FINITE MONOID-VALUED MEASURE ALGEBRAS

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We fix an abelian semigroup $\langle S, +\rangle$. We say that S is *positive* if it contains no additive identity. For $m, n \in \mathbb{N}$, an $m \times n$ *S-matrix* is an $m \times n$ matrix $A = (a_{i,j})$ whose entries are elements of S. If $A = (a_{i,j})$ is an $m \times n$ S-matrix, let

$$\mathbf{r}_A = \left(\sum_{j<n} a_{0,j}, \sum_{j<n} a_{1,j}, \ldots, \sum_{j<n} a_{m-1,j}\right)$$

denote its sequence of *row sums*, and let

$$\mathbf{c}_A = \left(\sum_{i<m} a_{i,0}, \sum_{i<m} a_{i,1}, \ldots, \sum_{i<m} a_{i,n-1}\right)$$

denote its sequence of *column sums*.

We say that S *splits four ways* if for every $r_0, r_1, c_0, c_1 \in S$ with $r_0+r_1 = c_0+c_1$, there is a 2×2 S-matrix A with $\mathbf{r}_A = (r_0, r_1)$ and $\mathbf{c}_A = (c_0, c_1)$.

Example 1. Suppose that $\langle G, +, <\rangle$ is an abelian group with identity 0_G and a translation-invariant partial order. We use G^+ to denote the positive semigroup $\{g : 0_G < g\}$. Denoting by $\exists^+$, $\forall^+$ quantification over G^+, we say G^+ *splits under sums* if

$$\forall^+ g_0, g_1\ \forall^+ k < g_0 + g_1\ \exists^+ h_0 < g_0\ \exists^+ h_1 < g_1\ (h_0 + h_1 = k).$$

It is not hard to see that G^+ splits four ways if and only if it splits under sums.

Example 2. As a special case of Example 1, suppose that $\langle G, +, <\rangle$ is an abelian group with a translation-invariant linear order. In this case, G^+ splits four ways if and only if G^+ has no $<$-minimal element.

Example 3. If $\langle L, \wedge, \vee\rangle$ is a lattice, we may view it as an abelian semigroup under the operation $\vee$. A semigroup arising in this fashion always splits four ways: suppose $r_0, r_1, c_0, c_1 \in L$ with $r_0 \vee r_1 = c_0 \vee c_1$. Then the matrix

$$\begin{pmatrix} r_0 \wedge c_0 & r_0 \wedge c_1 \\ r_1 \wedge c_0 & r_1 \wedge c_1 \end{pmatrix}$$

has the required row and column sums. Additionally, such a lattice is a positive semigroup if and only if it contains no bottommost element (e.g, the cofinite subsets of $\mathbb{N}$).

Lemma 4. *Suppose that S is an abelian semigroup that splits four ways. Suppose further that $m, n \in \mathbb{N}$ and $\mathbf{r} = (r_0, \ldots, r_{m-1})$, $\mathbf{c} = (c_0, \ldots, c_{n-1})$ are sequences of elements of S with $\sum_{i<m} r_i = \sum_{j<n} c_j$. Then there exists an $m \times n$ S-matrix A such that $\mathbf{r}_A = \mathbf{r}$ and $\mathbf{c}_A = \mathbf{c}$.*

Proof. We proceed by induction on $m+n$. The lemma is trivial when either of m, n is less than 2, and the case $m = n = 2$ is granted by the assumption that S splits four ways. By interchanging rows and columns if necessary, we may assume $m > 2$.

Suppose that $\mathbf{r} = (r_0, \ldots, r_{m-1})$ and $\mathbf{c} = (c_0, \ldots, c_{n-1})$ are as in the statement of the lemma. By the inductive hypothesis, we know there exists a $2 \times n$ S-matrix

$$A = \begin{pmatrix} a_{0,0} & \cdots & a_{0,n-1} \\ a_{1,0} & \cdots & a_{1,n-1} \end{pmatrix}$$

with $\mathbf{r}_A = (\sum_{i<m-1} r_i, r_{m-1})$ and $\mathbf{c}_A = (c_0, \ldots, c_1)$. Again using the inductive hypothesis, there exists a $(m-1) \times n$ S-matrix

$$B = \begin{pmatrix} b_{0,0} & \cdots & b_{0,n-1} \\ \vdots & \ddots & \vdots \\ b_{m-2,0} & \cdots & b_{m-2,n-1} \end{pmatrix}$$

with $\mathbf{r}_B = (r_0, \ldots, r_{m-2})$ and $\mathbf{c}_B = (a_{0,0}, \ldots, a_{0,1})$. We then simply observe that the matrix

$$\begin{pmatrix} b_{0,0} & \cdots & b_{0,n-1} \\ \vdots & \ddots & \vdots \\ b_{m-2,0} & \cdots & b_{m-2,n-1} \\ a_{1,0} & \cdots & a_{1,n-1} \end{pmatrix}$$

has the required row and column sumes. □

Remark 5. Lemma 4 remains true for nonabelian semigroups, with the same proof, provided that row and column sums are reinterpreted in the obvious way.

We say that a monoid $\langle G, + \rangle$ with identity 0_G is *nonnegative* if $G^+ = G \setminus \{0_G\}$ is a (positive) semigroup. Equivalently, if $g_0 + g_1 = 0_G$, then $g_0 = g_1 = 0_G$. We fix such a monoid.

We now turn our attention to the main focus of the paper, the class of naturally ordered finite measure algebras equipped with a measure taking values in G. Given a Boolean algebra $\langle B, \wedge, \vee, 0, 1 \rangle$, a *positive G-valued measure* on B is a function $\mu : B \to G$ such that for all $b_0, b_1 \in B$:

1. $\mu(b_0) = 0_G \Leftrightarrow b_0 = 0_G$;
2. if $b_0 \wedge b_1 = 0$, then $\mu(b_0 \vee b_1) = \mu(b_0) + \mu(b_1)$.

Fix a positive element $g_1 \in G$. The class $\mathcal{OMBA}_{G,g_1}$ consists of structures of the form $\mathbf{B} = \langle B, \wedge, \vee, 0, 1, \mu_\mathbf{B}, <_\mathbf{B} \rangle$, where $\langle B, \wedge, \vee, 0, 1 \rangle$ is a finite Boolean algebra, $\mu_\mathbf{B} : B \to G$ is a positive G-valued measure with $\mu_\mathbf{B}(1) = g_1$, and $<_\mathbf{B}$ is an order induced antilexicographically by an ordering of the atoms of B.

Theorem 6. *Suppose that G is a countable, nonnegative abelian monoid such that G^+ splits four ways, and that g_1 is a positive element of G. Then the class $\mathcal{OMBA}_{G,g_1}$ is a Fraïssé order class.*

Proof. We prove only that $\mathcal{OMBA}_{G,g_1}$ satisfies the AP, since the other properties are routinely verified (in particular, JEP follows from AP upon considering the $\{0,1\}$ Boolean algebra). Towards this end, fix $\mathbf{B},\mathbf{C},\mathbf{D} \in \mathcal{OMBA}_{G,g_1}$ as well as embeddings $f : \mathbf{B} \to \mathbf{C}$ and $g : \mathbf{B} \to \mathbf{D}$. Our goal is to find some $\mathbf{E} \in \mathcal{OMBA}_{G,g_1}$ and embeddings $r : \mathbf{C} \to \mathbf{E}$ and $s : \mathbf{D} \to \mathbf{E}$ satisfying $r \circ f = s \circ g$.

Let $b_0 >_{\mathbf{B}} \cdots >_{\mathbf{B}} b_{l-1}$ list the atoms of B. For each $k < l$, let $c_{0,k} >_{\mathbf{C}} \cdots >_{\mathbf{C}} c_{m_k-1,k}$ list the atoms below $f(b_k)$ in C. Similarly, let $d_{0,k} >_{\mathbf{D}} \cdots >_{\mathbf{D}} d_{n_k-1,k}$ list the atoms below $g(b_k)$ in D. In particular,

$$\sum_{i<m_k} \mu_{\mathbf{C}}(c_{i,k}) = \sum_{j<n_k} \mu_{\mathbf{D}}(d_{j,k}) = \mu_{\mathbf{B}}(b_k).$$

For each $k < l$, we define two sequences of positive elements of G by

$$\begin{aligned} \mathbf{r}_k &= (\mu_{\mathbf{C}}(c_{0,k}), \ldots, \mu_{\mathbf{C}}(c_{m_k-1,k})) \text{ and} \\ \mathbf{c}_k &= (\mu_{\mathbf{D}}(d_{0,k}), \ldots, \mu_{\mathbf{D}}(d_{n_k-1,k})). \end{aligned}$$

These sequences satisfy the hypotheses of Lemma 4, so we may find a G^+-matrix $A_k = (a_{i,j,k})$ with $\mathbf{r}_{A_k} = \mathbf{r}_k$ and $\mathbf{c}_{A_k} = \mathbf{c}_k$.

Intuitively, we identify the atoms of B with the collection of these matrices, the atoms of C with the rows of these matrices, and the atoms of D with their columns. Towards that end, let E be the Boolean algebra generated by some set of distinct atoms indexed as $\{e_{ijk} : k < l, i < n_k, \text{ and } j < m_k\}$. Let $\mu_{\mathbf{E}}$ be the unique positive G-valued measure on E such that for all i, j, k, $\mu_{\mathbf{E}}(e_{ijk}) = a_{i,j,k}$; such a measure exists by the nonnegativity of G.

We define embeddings $r : C \to E$ and $s : D \to E$ as the unique maps satisfying

$$r(c_{i,k}) = \bigvee_j e_{ijk} \text{ and } s(d_{j,k}) = \bigvee_i e_{ijk}.$$

Certainly

$$\begin{aligned} \mu_{\mathbf{E}}(r(c_{i,k})) &= \sum_j \mu_{\mathbf{E}}(e_{ijk}) = \sum_j a_{i,j,k} = \mu_{\mathbf{C}}(c_{i,k}) \text{ and} \\ \mu_{\mathbf{E}}(s(d_{j,k})) &= \sum_i \mu_{\mathbf{E}}(e_{ijk}) = \sum_i a_{i,j,k} = \mu_{\mathbf{D}}(d_{j,k}), \end{aligned}$$

by the conditions on the row and column sums of the G^+-matrices A_k. Furthermore, for all $k < l$,

$$r \circ f(b_k) = s \circ g(b_k) = \bigvee_{i,j} e_{i,j,k}$$

so $r \circ f = s \circ g$. To complete the proof of AP, it remains only to define an ordering of the atoms of E so that r and s preserve the orders of the atoms of C and D.

We desire to order the union of the sets of *leading atoms* $X = \{e_{i0k} : k < l \text{ and } i < m_k\}$ and $Y = \{e_{0jk} : k < l \text{ and } j < n_k\}$ in a way that induces an order compatible with the orders $<_{\mathbf{C}}$ and $<_{\mathbf{D}}$. Once we have ordered the leading atoms, we may order the remaining atoms however we like, so long as they are smaller than the leading atoms.

Let X be ordered by $e_{i0k} <_X e_{i'0k'} \Leftrightarrow c_{i,k} <_{\mathbf{C}} c_{i',k'}$. Similarly, let Y be ordered by $e_{0jk} <_Y e_{0j'k'} \Leftrightarrow d_{j,k} <_{\mathbf{D}} d_{j',k'}$. Notice that these two orderings coincide on $X \cap Y = \{e_{00k} : k < l\}$ since

$$e_{00k} <_X e_{00k'} \Leftrightarrow c_{0,k} <_{\mathbf{C}} c_{0,k'} \Leftrightarrow b_k <_{\mathbf{B}} b'_k \Leftrightarrow d_{0,k} <_{\mathbf{D}} d_{0,k'} \Leftrightarrow e_{00k} <_Y e_{00k'}.$$

Thus, by the amalgamation property for finite linear orderings, there is an order on $X \cup Y$ extending both $<_X$ and $<_Y$, so we have completed the proof. □

Remark 7. Continuing the analysis of Example 2, the assumption that G^+ has no minimal element is necessary. Indeed, suppose that g is the minimal element of G^+. Let $\mathbf{B} = \langle B, \wedge, \vee, 0, 1, \mu_{\mathbf{B}}, <_{\mathbf{B}} \rangle$, where B is the 4-element Boolean algebra with atoms $\{b_0, b_1\}$, $\mu_{\mathbf{B}}(b_i) = 2g$ for all $i < 2$, and $b_0 <_{\mathbf{B}} b_1$. Let $\mathbf{C} = \langle C, \wedge, \vee, 0, 1, \mu_{\mathbf{C}}, <_{\mathbf{C}} \rangle$ and $\mathbf{D} = \langle D, \wedge, \vee, 0, 1, \mu_{\mathbf{D}}, <_{\mathbf{D}} \rangle$, where C and D both equal the 16-element Boolean algebra with atoms $\{a_0, a_1, a_2, a_3\}$, $\mu_{\mathbf{C}}(a_i) = \mu_{\mathbf{D}}(a_i) = g$ for all $i < 4$. Finally, the orders are given by

$$a_0 <_{\mathbf{C}} a_1 <_{\mathbf{C}} a_2 <_{\mathbf{C}} a_3,$$
$$a_0 <_{\mathbf{D}} a_2 <_{\mathbf{D}} a_1 <_{\mathbf{D}} a_3.$$

Let $f : \mathbf{B} \to \mathbf{C}$ and $g : \mathbf{B} \to \mathbf{D}$ be the embeddings extending $f(b_0) = g(b_0) = a_0 \vee a_1$, $f(b_1) = g(b_1) = a_2 \vee a_3$. A moment's reflection reveals that the minimality of g and the particular orders on $\mathbf{C}$ and $\mathbf{D}$ prevent the amalgamation of these structures.