PARTITION RELATIONS VIA IDEAL PRODUCTS

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Abstract. We analyze how a simple splitting condition relating an ideal with a linear order allows the construction of certain embeddings of the rationals into the order. We extract as consequences proofs of some well known partition relations, including the Baumgartner-Hajnal theorem ($\varphi \to (\omega)^1_\omega$ implies $\varphi \to (\alpha)^2_2$ for all countable ordinals α) for uncountable orders φ not containing ω_1, the result of Erdős-Rado that $\eta \to (\eta, \aleph_0)^2$, and the standard canonization of colorings of pairs of rationals.

We fix a linearly ordered set $(X, <)$. For sets $A, B \subseteq X$, $A < B$ means $a < b$ for all $a \in A$ and $b \in B$. We also fix some ideal $\mathcal{I}$ of subsets of X, and use terms like *small*, *positive* (or *nonsmall*), and *cosmall* in the obvious way. We say *positive sets split over the order* if any positive set $A \subseteq X$ contains positive sets A_0 and A_1 with $A_0 < A_1$. Finally, $\bar{\imath}$ is shorthand for $1 - i$.

Definition 1. Given a coloring $c : [X]^2 \to 2$ and $i \in 2$, we say that (A, B) is an *i-compatible pair* if A and B are both positive subsets of X, and moreover for every positive $B' \subseteq B$ the set

$$\{a \in A : \text{the set } \{b \in B' : c(\{a, b\}) = i\} \text{ is positive}\}$$

is co-small in A.

Definition 2. Given a coloring $c : [X]^2 \to 2$ and $i \in 2$, we say that (A, B) is an *i-focused pair* if A and B are both positive subsets of X, and moreover for all $a \in A$ the set

$$\{b \in B : c(\{a, b\}) = i\}$$

is co-small in B.

Remark 3. If (A, B) is an i-compatible pair (respectively, an i-focused pair) and $A' \subseteq A$ and $B' \subseteq B$ are both positive, then (A', B') is an i-compatible (resp., i-focused) pair. Also, if (A, B) is an i-focused pair, then it is an i-compatible pair.

We first establish a lemma granting structure similar to the sort bequeathed by localization and Kuratowski-Ulam in the special case of the meager ideal (or, if you prefer, density and Fubini in the case of the null ideal).

Lemma 4. *Suppose that $c : [X]^2 \to 2$ is an arbitrary coloring, and suppose further that $A, B \subseteq X$ are positive sets. Then there exist positive sets $A^\star \subseteq A$ and $B^\star \subseteq B$ such that one of the following holds:*

1. *$(A^\star, B^\star)$ is an i-focused pair for some $i \in 2$, or*
2. *$(A^\star, B^\star)$ is an i-compatible pair for all $i \in 2$.*

Proof. We simply consider two exhaustive cases, determined by the truth value of the sentence

$$\exists i \in 2\ \exists^{\mathcal{I}} B' \subseteq B\ \exists^{\mathcal{I}} A' \subseteq A\ (\{a \in A' : \{b \in B' : c(\{a,b\}) = i\} \text{ is positive}\} \text{ is small}),$$

where $\exists^{\mathcal{I}}$ is shorthand for "there exists a positive set."

Case 1: it is true. In this case, we may choose i, A', and B' witnessing the statement's truth, and let

$$A^\star = A' \setminus \{a \in A' : \{b \in B' : c(\{a,b\}) = i\} \text{ is positive}\}.$$

Subsequently, we observe that $(A^\star, B')$ is an $\bar{\imath}$-focused pair.

Case 2: it is false. We then have

$$\forall i \in 2\ \forall^{\mathcal{I}} B' \subseteq B\ \forall^{\mathcal{I}} A' \subseteq A\ (\{a \in A' : \{b \in B' : c(\{a,b\}) = i\} \text{ is positive}\} \text{ is positive}),$$

where $\forall^{\mathcal{I}}$ is shorthand for "for all positive sets." But this is equivalent to

$$\forall i \in 2\ \forall^{\mathcal{I}} B' \subseteq B\ (\{a \in A : \{b \in B' : c(\{a,b\}) = i\} \text{ is positive}\} \text{ is cosmall in } A).$$

Consequently, (A, B) is i-compatible for all $i \in 2$. $\square$

It would be nice if we could symmetrize our notion of compatibility, strengthening the conclusion of Lemma 4. Unfortunately, we can't, but we can do the next best thing.

Definition 5. Given a coloring $c : [X]^2 \to 2$ and $i, j \in 2$, we say that (A, B) is an *(i, j)-compatible pair* if (A, B) is an i-compatible pair, and (B, A) is a j-compatible pair.

Using this notion, it is easy to prove the following two corollaries by first applying Lemma 4 to the pair (A, B), and then to the pair $(B^\star, A^\star)$ obtained by flipping the pair granted by the lemma.

Corollary 6. *Suppose that $c : [X]^2 \to 2$ is an arbitrary coloring, and suppose further that $A, B \subseteq X$ are positive sets. Then there exist positive sets $A^\star \subseteq A$ and $B^\star \subseteq B$ such that one of the following holds:*

1. *$(A^\star, B^\star)$ is an i-focused pair for some $i \in 2$, or*
2. *$(A^\star, B^\star)$ is an (i, i)-compatible pair for some $i \in 2$.*

Corollary 7. *Suppose that $c : [X]^2 \to 2$ is an arbitrary coloring, and suppose further that $A, B \subseteq X$ are positive sets. Then there exist positive sets $A^\star \subseteq A$ and $B^\star \subseteq B$ and $i, j \in 2$ such that $(A^\star, B^\star)$ is an (i, j)-compatible pair.*

In particular, we can apply these corollaries to the pairs obtained by splitting positive sets, as we see in the lemma below.

Lemma 8. *Suppose that positive sets split over the order and $c : [X]^2 \to 2$ is an arbitrary coloring. Then there is a positive $A \subseteq X$ and $i, j \in 2$ such that whenever $A' \subseteq A$ is positive, then we may find $A'_0 < A'_1$, both positive subsets of A', so that the pair (A'_0, A'_1) is (i, j)-compatible.*

Proof. This follows from Corollary 7 and an easy density argument. □

The upshot of all this is that, passing down to a positive set if necessary, we may assume that there is a universal choice of $i, j \in 2$ so that we can always split positive sets into (i, j)-compatible pairs. For convenience, we will refer to this situation by saying that the coloring is (i, j)-*splitting*. In fact, we can do slightly better than merely splitting — we can split with a point in the middle!

Lemma 9. *Suppose that $c : [X]^2 \to 2$ is an (i, j)-splitting coloring. Then there exist $X_0, X_1 \subseteq X$ and $x \in X$ with $X_0 < \{x\} < X_1$ such that (X_0, X_1) is an (i, j)-compatible pair. Moreover, for all $x_0 \in X_0$ and $x_1 \in X_1$ we have $c(\{x_0, x\}) = j$ and $c(\{x, x_1\}) = i$.*

Proof. Since c is (i, j)-splitting, we may split twice to find positive subsets $A < B < D$ of X such that (A, B), (A, D), and (B, D) are all (i, j)-compatible pairs. Since (B, A) is j-compatible, we know that the set

$$B_A := \{b \in B : \text{the set } \{a \in A : c(\{a, b\}) = j\} \text{ is positive}\}$$

is cosmall in B. Similarly, since (B, D) is i-compatible, we know that the set

$$B_D := \{b \in B : \text{the set } \{d \in D : c(\{b, d\}) = i\} \text{ is positive}\}$$

is cosmall in B. We may thus choose $x \in B_A \cap B_D$, and we are done once we set

$$\begin{aligned} X_0 &= \{a \in A : c(\{a, x\}) = j\} \\ X_1 &= \{d \in D : c(\{x, d\}) = i\} \end{aligned}$$

□

The complete binary tree, $2^{<\omega}$, plays a central role in the remainder of the note. We extend the lexicographical order on each 2^n to a linear order on $2^{<\omega}$ by setting $s < t$ iff $s(n) < t(n)$ where n is the first coordinate on which they differ (adopting the convention that $0 <$ undefined < 1). Clearly, the order type of $2^{<\omega}$ under this ordering is η. For $s \in 2^{<\omega}$, we denote by $|s|$ the *length* of s.

We also define, using this ordering, four colorings of $[2^{<\omega}]^2$. For $i, j \in 2$, we define $c_{ij} : [2^{<\omega}]^2 \to 2$ by

$$c_{ij}(\{s, t\}) = \begin{cases} i & \text{if } s < t \text{ and } |s| \le |t| \\ j & \text{if } s < t \text{ and } |s| > |t|. \end{cases}$$

These can be viewed as colorings of $[\mathbb{Q}]^2$ in the obvious way. The two colorings c_{ii} and c_{jj} simply correspond to constant colorings, while the other two are the standard impediments to Ramsey's theorem on order type η. The main result of this note is that these four colorings form a basis for colorings of pairs in well behaved spaces.

Theorem 10. *Suppose that positive sets split over the order and $c : [X]^2 \to 2$ is an arbitrary coloring. Then there exists $i, j \in 2$ and an order-preserving injection $\varphi : 2^{<\omega} \to X$ such that*

$$\forall s, t \in 2^{<\omega} \ c(\{\varphi(s), \varphi(t)\}) = c_{ij}(\{s, t\}).$$

Proof. By Lemma 8, we may assume that $c : [X]^2 \to 2$ is an (i, j)-splitting coloring. We will embed the coloring c_{ij} corresponding to these values of i and j.

We recursively construct for each $n \in \omega$ a function $\varphi_n : 2^n \to X$ approximating the desired function. In addition, we construct for each $s \in 2^{n+1}$ a positive set $A_s \subseteq X$ such that for all $s < t \in 2^{n+1}$, (A_s, A_t) is an (i, j)-compatible pair. Moreover, for all $s \in 2^{<n+1}$ and $t \in 2^{<n+1}$,

$$a \in A_t \Rightarrow c(\{\varphi_{|s|}(s), a\}) = c_{ij}(\{s, t\}).$$

At stage $n = 0$ of the construction, simply use the splitting assumption and Lemma 9 to find $x \in X$, and positive sets A_0 and A_1 satisfying $A_0 < \{x\} < A_1$ such that $c(\{a_0, x\}) = j$ and $c(\{x, a_1\}) = i$ for all $a_0 \in A_0$ and $a_1 \in A_1$, and additionally that (A_0, A_1) is an (i, j)-compatible pair. Set $\varphi(\emptyset) = x$.

Now suppose that we have completed the construction up through stage $n - 1$. We complete stage n from left to right. By the assumption of compatibility, we may assume that there is a set A'_{0^n} cosmall in A_{0^n} such that for all $x \in A'_{0^n}$ and $t \in 2^n$ with $0^n < t$, the set

$$\{a \in A_t : c(\{a, x\}) = c_{ij}(\{0^n, t\}) = i\}$$

is positive. We apply Lemma 9 to A'_{0^n} as before to obtain $\varphi_n(0^n) \in A'_{0^n}$, and an (i, j)-compatible pair (A_{0^n0}, A_{0^n1}). Then, replace each A_t with the set

$$\{a \in A_t : c(\{\varphi_n(0^n), a\}) = c_{ij}(\{0^n, t\})\},$$

which is guaranteed to be positive.

Continuing from left to right, fix $s \in 2^n$ and suppose we have defined φ_n, A_{s0}, and A_{s1} for all elements of 2^n less than s. By the assumption of compatibility, we may assume (discarding a small set if necessary) that for all $x \in A_s$ and $t \in 2^n$ with $s < t$, the set

$$\{a \in A_t : c(\{x, a\}) = c_{ij}(\{x, a\}) = i\}$$

is positive. Moreover, we may assume that for all $x \in A_s$ and $t \in 2^{n+1}$ with $t < s$, the set

$$\{a \in A_t : c(\{a, x\}) = c_{ij}(\{a, x\}) = j\}$$

is positive. We apply Lemma 9 to A_s as before to obtain $\varphi_n(s) \in A_s$, $f_n(s) \in 2$, and a (i, j)-compatible pair (A_{s0}, A_{s1}). For each $t \in 2^n$ with $s < t$, replace A_t with the set

$$\{a \in A_t : c(\{\varphi_n(s), a\}) = c_{ij}(\{s, t\})\},$$

and, analogously, for each $t \in 2^{n+1}$ with $t < s$, replace A_t with the set

$$\{a \in A_t : c(\{a, \varphi_n(s)\}) = c_{ij}(\{t, s\})\},$$

Now suppose that we have completed the construction for all $n \in \omega$; we set $\varphi = \bigcup_n \varphi_n$. We just need to check that this function works. Fix $s, t \in 2^{<\omega}$.

Without loss of generality, we may assume that $\varphi(s)$ was determined before $\varphi(t)$. Then, since $\varphi(t)$ belongs to the refined version of A_t constructed when $\varphi(s)$ was decided, we know that $c(\{\varphi(s), \varphi(t)\}) = c_{ij}(\{s,t\})$. □

We close the note with a couple of applications of the main theorem.

Corollary 11 (Devlin, Galvin, Vuksanovic). *Suppose that $c : [\mathbb{Q}]^2 \to 2$ is an arbitrary coloring. Then we may find $i, j \in 2$ and $A \subseteq \mathbb{Q}$ of order type η such that $c|A = c_{ij}|A$.*

Proof. Simply apply Theorem 10 to $\mathbb{Q}$ equipped with the ideal of sets not containing a set of order type η. □

Remark 12. The above result (and everything else in the note) holds for any finite number of colors. In this setting, you get a larger basis of colorings, but they are all either constant functions or c_{ij} for $i \neq j$ (in particular, they all use at most two colors). Also, this corollary yields the result (due to Erdős-Rado) that $\eta \to (\eta, \aleph_0)^2$, meaning that any coloring of pairs of rationals by $\{0, 1\}$ admits either a 0-homogeneous set of order type η or an infinite 1-homogeneous set.

Corollary 13. *Suppose that $\mathcal{I}$ is a σ-additive ideal on X, that positive sets split over the order, and $c : [X]^2 \to 2$ is an arbitrary coloring. Then for all $\alpha < \omega_1$ there exists a c-homogeneous set $A \subseteq X$ of order type α.*

Proof. Note that if we can find an (i, i)-splitting, positive $X' \subseteq X$, Theorem 10 would let us construct a c-homogeneous set of order type η, which is more than enough to get c-homogeneous sets of order type α.

By Corollary 6 and a standard density argument, we may assume that there exists $i \in 2$ such that every positive set $X' \subseteq X$ can split into $X_0 < X_1$ with (X_0, X_1) an i-focused pair. We use transfinite induction to argue that in any such set we may find a homogeneous set (of color i) for all $\alpha < \omega_1$.

Suppose first that $\alpha = \beta + 1$. Simply split X into $X_0 < X_1$ with (X_0, X_1) an i-focused pair. We know we may find a homogeneous set A_β of order type β inside X_0. For each $x_0 \in A_\beta$, the set

$$\{x_1 \in X_1 : c(\{x_0, x_1\}) = i\}$$

is cosmall in X_1. Since $\mathcal{I}$ is σ-additive and A_β is countable, we may find an x_β in the intersection of all these sets. The set $A_\beta \cup \{x_\beta\}$ is as desired.

Suppose now that $\alpha = \bigcup_n \beta_n$ with each $\beta_n < \alpha$. Splitting X several times, we may find $X_0 < X_1 < \cdots$ such that (X_{n_0}, X_{n_1}) is an i-focused pair whenever $n_0 < n_1$. By the inductive hypothesis, we may find a homogeneous set A_{β_0} of order type β_0 inside X_0. As before, we may refine X_n for all $n > 0$ so that for all $x_0 \in A_{\beta_0}$ and $x_n \in X_n$, $c(\{x_0, x_n\}) = i$. We continue in this fashion, finding a homogeneous A_{β_n} of order type β_n within X_n, refining after each step. In the end, $\bigcup_n A_{\beta_n}$ is a homogeneous set of order type $\sum_n \beta_n \geq \alpha$. □

Corollary 14 (Baumgartner-Hajnal, Galvin). *Suppose that φ is an order type such that $\varphi \to (\omega)^1_\omega$ and, moreover, that ω_1 does not embed into φ. Then for all $\alpha < \omega_1$, $\varphi \to (\alpha)^2_2$.*

Proof. Suppose that φ is an order type satisfying the hypotheses of the corollary, and let $(X, <)$ be a linearly ordered set of order type φ. We equip X with the ideal $\mathcal{I}$ defined by

$$A \in \mathcal{I} \Leftrightarrow A \not\rightarrow (\omega)^1_\omega.$$

It is clear that $\mathcal{I}$ is a σ-additive ideal (indeed, it is the σ-ideal generated by subsets of X not containing a copy of ω), so once we check that positive sets split over the order we may appeal to Corollary 13.

Towards that end, suppose that $A \subseteq X$ is not in $\mathcal{I}$. For each $x \in A$, define the sets $A_{<x}$ and $A_{>x}$ by

$$\begin{aligned} A_{<x} &= \{a \in A : a < x\} \\ A_{>x} &= \{a \in A : a > x\}. \end{aligned}$$

We argue that there is a set A' cosmall in A such that for all $x \in A'$, both $A_{<x}$ and $A_{>x}$ are positive, which is more than enough to show that positive sets split over the order.

First, let $B_0 = \{x \in A : A_{<x} \text{ is small}\}$. Let $(x_\beta)_{\beta<\kappa}$ be an increasing, cofinal sequence in B_0. Since ω_1 does not embed into φ, we may assume that κ is countable. Then $B_0 \subseteq \bigcup_{\beta<\kappa} A_{<x_\beta}$, and consequently B_0 is small.

Next, let $B_1 = \{x \in A : A_{>x} \text{ is small}\}$. Parallel to the earlier argument, let $(x_\beta)_{\beta<\kappa}$ be a decreasing, coinitial sequence in B_1. We may no longer assume κ is countable, but we can directly argue that B_1 is in $\mathcal{I}$. Define a function $f : B_1 \to \kappa$ by $f(x) = \min\{\beta < \kappa : x \in A_{>x_\beta}\}$, and fix for each $\beta < \kappa$ colorings $c_\beta : A_{>x_\beta} \to \omega$ witnessing $A_{>x_\beta} \not\rightarrow (\omega)^1_\omega$. Define a coloring $c : B_1 \to \omega$ by

$$c(x) = c_{f(x)}(x).$$

We claim that this coloring witnesses $B_1 \not\rightarrow (\omega)^1_\omega$.

Suppose, towards a contradiction, that we have an increasing sequence $(b_n)_{n\in\omega}$ in B_1 such c is constant on $\{b_n : n \in \omega\}$. Since the sequence $(f(b_n))_{n\in\omega}$ is a nondecreasing sequence of ordinals, it must be eventually constant. We may thus assume without loss of generality that there exists $\beta < \kappa$ such that $f(b_n) = \beta$ for all $n \in \omega$. But we would then have that c_β is constant on $\{b_n : n \in \omega\}$, contradicting the choice of c_β.

Consequently, both B_0 and B_1 are small, and thus the set $A \setminus (B_0 \cup B_1)$ is cosmall in A as desired. $\square$

Remark 15. In fact, for all countable α the statement $\omega_1 \rightarrow (\alpha)^2_2$ is true, so the hypothesis in Corollary 14 that ω_1 does not embed into φ is unnecessary. Todorcevic has shown that an analog holds in the greater generality of partial orders.