Brooks' theorem for Bernoulli shifts

Clinton T. Conley

August 20, 2013

Abstract

If Γ is an infinite group with finite symmetric generating set S, we consider the graph $G(\Gamma, S)$ on $[0,1]^\Gamma$ by relating two distinct points if an element of s sends one to the other via the shift action. We show that, aside from the cases $\Gamma = \mathbb{Z}$ and $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, $G(\Gamma, S)$ satisfies a measure-theoretic version of Brooks' theorem: there is a $G(\Gamma, S)$-invariant conull Borel set $B \subseteq [0,1]^\Gamma$ and a Borel coloring $c\colon B \to d$ of $G(\Gamma, S) \restriction B$, where $d = |S|$ is the degree of $G(\Gamma, S)$. As a corollary we obtain a translation-invariant random d-coloring of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ which is a factor of IID, addressing a question from [9].

1 Introduction

By a *graph* on a vertex set X we mean a symmetric, irreflexive subset of X^2; to ease reading we write $x \mathrel{G} y$ instead of $(x, y) \in G$. The *restriction* $G \restriction A$ of G to some subset $A \subseteq X$ is simply $G \cap A^2$. A set $A \subseteq X$ is (G-)*independent* if $G \restriction A = \emptyset$, i.e., no two vertices in A are G-related. A (proper) (Y-)*coloring* of G is a function $c\colon X \to Y$ so that $c^{-1}(y)$ is G-independent for all $y \in Y$. The (G-)*degree* of a vertex $x \in X$ is the cardinality of the set $\{y \in X : x \mathrel{G} y\}$.

We recall a classic theorem of Brooks from finite combinatorics: if G is a connected graph on finite vertex set X in which every vertex has degree at most d, and moreover G is neither a clique nor an odd cycle, then G admits a d-coloring. A straightforward compactness argument extends the conclusion to infinite graphs, but there is no guarantee that such a coloring will satisfy various measurability properties.

If G is a Borel graph on a standard Borel space X equipped with a Borel probability measure μ we may consider analogs of Brooks' theorem. For instance, if G has vertex degree bounded by d and has no cliques or odd cycles as connected components, then [5, Theorems 2.19, 2.20] allow us to find Borel sets $A \subseteq X$ of arbitrarily small measure such that $G \restriction (X \setminus A)$ admits a Borel d-coloring. In this paper we consider whether we can get away with discarding only a null set.

Given a countable group Γ we consider the space $[0,1]^\Gamma$ equipped with the product Lebesgue measure μ. The *Bernoulli shift* action of Γ on $[0,1]^\Gamma$ is given by $(\gamma \cdot x)(\delta) = x(\gamma^{-1}\delta)$. Given a symmetric generating set S of Γ, we may associate with the Bernoulli shift a graph $G(\Gamma, S)$ on $[0,1]^\Gamma$ by relating vertices x, y iff $x \neq y$ and there exists $s \in S$ such that $s \cdot x = y$. Note that because the action preserves μ, every μ-null set is contained in a $G(\Gamma, S)$-invariant null set. On the μ-conull set on which the action is free every vertex has degree $d = |S|$, assuming of course that S does not contain the group's identity element. We then have

Theorem 1.1. *Suppose that Γ is an infinite group with finite symmetric generating set S isomorphic neither to $\mathbb{Z}$ nor to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Then there is a μ-conull, $G(\Gamma, S)$-invariant Borel set B such that $G(\Gamma, S) \restriction B$ admits a Borel d-coloring.*

The exceptions are necessary. If $\Gamma = \mathbb{Z}$ with $S = \{\pm 1\}$ the graph $G(\mathbb{Z}, S)$ can not be Borel 2-colored on a conull set, as each color would be a set of measure $1/2$ invariant under the action of $2\mathbb{Z}$, contradicting the mixing properties of the action. An analogous argument works for $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ with the usual generators.

Acknowledgments. The author would like to thank Alexander Kechris, Russ Lyons, Andrew Marks, Scott Messick, and Robin Tucker-Drob for many pleasant and illuminating conversations.

2 The proof

It will be useful to isolate a certain type of acyclic subgraph which will provide a skeleton along which to color the vertices of the graph. Given a graph G on X, a *G-ray* is an injective sequence $\rho \in X^{\mathbb{N}}$ such that $\rho(i) \mathrel{G} \rho(i+1)$ for all $i \in \mathbb{N}$. Two such rays ρ and σ are *end-equivalent* if for every finite set

$F \subseteq X$ there are $i, j \in \mathbb{N}$ such that $\rho(i)$ and $\sigma(j)$ lie in the same connected component of $G \restriction (X \setminus F)$; sometimes this is defined by deleting edges instead of vertices, but in the bounded degree case the definitions coincide. Finally, an *end* is an end-equivalence class of G-rays.

It follows that a connected acyclic graph G has one end iff there exists a G-ray ρ such that every G-ray σ is tail-equivalent to ρ in the sense that there exist $i, j \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $\rho(i + k) = \sigma(j + k)$.

Proposition 2.1. *Suppose that G is a Borel graph on a standard Borel space X with degree bounded by d. Suppose moreover that $T \subseteq G$ is an acyclic Borel graph in which each component has one end. Then there is a Borel coloring $c\colon X \to d$ of G.*

Proof. Let $X_0 \subseteq X$ be the (Borel) set of T-monovalent vertices. Define inductively $X_{i+1} \subseteq X$ as the set of $T \restriction (X \setminus \bigcup_{j \leq i} X_i)$-monovalent vertices. The one-endedness of T ensures that $X = \bigsqcup_{i \in \mathbb{N}} X_i$. Moreover, each vertex in X_i is T- (and hence G-) adjacent to at least one vertex in X_{i+1}.

In particular, $G \restriction X_0$ has degree bounded by $d - 1$, so by [8, Proposition 4.6] there is a Borel coloring $c_0\colon X_0 \to d$ of $G \restriction X_0$. The following lemma is a special case of [5, Lemma 2.18], but we include its short proof in the interest of self-containment.

Lemma 2.2. *Any Borel coloring $c_{i-1}\colon \bigcup_{j<i} X_j \to d$ of $G \restriction \bigcup_{j<i} X_j$ extends to a Borel coloring $c_i\colon \bigcup_{j \leq i} X_j \to d$ of $G \restriction \bigcup_{j \leq i} X_j$.*

Proof of the lemma. Again by [8, Proposition 4.6] there is a partition $X_i = X_i^1 \sqcup \cdots \sqcup X_i^d$ of X_i into Borel G-independent sets. First extend c_{i-1} to a coloring $c'\colon \bigcup_{j<i} \cup X_i^1$ by coloring each vertex in X_j^1 the least color not used among its (fewer than d many) colored neighbors. Similarly extend in turn to $X_i^2, \ldots, X_i^d$. ☐

Now iteratively apply the lemma to obtain a coherent sequence of colorings $c_i\colon \bigcup_{j \leq i} X_j \to d$. Then $c = \bigcup_{i \in \mathbb{N}} c_i$ is the desired Borel coloring of G. ☐

We next examine situations in which $G(\Gamma, S)$ contains such a nice acyclic subgraph. For convenience we work instead with the shift action of Γ on $[0, 1]^E$, where E is the edge set of the (right) Cayley graph $\mathrm{Cay}(\Gamma, S)$ (and as usual Γ acts by left translation on the Cayley graph). We denote the corresponding graph on $[0, 1]^E$ by $G'(\Gamma, S)$. Since the shift action on $[0, 1]^E$

is measure-theoretically isomorphic to the Bernoulli shift on $[0,1]^\Gamma$, we lose nothing by working with $G'(\Gamma, S)$ rather than $G(\Gamma, S)$.

We may use each $x \in [0,1]^E$ to label the edges of its connected component in $G'(\Gamma, S)$, assigning $(\gamma \cdot x, s\gamma \cdot x)$ the label $x(\gamma^{-1}, \gamma^{-1}s^{-1})$. The structure of the action ensures that this labeling is independent of the particular choice of x, and in particular this labeling is a Borel function from $G'(\Gamma, S)$ to $[0,1]$. Following [10] we obtain the *wired minimal spanning forest*, $\mathrm{WMSF}(G'(\Gamma, S))$, by deleting those edges from $G'(\Gamma, S)$ which receive a label which is maximal in some simple cycle or bi-infinite path. By construction, $\mathrm{WMSF}(G'(\Gamma, S))$ is acyclic.

Theorem 2.3 (Lyons-Peres-Schramm)**.** *Suppose that Γ is a nonamenable group with finite symmetric generating set S, and consider the graph $G'(\Gamma, S)$ defined above. There is a conull, $G'(\Gamma, S)$-invariant Borel set $B \subseteq [0,1]^E$ on which each connected component of* $\mathrm{WMSF}(G'(\Gamma, S))$ *has one end.*

Proof. See [10, 3.12], recalling that by [2, Theorem 1.1] nonamenable Cayley graphs have no infinite clusters at critical percolation. □

Consequently, after discarding an invariant null set, for nonamenable Γ there is an acyclic Borel subgraph T of $G(\Gamma, S)$ such that each connected component of T has one end.

Question 2.4. *For which groups Γ is there such an acyclic Borel subgraph of $G(\Gamma, S)$ in which each connected component has one end (after discarding a null set)? More generally, which graphs admit such subgraphs?*

Remark 2.5. Russ Lyons (private communication) points out that Question 2.4 has a positive answer for finitely generated groups of more than linear growth by using the wired *uniform* spanning forest (WUSF); see §10 of [3]. The identification of the WUSF with a subgraph of $G(\Gamma, S)$ follows from Wilson's algorithm rooted at infinity (see [7, Proof of Proposition 9]) in the transient case and Pemantle's strong Følner independence [12] in the amenable case.

We will also use the following theorem. Recall that the number of ends of a finitely generated group is the number of ends of its Cayley graph—this quantity is the same for any finite generating set.

Theorem 2.6 ([5, Theorem 5.12]). *Suppose that Γ is a finitely generated infinite group isomorphic neither to $\mathbb{Z}$ nor to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Suppose further that Γ has finitely many ends. Let S be a finite symmetric generating set for Γ, and put $d = |S|$. Then for any free Borel action of Γ on a standard Borel space X the graph $G(\Gamma, S)$ admits a Borel d-coloring.*

Now we have all the necessary ingredients to prove the main theorem.

Proof of Theorem 1.1. If Γ is nonamenable, then by Theorem 2.3 there is a $G(\Gamma, S)$-invariant μ-conull Borel set B on which every connected component of the acyclic graph arising from the wired minimal spanning forest has one end. Then Proposition 2.1 grants a Borel d-coloring of $G(\Gamma, S) \restriction B$.

On the other hand, if Γ is amenable then by Stallings' theorem (see [13]) Γ has finitely many ends. Letting B be a $G(\Gamma, S)$-invariant μ-conull Borel set on which the Bernoulli shift action is free, Theorem 2.6 grants the desired Borel d-coloring of $G(\Gamma, S) \restriction B$. $\square$

Remark 2.7. The ability to discard a μ-null set is crucial in Theorem 1.1. Indeed, from [11, Theorem 3.1] it follows that there is no Borel $2n$-coloring of the graph associated with the Bernoulli action of the free group $\mathbb{F}_n$ on $[0, 1]^{\mathbb{F}_n}$ with free generating set, even after restricting to the free part of the action.

Remark 2.8. By [4, Theorem 1.1], when Γ contains a nonabelian free group and X is any nontrivial standard probability space, the Bernoulli shift of Γ on $[0, 1]^\Gamma$ is a factor of the Bernoulli shift of Γ on X^Γ. Pulling back the coloring, we see that for such groups the analog of Theorem 1.1 holds for the corresponding graph on X^Γ. For general groups, however, we do not know if the Brooks bound is attainable almost everywhere even for the graph associated with the Bernoulli shift on 2^Γ (with say the product $(1/2, 1/2)$ measure).

Remark 2.9. Robin Tucker-Drob (private communication) points out that for $\Gamma \neq \mathbb{Z}, (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ Theorem 1.1 provides an alternate proof of the fact that every free μ-preserving action b of (Γ, S) on (X, μ) is weakly equivalent to one whose associated graph admits a Borel d-coloring μ-a.e. [6, Theorem 6.1]. Indeed, letting s_Γ denote the Bernoulli shift of Γ on $[0, 1]^\Gamma$, we have by [14, Corollary 1.6] that the diagonal product $b \times s_\Gamma$ is weakly equivalent to b. Pulling back the coloring granted by Theorem 1.1 through the projection onto the second factor yields the desired coloring.

3 Random d-colorings

Given a group Γ with generating set S and a natural number k, we may view the space $\mathrm{Col}(\Gamma, S, k)$ of k-colorings of the (right) Cayley graph $\mathrm{Cay}(\Gamma, S)$ as a closed (thus Polish) subset of k^Γ. The action of Γ by left translations on $\mathrm{Cay}(\Gamma, S)$ induces an action on $\mathrm{Col}(\Gamma, S, k)$. A *translation-invariant random k-coloring* of $\mathrm{Cay}(\Gamma, S)$ is a Borel probability measure on $\mathrm{Col}(\Gamma, S, k)$ invariant under this Γ action.

In §5 of [9] it is asked for which k can translation-invariant random k-colorings of Cayley graphs be attained as IID factors (see also [1, Question 10.5]). In particular, it is asked whether such random d-colorings can be found, where as before d is the degree of the graph. In [6, Corollary 6.4] translation-invariant random d-colorings of Cayley graphs are constructed, but this involves passing to actions weakly equivalent to the Bernoulli shift (or alternatively taking weak limits of IID factors). We now answer the question for d-colorings affirmatively aside from the two problematic groups.

Corollary 3.1. *Suppose that Γ is an infinite group isomorphic neither to $\mathbb{Z}$ nor to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Suppose that S is a finite symmetric generating set for Γ with $|S| = d$. Then there is a translation-invariant random d-coloring of $\mathrm{Cay}(\Gamma, S)$ which is an IID factor.*

Proof. Fix by Theorem 1.1 a μ-conull $G(\Gamma, S)$-invariant Borel subset $B \subseteq [0,1]^\Gamma$ and a Borel coloring $c\colon B \to d$ of $G(\Gamma, S) \restriction B$. Define $\pi\colon B \to \mathrm{Col}(\Gamma, S, d)$ by $(\pi(x))(\gamma) = c(\gamma^{-1} \cdot x)$. Then $\pi_*\mu$ is a translation-invariant random d-coloring which is a factor of IID by construction, where as usual $\pi_*\mu(A) = \mu(\pi^{-1}(A))$. $\square$

References

[1] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electr. J. Probab.*, 12:1454–1508, 2007.

[2] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm. Critical percolation on any nonamenable group has no infinite clusters. *Ann. Probab.*, 27(3):1347–1356, 1999.

[3] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm. Uniform spanning forests. *Ann. Probab.*, 29(1):1–65, 2001.

[4] L. Bowen. Weak isomorphisms between Bernoulli shifts. *Israel J. Math.*, 183(1):93–102, 2011.

[5] C.T. Conley and A.S. Kechris. Measurable chromatic and independence numbers for ergodic graphs and group actions. *Groups Geom. Dyn.*, 7(1):127–180, 2013.

[6] C.T. Conley, A.S. Kechris, and R.D. Tucker-Drob. Ultraproducts of measure preserving actions and graph combinatorics. *Erg. Theory and Dynam. Systems*. To appear.

[7] D. Gaboriau and R. Lyons. A measurable-group-theoretic solution to von Neumann's problem. *Invent. Math.*, 177(3):533–540, 2009.

[8] A.S. Kechris, S. Solecki, and S. Todorcevic. Borel chromatic numbers. *Adv. Math.*, 141(1):1–44, 1999.

[9] R. Lyons and F. Nazarov. Perfect matchings as IID factors on non-amenable groups. *European J. Combin.*, 32(7):1115–1125, 2011.

[10] R. Lyons, Y. Peres, and O. Schramm. Minimal spanning forests. *Ann. Probab.*, 34(5):1665–1692, 2006.

[11] A. Marks. A determinacy approach to Borel combinatorics. Preprint available at http://www.its.caltech.edu/∼marks/.

[12] R. Pemantle. Towards a theory of negative dependence. *J. Math. Phys.*, 41(3):1371–1390, 2000. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.

[13] J.R. Stallings. *Group Theory and Three-Dimensional Manifolds*, volume 4 of *Yale Mathematical Monographs*. Yale Univ. Press, 1971.

[14] R.D. Tucker-Drob. Weak equivalence and non-classifiability of measure preserving actions. Preprint available at http://www.its.caltech.edu/∼rtuckerd/.