $m > n \geq 3$ is not a MSG when $m < 2n$.

**Lemma 2.7** Suppose $K_{n_1,n_2,...,n_m}$ is a MSG, i.e. $V = \mathcal{F}$, with $V_1$ and $V_2$ partite sets, and $|V_2| = n_2 \geq |V_1| = n_1 \geq 3$. Let $V_1 = \{a_1, a_2, ..., a_{n_1}\}$ with $\gcd(a_1, a_2, ..., a_{n_1}) = d$. If $n_2 > n_1$, there exist $b \in V_2$ such that for all $i \in \mathbb{Z}$, $b + id \in V_2$.

**Proof.** Since for all $b \in V_2$, $0 < b < \mathcal{Z}$, then $2b \equiv 0$ implies $b = \frac{2}{2}$. Since $n_2 \geq 3$, we can chose a $b \in V_2$ such that $2b \neq 0$. Assume $V_1 = \{C_1d, C_2d, ..., C_{n_1}d\}$. By Lemma 2.6, for all $i \in \mathbb{Z}$, $b + id \in V - V_1$ and there exist a $L \in \mathbb{Z}^+$ such that $(L+1)d \equiv 0$. Let $L$ be the smallest such positive integer. Suppose there exist $j$ such that $b + jd \notin V_2$.

**THEN WE ARE GOING TO SHOW** $n_1 = n_2$. We first establish $2b + C_1d \in V_1$, $b + C_1d \notin V_2$, and $b + 2C_1d, 3b + 2C_1d \in V_2$. Since $|R| = 0$, $\{b, b + jd\} \in E$. So $2b + jd \in V$. Assume for $k \in \mathbb{Z}^+$, there exist an $i$ such that $kb + id \in V$. Now either $kb + id \in V_2$ or $kb + id \notin V_2$. Hence either $\{kb + id, b + jd\} \in E$ or $\{kb + id, b\} \in E$. Thus either $(k + 1)b + (i + j)d \in V$ or $(k + 1)b + id \in V$. Therefore by induction for all $k \in \mathbb{Z}^+$ there exist a $i$ such that $kb + id \in V$. This implies that there must exist $K > N > 0$, and $i, j \in \mathbb{Z}$ such that $Kb + id \equiv Nb + jd$.

We wish to show that there exist a $J \in \mathbb{Z}^+$ such that $(J + 1)b \equiv 0$. So $(K - N)b \equiv (j - i)d$. If $j = i$, since $b \neq 0$, we have a $J > 0$ such that $(J + 1)b \equiv 0$. If $j \neq i$, $b + k(K - N)b \equiv b + k(j - i)d \in V - V_1$ for all $k \in \mathbb{Z}$. Hence there must exist $r, s \in \mathbb{Z}^+$ with $r > s$, such that $b + r(K - N)b \equiv b + s(K - N)b$. Thus $(r - s)(K - N)b \equiv 0$. Since $r > s$ and $K > N$, there a exist a $J \in \mathbb{Z}^+$ such that $(J + 1)b \equiv 0$. Let $J$ be the smallest such positive integer. By the above argument there exist a $k$ such that $Jb + kd \in V$. Hence $(J + 1)b \equiv (L + 1)d \equiv 0$. 


Suppose for any \( l \in \mathbb{Z} \), that \( Jb + ld \in V_1 \). Now \( b + (C_1 + C_2 - l)d \in V - V_1 \). Then \( \{Jb + ld, b + (C_1 + C_2 - l)d\} \in E \) which implies \((C_1 + C_2)d \in V\), a contradiction. So \( Jb + ld \in V - V_1 \). Thus \( Jb + kd \notin V_1 \). Whence \( \{Jb + kd, C_1d\} \in E \), and \( Jb + kd + C_1d \in V \). By the previous argument \( Jb + kd + C_1d \in V - V_1 \). Repeating one can see that \( Jb + kd + \sum_{i=1}^{n_1} K_i C_i d \in V - V_1 \) for all \( K_i \geq 0 \). Since \( d = \gcd(a_1, \ldots, a_{n_1}) \), by Observation 2.1, there exist nonnegative integers \( K'_i \) for \( 1 \leq i \leq n_1 \) such that \( d \equiv \sum_{i=1}^{n_1} K'_i a_i \). Therefore for \( l \geq 0 \), \( Jb + kd + l \sum_{i=1}^{n_1} K'_i a_i \equiv Jb + kd + ld \in V - V_1 \). Since we have \( L \) as the smallest positive integer such that \((L+1)d \equiv 0 \), then \( Jb + id \in V - V_1 \) for \( 0 \leq i \leq L \) are \( L + 1 \) distinct elements.

We wish to show for all \( C_i d, C_j d \in V_1 \), that \( b + C_i d, J + C_j d \) are in the same partite set distinct from \( V_2 \). Denote \( V_3 \), the partite set containing \( b + C_i d \). Now if \( b + C_i d \equiv Jb + C_i d \) for some \( i \) then it follows that \( Jb + C_i d \in V_3 \). Suppose \( b + C_i d \neq Jb + C_i d \) for \( i \neq 1 \). Then \( Jb + C_i d \in V_3 \) since otherwise \( \{Jb + C_i d, b + C_1 d\} \in E \) implies \( C_i d + C_1 d \in \mathcal{F} \) for \( i \neq 1 \). Hence \( Jb + C_i d \in V_3 \) for \( i \neq 1 \). Since \( n_1 \geq 3 \), for \( b + C_j d \) where \( j \neq 1 \), there exist \( C_i d \) such that \( C_i d \neq C_j d \), \( C_i d \neq C_1 d \). So \( Jb + C_i d \in V_3 \), by the previous argument, implies \( b + C_j d \in V_3 \). Hence \( b + C_j d \in V_3 \) for all \( C_j d \in V_1 \). Finally, we have \( b + C_2 d \in V_3 \) implies \( Jb + C_1 d \in V_3 \). Now there exist an \( i \) such that \( Jb + C_i d \neq b \). Since \( Jb + C_i d + b \equiv C_i d \), it follows that \( \{Jb + C_i d, b\} \in E \). So \( V_3 \) is a different partite set then \( V_2 \) and \( V_3 \) contains \( b + C_i d, Jb + C_j d \) for all \( C_i d, C_j d \in V_1 \). Therefore we have \( \{b, b + C_i d\} \in E \) implying that \( 2b + C_i d \in V \). Now \( \{b + C_i d, b + C_j d\} \notin E \) for \( i \neq j \). So \( 2b + C_i d + C_j d \notin V \), for \( i \neq j \). So \( 2b + C_i d \in V_1 \) for all \( C_i d \in V_1 \).

The vertices \( Jb, Jb + (C_i + C_j)d \in V_2 \) for \( i \neq j \), since these vertices must not be
adjacent to \( b \). So \( b + C_i d + C_j d \in V_2 \) for \( i \neq j \), because they can not be adjacent to \( Jb \). Now we established \( b + C_1 d \notin V_1 \). Then \( \{b + C_1 d, C_i d\} \in E \), and so \( b + 2C_1 d \in V \). Since \( 2b \neq 0 \), then \( C_1 d \neq 2b + C_1 d \). Notice \( 2b + 2C_1 d \notin F \) implies \( \{b + 2C_1 d, b\} \notin E \) implying \( b + 2C_1 d \in V_2 \). Also since \( C_1 d \neq 2b + C_1 d \), we have \( b + (C_1 d) + (2b + C_1 d) \equiv 3b + 2C_1 d \in V_2 \). So we have established \( 2b + C_1 d \in V_1 \), \( b + C_1 d \notin V_2 \), and \( b + 2C_1 d, 3b + 2C_1 d \in V_2 \).

We wish to establish \( n_1 = n_2 = |V_3| \). By assumption \( n_2 \geq n_1 \). Let \( f : V_2 \to V_1 \) by \( f(x) = x + b + C_1 d \). Pick \( x \in V_2 \). Then \( \{x, b + C_1 d\} \in E \), and so \( x + b + C_1 d \in V \). Now if \( x + b + C_1 d \notin V_1 \), then \( \{x + b + C_1 d, C_i d\}, \{x + b + C_1 d, 2b + C_1 d\} \in E \). Therefore \( x + b + 2C_1 d, x + 3b + 2C_1 d \in V \). Well \( 2b \neq 0 \) implies \( b + 2C_1 d \neq 3b + 2C_1 d \). Hence either \( \{x, b + 2C_1 d\} \in E \) or \( \{x, 3b + 2C_1 d\} \in E \), which can not happen. Therefore \( x + b + C_1 d \in V_1 \). If \( f(x) \equiv f(y) \) then \( x + b + C_1 d \equiv y + b + C_1 d \), and so \( x \equiv y \). So \( f \) is an injection. Thus \( n_1 = |V_1| \geq |V_2| = n_2 \), a contradiction. \( \Box \)

The next lemma is cited in Theorem 2.12 and Theorem 2.13. These theorems \( \rho(K_{n,m}) \) when \( m > n \geq 3 \) and \( m \) is even.

**Lemma 2.8** If \( K_{n_1,n_2,...,n_m} \) is a MSG, i.e. \( V = F \), with \( V_1 \) and \( V_2 \) partite sets \( |V_2| = n_2 > |V_1| = n_1 \geq 3 \), then \( n_2 \geq 2n_1 \). If \( a_1, 2a_1 \in V_1 \), then \( n_2 \geq 3n_1 - 3 \).

**Proof.** Denote \( V_1 = \{C_1 d, C_2 d, ..., C_n d\} \) where \( d = gcd(C_1 d, C_2 d, ..., C_n d) \). By Lemma 2.7, there exist \( b \in V_2 \) such that \( b + id \in V_2 \) for \( 0 \leq i \leq L \) where \( L \) is the smallest positive integer such that \( (L + 1)d \equiv 0 \). Thus \( b + id \in V_2 \) for \( 0 \leq i \leq L \) are \( L + 1 \) distinct vertices implying \( n_2 \geq L + 1 \).

Since \( L \) is the smallest positive integer such that \( (L + 1)d \equiv 0 \) we have at most \( L + 1 \) distinct multiples of \( d \) modulo \( Z \). Now \( \{C_1 d, C_i d\} \notin E \), so \( C_1 d + C_i d \notin F \).
for $2 \leq i \leq n_1$. If $C_i d + C_j d \equiv C_i d + C_j d$, then $C_i d \equiv C_j d$. So $C_j d, \{C_1 + C_i\} d$ for $1 \leq j \leq n_1$, $2 \leq i \leq n$ are $2n_1 - 1$ distinct multiples of $d$ modulo $\mathcal{Z}$. So $n_2 \geq L + 1 \geq 2n_1 - 1$.

If $C_1 d + C_j d \equiv 2C_1 d$, then $C_j d \equiv C_1 d$. Hence if $2C_1 d \notin V_1$, then $n_2 \geq L + 1 \geq 2n_1$.

Suppose $2C_1 d \equiv C_2 d \in V_1$. We will establish $C_1 d$, $2C_1 d$, $3C_1 d$, $C_i d$, $C_1 d + C_i d$, $2C_1 d + C_i d$ for $3 \leq i \leq n_1$ are $3n_1 - 3$ distinct multiples of $d$ modulo $\mathcal{Z}$. Now $C_i d \not\equiv C_1 d$, $2C_i d \not\equiv C_i d$ for $3 \leq i \leq n_1$. Also $3C_1 d$, $C_1 d + C_i d$, $2C_1 d + C_i d \notin \mathcal{F}$ for $3 \leq i \leq n_1$. Likewise $C_1 d$, $2C_1 d$, $C_i d \notin \{3C_1 d, C_1 d + C_j d, 2C_1 d + C_j d\}$ (mod $\mathcal{Z}$) for $3 \leq j \leq n_1$, $3 \leq i \leq n_1$. If $3C_1 d \equiv C_1 d + C_i d$, then $2C_1 d \equiv C_i d$. Notice if $3C_1 d \equiv 2C_1 d + C_i d$ then $C_1 d \equiv C_i d$. So $3C_1 d \notin \{C_1 d + C_i d, 2C_1 d + C_i d\}$ (mod $\mathcal{Z}$) for $3 \leq i \leq n_1$. Finally $C_1 d + C_i d \not\equiv 2C_1 d + C_j d$, since $C_i d \not\equiv C_1 d + C_j d$. Hence we have $1 + 1 + 1 + (n_1 - 2) + (n_1 - 2) + (n_1 - 2) = 3n_1 - 3$ distinct multiples of $d$ modulo $\mathcal{Z}$. So $n_2 \geq L + 1 \geq 3n_1 - 3 \geq 2n_1$ when $n_1 \geq 3$. □

**Corollary 2.9** If $K_{n_1, n_2, \ldots, n_m}$ is a MSG with $n_1 > n_2 > \ldots > n_{m-1} > n_m \geq 3$, then $n_1 \geq 2n_2 \geq 2^2n_3 \geq \ldots \geq 2^{m-2}n_{m-1} \geq 2^{m-1}n_m$.

**Proof.** The result follows by induction using Lemma 2.8. □

The next theorem is a special case needed for Theorem 2.12. This result was first proved by Miller, Ryan, Sutton, and Slamin in [9].

**Theorem 2.10** $K_{n,n}$ is not a MSG for $n \geq 3$.

**Proof.** Denote partitions $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_n\}$. By Lemma 2.5, if $K_{n,n}$ is a MSG, with $a_j \in V_1$ then for all $b_i \in V_2$, $a_j + b_i \in V - V_2$. Likewise
for $b_j \in V_2$, for all $a_i \in V_1$, $a_i + b_j \in V - V_1$. Since both can not be true $a_j + b_i \in R$, for all $a_j \in V_1$, $b_i \in V_2$. □

The following theorem is used in Theorem 2.12.

**Theorem 2.11** If $K_{n,m}$ is not a MSG, with $n \leq m$ then $n \leq \rho(K_{n,m}) \leq m$.

**Proof.** Denote partitions $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$. By assumption there exist $a_i$, $b_j$ such that $a_i + b_j \in R$. Now by Lemma 2.5, $b_j + a_i \in R$ for $1 \leq i \leq n$. So $\rho(K_{n,m}) \geq n$.

We now give a labelling for which $|R| = m$. Let $a_i = (i-1)10+7$, $b_i = (i-1)10+9$, $r_i = (i-1)10+6$ with $\mathcal{Z} = 10m$. Then $R = \{6, 16, \ldots, 10(m-1)+6\}$. Notice $a_i + b_j = (i+j-2)10+16 \in R$, $a_i + a_j = (i+j-2)10+14 \notin \mathcal{F}$, $b_i + b_j = (i+j-2)10+18 \notin \mathcal{F}$, $r_i + r_j = (i+j-2)10+12 \notin \mathcal{F}$, $a_i + r_j = (i+j-2)10+13 \notin \mathcal{F}$, and finally, $b_i + r_j = (i+j-2)10+15 \notin \mathcal{F}$. Thus $\rho(K_{n,m}) \leq m$. □

The next three theorems, Theorem 2.12, Theorem 2.13, and Theorem 2.14, show $K_{n,m}$ is a mod sum graph when $m$ is even and $m \geq 2n$ or when $m$ is odd and $m \geq 3n - 3$ and $\rho(K_{n,m}) = n$ when $3 \leq n \leq m < 2n$.

**Theorem 2.12** $\rho(K_{n,m}) = n$, if $3 \leq n \leq m < 2n$.

**Proof.** By Lemma 2.8, $K_{n,m}$ is not a MSG since $m < 2n$. By Theorem 2.11, $\rho(K_{n,m}) \geq n$. Denote partitions $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$.

We now give a labelling for which $|R| = n$. Let $a_i = (i-1)10+9$, $b_i = (i-1)10+6$ for $1 \leq i \leq n$, $b_i = (i-1)10+7$ for $n < i \leq m$. Let $r_i = (i-1)10+5$ with $\mathcal{Z} = 10n$. Then $R = \{5, 15, \ldots, 10(n-1)+5\}$. Notice if $j \leq n$, then $a_i + b_j = (i+j-2)10+15 \in R$, and if $j > n$, then $a_i + b_j = (i+j-2)10+16 \in V_2$. Now for any $i,j$, $a_i + a_j =
\[ (i + j - 2)10 + 18 \notin \mathcal{F}, \text{ for } i, j \leq n, \quad b_i + b_j = (i + j - 2)10 + 12 \notin \mathcal{F}, \text{ for } i \leq n, j > n, \]
\[ b_i + b_j = (i + j - 2)10 + 13 \notin \mathcal{F}, \text{ and for } i, j > n, \quad b_i + b_j = (i + j - 2)10 + 14 \notin \mathcal{F}. \]
Also, \( r_i + r_j = (i + j - 2)10 + 10 \notin \mathcal{F}, \quad a_i + r_j = (i + j - 2)10 + 14 \notin \mathcal{F}, \) and finally, for \( i \leq n, \quad b_i + r_j = (i + j - 2)10 + 11 \notin \mathcal{F}, \) for \( i > n, \quad b_i + r_j = (i + j - 2)10 + 12 \notin \mathcal{F}. \)
Thus \( \rho(K_{n,m}) \leq n. \]

**Theorem 2.13** If \( m \) is even and \( m \geq 4 \), then \( K_{n,m} \) with \( m \geq n \) is a MSG if and only if \( m \geq 2n \).

**Proof.** \( \implies \) By Theorem 2.10, \( K_{n,m} \) is not a MSG if \( n = m \). By Lemma 2.8, if \( K_{n,m} \) is MSG with \( m > n \), then \( m \geq 2n \).

\( \iff \) Assume \( m \geq 2n \) with \( m \) even. Let \( V_1 \) and \( V_2 \) be partite sets. Let \( V_1 \subset \{d, 3d, \ldots, (m-1)d\} \), \( V_2 = \{1, 1+d, 1+2d, \ldots, 1+(m-1)d\} \), \( d = 4 \), and \( Z = dm = 4m \). Pick \( (2i+1)d \in V_1 \) and \( 1+jd \in V_2 \), then \( \{(2i+1)d, 1+jd\} \in E \) since
\[ (2i+1)d + 1 + jd = 1 + (jd + 2(i+1)d) \pmod{dm} = 1 + kd \] for some k. Pick \((2i+1)d, (2j+1)d \in V_1\). Now \([(2i+1)d + (2j+1)d] = 2(i + j + 1)d \pmod{dm} = 2kd\) when \(m\) is even. Since \(2kd \notin \mathcal{F}, \{(2i + 1)d, (2j + 1)d\} \notin E\). Pick \(1 + id, 1 + jd \in V_2\).

Now \((1 + id + 1 + jd) \pmod{dm} = 2 + kd\) for some k. Also \(2 + kd \neq ld\), since \(2 = (k-l)d \pmod{dm}\) implies \(d < 4\). Likewise \(2 + kd \neq 1 + ld\), since \(1 = (k-l)d \pmod{dm}\) implies \(d < 4\). Thus \(1 + id + 1 + jd \notin \mathcal{F}\) which in turn means \(\{1 + id, 1 + jd\} \notin E\). \(\square\)

**Theorem 2.14** If \(m\) is odd and \(m \geq 3\), then \(K_{n,m}\) with \(m \geq 3n - 3\) is a MSG.

**Proof.** Let \(L = \lfloor \frac{m}{3} \rfloor + 1\). Then \(L\) is the largest integer such that \(3L - 3 \leq m\). So if \(3n - 3 \leq m\), then \(3n - 3 \leq 3L - 3\), which implies \(n \leq L\). Either \(L = 2K + 1\) or \(L = 2K\).

Case 1: \((L=2K+1)\) Let \(V_1 \subset \{d, 3d, ..., (l-2)d, (m-l)d, (m-l+2)d, ..., (m-3)d, (m-1)d\}\), and \(V_2 = \{1, 1+d, 1+2d, ..., 1+(m-1)d\}\) where \(d = 4\) and \(dm = Z\). So for any \(s \in V_1, r \in V_2, s+r \in V_2\), and for any \(s, r \in V_2, s+r \notin V\). Denote \(a_{i} = (2i-1)d\)
for $1 \leq i \leq K$, $b_i = (m - 2i + 1)d$ for $1 \leq i \leq K + 1$. Then the $a_i$ are odd multiples of $d$ and the $b_i$ are even multiples of $d$. Now $a_1 < a_2 < ... < a_K < b_{K+1} < b_1$. Therefore for any $a_i, a_j \in V_1$ for $i \neq j$, $2(i+j-1)d = a_i + a_j \leq a_{K-1} + a_K \leq (2L - 6)d < (2L - 2)d \leq (m - L + 1)d < dm$. So $a_i + a_j = 2(i + j - 1)d \neq a_h = (2h - 1)d$ for any $h$, and $a_i + a_j < (m - L)d = b_{K+1}$. Hence $a_i + a_j \notin V_1$, and $a_i + a_j \notin V_2$. Pick $a_i, b_j \in V_1$. Now $a_i + b_j = (2i - 1)d + (m - 2j + 1)d \equiv (m - 2(i - j))d$. If $a_i + b_j < dm$, then $a_i + b_j$ is an odd multiple of $d$, and $a_K < a_i + b_j$, so $a_i + b_j \notin V_1$. If $a_i + b_j > dm$, then $a_i + b_j < a_i (mod dm) < b_{K+1}$, and $a_i + b_j$ is an even multiple. So $a_i + b_j \notin V_1$, and $a_i + b_j \notin V_2$. Pick $b_i, b_j \in V_1$ with $i \neq j$. Then $dm < (2m - 2L + 2)d = (2m - 4K)d = b_K + b_{K+1} \leq b_i + b_j < 2md$. So $(2m - 2L + 2)d \equiv (m - 2L + 2)d$, and $a_K = (L - 2)d < (L - 1)d \leq (m - 2L + 2)d < dm$. So $b_i + b_j \notin V_1$, and $b_i + b_j \notin V_2$.

Case 2: (L=2K) Let $V_1 \subset \{d, 3d, ..., (l - 1)d, (m - l + 1)d, (m - l + 3)d, ..., (m - 3)d, (m - 1)d\}$, and $V_2 = \{1, 1 + d, 1 + 2d, ..., 1 + (m - 1)d\}$ where $d = 4$ and $dm = \mathbb{Z}$. Denote $a_i = (2i - 1)d$ for $1 \leq i \leq K$, $b_i = (m - 2i + 1)d$ for $1 \leq i \leq K$. Now
$a_1 < a_2 < \ldots < a_K < b_K < b_{K-1} < \ldots < b_1$. Similarly to the previous argument, we can show the desired properties results. $\square$

![Graph](image)

Figure 4: $K_{6,15} \ (mod \ 60)$
CHAPTER 3
CONCLUSIONS and CONJECTURES

Conclusions

In this paper we have extended the classes of graphs for which we know the mod sum number. These are

- \( \rho(K_{n_1,n_2,...,n_m}) > 0 \) if there exist \( n_i \) and \( n_j \) such that \( n_i < n_j < 2n_i \).

- \( \rho(K_{n,m}) = 0 \) when \( m \geq 2n, n \geq 3 \) and \( m \) even.

- \( \rho(K_{n,m}) = 0 \) when \( m \geq 3n - 3, n \geq 3 \) and \( m \) odd.

- \( \rho(K_{n,m}) = n \) when \( 2n > m \geq n \geq 3 \).

So we conclude with conjectures that our research has shown may be possible, but for lack of time we were not able to pursue.

Conjectures

Lemma 2.8 and Theorem 2.11 might be used to show the following.

Conjecture 3.15 \( \rho(K_{n,m}) = m \) when \( 3n - 3 > m \geq n \geq 3 \).

The following would be an extension to Theorem 2.11.

Conjecture 3.16 If \( K_{n_1,n_2,...,n_m} \) with \( n_1 > n_2 > ... > n_m \) is not a MSG, then
\[
(m - 1)n_m \leq \rho(K_{n_1,n_2,...,n_m}) \leq (m - 1)n_1.
\]
The following shows good promise and would be an extension of Lemma 2.8.

**Conjecture 3.17** If $K_{n_1,n_2,...,n_m}$ with $n_1 > n_2 > ... > n_m$ is a MSG, then $n_i \geq 2n_{i+1} + \sum_{j=i+2}^{m} n_j$.

The following would be extension to work by [7] and [9].

**Conjecture 3.18** $\rho(W_n) = 3$ when for $n > 4$ and $n$ even.

**Conjecture 3.19** $\rho(W_n) = n$ when for $n > 4$ and $n$ odd.

The following conjectures appear to be true, but seem difficult to prove or disprove.

**Conjecture 3.20** Let $G = (V, E)$ be a graph with $|V| = n$, then $\rho(G) \leq n$.

**Conjecture 3.21** Let $G = (V, E)$ be a graph, then finding the mod sum number is NP-complete.

**Conjecture 3.22** Let $G = (V, E)$ be a graph. There exist $L$ based on $n$ such that if $G$ is not MSG modulo $Z \leq L$ then $G$ is not a MSG.
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