Generating Functions
September 21, 2017

Useful Identities
For $|x| < 1$, the following identities hold:

1. (Taylor series) $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$
   need $f$ to be analytic at 0

2. $\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{k + (n-1)}{n-1} x^k$ and $\prod_{k=0}^{\infty} \left(1 + x^k\right) = \frac{1}{1-x}$
   need $|x| < 1$

3. $\sum_{i \in A, j \in B} (\alpha_i x^i)(\beta_j x^j) = \left(\sum_{i \in A} \alpha_i x^i\right) \left(\sum_{j \in B} \beta_j x^j\right)$
   need absolute convergence

Warmup
1. How to multiply: Expand $(1 + x^2 + x^7 + x^{20})(x + x^3 + x^4)$. Expand $(1 + 3x^2 + x^7 + 4x^{20})(x - x^3 + x^4)$.

Things you can do to a GF
1. $(a_0, a_1, a_2, \ldots), (b_0, b_1, b_2, \ldots) \mapsto (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots): G_1, G_2 \mapsto G_1 + G_2$
2. $(a_0, a_1, a_2, \ldots) \mapsto (\alpha a_0, \alpha a_1, \alpha a_2, \ldots): G \mapsto \alpha G$
3. $(a_0, a_1, a_2, \ldots) \mapsto (0, a_0, a_1, \ldots): G(x) \mapsto xG(x)$
4. $(a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, a_3, \ldots): G(x) \mapsto \frac{G(x)-a_0}{x}$
5. $(a_0, a_1, a_2, \ldots) \mapsto (a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots): G(x) \mapsto \frac{G(x)}{1-x}$
6. $(a_0, a_1, a_2, \ldots) \mapsto (a_1, 2a_2, 3a_3, \ldots): G \mapsto G'$
7. $(a_0, a_1, a_2, \ldots) \mapsto (C, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots)$
8. Prove that the GF of $H_n$ is $-\frac{\log(1-x)}{1-x}$.

Recurrences
1. Solve the recurrence $a_0 = 1, a_1 = 2, a_n = 3a_{n-1} - 2a_{n-2}$.
2. Find the value of $1^2 + 2^2 + \cdots + n^2$.
3. The Catalan numbers are defined by $C_0 = 1$ and
   
   $C_n = C_{n-1}C_0 + C_{n-2}C_1 + \cdots + C_0C_{n-1}$
   
   for $n \geq 1$. Find the generating function for $C_n$, and use it to find an explicit formula for $C_n$. 

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Summing

1. Find the value of \( \sum_{n=0}^{\infty} \frac{n}{2^n} \). What about \( \sum_{n=0}^{\infty} \frac{n^2}{2^n} \)?

2. Given \( f(x) = a_0 + a_1x + a_2x^2 + \ldots \), find \( a_0 + a_4x^4 + a_8x^8 + \ldots \).

3. Find the value of \( \sum_{n=0}^{\infty} \frac{1}{2^{2^n} - 2^{-2^n}} \).

Counting

1. How many solutions are there to \( x_1 + x_2 + \cdots + x_m = n \) such that \( 0 \leq x_1, x_2, \ldots, x_m \leq n \)? Such that \( 0 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq n \)?

2. Determine the number of \( k \)-element subsets of \([n]\) such that the \( i \)th largest element of the subset is congruent to \( i \mod 2 \).

Two fairly hard problems

1. Let \( (a_n)_{n \in \mathbb{N}} \) be the sequence defined by
   \[
   a_0 = 1, \quad a_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{n-k+1}
   \]
   Find the limit
   \[
   \lim_{n \to \infty} \sum_{k=0}^{n} \frac{a_k}{2^k}
   \]

2. Evaluate
   \[
   \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}
   \]

Problems

1. Prove that \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).

2. Find the value of \( \sum_{k=0}^{n} \binom{n}{k} (-1)^k \). Deduce that the number of subsets of \( \{1, 2, \ldots, n\} \) with odd size is equal to the number with even size.

3. By comparing the coefficient of \( x^n \) in \( (x + 1)^{a+b} \) and \( (x + 1)^a(x + 1)^b \), prove that
   \[
   \binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k}
   \]

4. Let \( \{1, 1, 2, 3, 5, 8, \ldots\} \) be the Fibonacci sequence. Prove that the number
   \[
   \frac{1}{10^3} + \frac{1}{10^6} + \frac{2}{10^9} + \frac{3}{10^{12}} + \cdots = 0.001001002003\ldots
   \]
   is a rational number. What is it in reduced form?

5. Find an explicit formula for the \( n \)th Fibonacci number. Hint: this formula will likely involve the number \( \frac{1 + \sqrt{5}}{2} \), a root of \( x^2 - x - 1 \).

6. How many \( n \)-digit numbers, whose digits are in the set \{2, 3, 7, 9\} are divisible by 3?