INCOMPARABLE ACTIONS OF FREE GROUPS

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ABSTRACT. Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is an E-invariant Borel probability measure on X. We consider the circumstances under which for every countable non-abelian free group Γ, there is a Borel sequence $(\cdot_r)_{r \in \mathbb{R}}$ of free actions of Γ on X, generating subequivalence relations E_r of E with respect to which μ is ergodic, with the further property that $(E_r)_{r \in \mathbb{R}}$ is an increasing sequence of relations which are pairwise incomparable under μ-reducibility. In particular, we show that if E satisfies a separability condition, then this is the case as long as there exists a free Borel action of a countable non-abelian free group on X, generating a subequivalence relation of E with respect to which μ is ergodic.

INTRODUCTION

A *Polish space* is a separable topological space admitting a compatible complete metric. A subset of such a space is *Borel* if it is in the σ-algebra generated by the underlying topology.

A *standard Borel space* is a set X equipped with the family of Borel sets associated with a Polish topology on X. Every subset of a standard Borel space inherits the σ-algebra consisting of its intersection with each Borel subset of the original space; this restriction is again standard Borel exactly when the subset in question is Borel (see, for example, [Kec95, Corollary 13.4 and Theorem 15.1]). The *product* of standard Borel spaces X and Y is the set $X \times Y$, equipped with the σ-algebra generated by the family of all sets of the form $A \times B$, where $A \subseteq X$ and $B \subseteq Y$ are Borel.

A function between standard Borel spaces is *Borel* if pre-images of Borel sets are Borel. We say that a sequence $(x_i)_{i \in I}$ of points of X is *Borel* if $\{(i, x_i) \mid i \in I\}$ is Borel, and more generally, a sequence $(B_i)_{i \in I}$ of subsets of X is *Borel* if $\{(i, x) \in I \times X \mid x \in B_i\}$ is Borel.

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Suppose that E and F are equivalence relations on X and Y. A *homomorphism* from E to F is a function $\phi\colon X \to Y$ sending E-equivalent points to F-equivalent points, a *reduction* of E to F is a homomorphism sending E-inequivalent points to F-inequivalent points, an *embedding* of E into F is an injective reduction, and an *isomorphism* of E with F is a bijective reduction.

A *Borel measure* on X is a function $\mu\colon \mathscr{B} \to [0,\infty]$, where $\mathscr{B}$ is the family of Borel subsets of X, with the property that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n)$, whenever $(B_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint Borel subsets of X. We say that μ is *σ-finite* if there are Borel sets $B_n \subseteq X$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$, μ is *finite* if $\mu(X) < \infty$, and μ is a *Borel probability measure* if $\mu(X) = 1$. A Borel set $B \subseteq X$ is *μ-null* if $\mu(B) = 0$, *μ-positive* if $\mu(B) > 0$, and *μ-conull* if its complement is μ-null.

The *orbit equivalence relation* generated by an action of a group Γ on a set X is the relation on X given by $x \mathrel{E^X_\Gamma} y \iff \exists \gamma \in \Gamma\ \gamma\cdot x = y$. Actions of groups Γ and Δ on standard Borel spaces X and Y equipped with Borel measures μ and ν are *orbit equivalent* if there is a measure-preserving Borel isomorphism of their orbit equivalence relations.

The primary result of [GP05] ensures that for every countable non-abelian free group Γ, there are uncountably many Borel-probability-measure-preserving free Borel actions of Γ on a standard Borel space which are pairwise non-orbit-equivalent. A number of simplifications of the proof and strengthenings of the result have subsequently appeared (see, for example, [Tör06, Hjo12]).

Following the usual abuse of language, we say that an equivalence relation is *countable* if its classes are all countable, and *finite* if its classes are all finite. Suppose now that E is a countable Borel equivalence relation on X. We say that a Borel probability measure μ on X is *E-ergodic* if every E-invariant Borel set is μ-null or μ-conull. We say that μ is *E-invariant* if $\mu(B) = \mu(T(B))$, for every Borel set $B \subseteq X$ and every Borel automorphism $T\colon X \to X$ whose graph is contained in E. And we say that μ is *E-quasi-invariant* if the family of μ-null Borel subsets of X is closed under E-saturation. As a result of Feldman-Moore (see [FM77, Theorem 1]) ensures that E is generated by a Borel action of a countable discrete group, it follows that every E-invariant Borel probability measure is also E-quasi-invariant.

We say that E is *μ-reducible* to F if there is a μ-conull Borel set on which there is a Borel reduction of E to F. Similarly, we say that E is *μ-somewhere reducible* to F if there is a μ-positive Borel set on which there is a Borel reduction of E to F. The Lusin-Novikov uniformization theorem for Borel subsets of the plane with countable vertical sections (see, for example, [Kec95, Theorem 18.10]) ensures that when μ is E-ergodic, these two notions are equivalent.

In [CM14], a variety of results were established concerning the nature of countable Borel equivalence relations at the base of the measure reducibility hierarchy, using arguments substantially simpler than those previously appearing. While the analogs of many of these results for equivalence relations generated by free Borel actions of countable non-abelian free groups trivially follow, the analog corresponding to the main results of [GP05, Hjo12] also requires a generalization of a stratification result utilized in [Hjo12, CM14]. Here we establish the latter, and incorporate it into ideas from [CM14] to obtain the former.

We say that a countable Borel equivalence relation is *hyperfinite* if it is the union of an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations, and *μ-nowhere hyperfinite* if there is no μ-positive Borel set on which it is hyperfinite. We say that a countable Borel equivalence relation F on a standard Borel space Y is *projectively separable* if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel probability measure on X for which E is μ-nowhere hyperfinite, the pseudo-metric $d_\mu(\phi, \psi) = \mu(\{x \in X \mid \phi(x) \neq \psi(x)\})$ on the space of all countable-to-one Borel homomorphisms $\phi \colon B \to Y$ from $E \restriction B$ to F, where B ranges over all Borel subsets of X, is separable.

In [CM14], it is shown that the family of such equivalence relations includes the orbit equivalence relation induced by the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$, and is closed downward under both Borel reducibility and Borel subequivalence relations. Moreover, it is shown that such relations possess many of the exotic properties of countable Borel equivalence relations which were previously known to hold only of relations relatively high in the Borel reducibility hierarchy. Our primary result here strengthens one such theorem.

Theorem. *Suppose that X is a standard Borel space, E is a projectively separable countable Borel equivalence relation on X, μ is an E-invariant Borel probability measure on X, and there is a free Borel action of a countable non-abelian free group on X generating a subequivalence relation of E with respect to which μ is ergodic. Then for every countable non-abelian free group Γ, there is a Borel sequence $(\cdot_r)_{r \in \mathbb{R}}$ of free actions of Γ on X, generating subequivalence relations E_r of E with respect to which μ is ergodic, with the further property that $(E_r)_{r \in \mathbb{R}}$ is an increasing sequence of relations which are pairwise incomparable under μ-reducibility.*

Although the existence of incomparable orbit equivalence relations is primarily of interest in the presence of Borel probability measures which are both ergodic and invariant, we also establish analogous results in which these assumptions are omitted.

In §1, we characterize the existence of subequivalence relations induced by free Borel actions of countable non-abelian free groups in which one of the generators acts ergodically. In §2, we use this to obtain new stratification results for free Borel actions of countable non-abelian free groups. And in §3, we prove our primary results.

1. Free actions with an ergodic generator

We begin this section by noting that if an invariant Borel probability measure is ergodic with respect to a subequivalence relation generated by a free Borel action of a countable non-abelian free group, then by passing to an appropriate subequivalence relation, we can assume that it is ergodic with respect to the subequivalence relation generated by one of the generators.

We say that an equivalence relation is *aperiodic* if all of its classes are infinite. If E is an aperiodic countable Borel equivalence relation and μ is an E-invariant Borel probability measure, then $\mu(A) = 0$ for every Borel set $A \subseteq X$ intersecting each E-class in finitely many points.

Proposition 1.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, μ is an E-invariant Borel probability measure on X, and there is a free Borel action of a countable non-abelian free group on X generating a subequivalence relation of E with respect to which μ is ergodic. Then for every non-abelian group Γ freely generated by a countable set S and for every $\gamma \in S$, there is a free Borel action of Γ on X generating a subequivalence relation of E such that μ is ergodic with respect to the equivalence relation generated by γ.*

Proof. Note that if (γ_0, γ_1) freely generates $\mathbb{F}_2$, then $(\gamma_1^n \gamma_0 \gamma_1^{-n})_{n \in \mathbb{N}}$ freely generates a copy of $\mathbb{F}_{\aleph_0}$ within $\mathbb{F}_2$. Thus we need only construct the desired action for $\mathbb{F}_2$, and can therefore freely discard E-invariant μ-null Borel sets in the course of the construction.

The *horizontal sections* of a set $R \subseteq X \times Y$ are the sets of the form $R^y = \{x \in X \mid x \mathrel{R} y\}$ for $y \in Y$, whereas the *vertical sections* of R are the sets of the form $R_x = \{y \in Y \mid x \mathrel{R} y\}$ for $x \in X$. A *graph* on X is an irreflexive symmetric subset G of $X \times X$. The equivalence relation *generated* by such a graph is the smallest equivalence relation on X containing G, and a *graphing* of E is a graph generating E.

The *μ-cost* of a locally countable Borel graph G on X is given by $C_\mu(G) = \frac{1}{2} \int |G_x| \, d\mu(x)$, and the *μ-cost* of E is given by

$$C_\mu(E) = \inf\{C_\mu(G) \mid G \text{ is a Borel graphing of } E\}.$$

As we can assume that E is itself generated by a free Borel action of a countable non-abelian free group, Gaboriau's formula for the cost of such a relation ensures that $C_\mu(E) \geq 2$ (see, for example, [KM04, Theorem 27.6]).

Fix an aperiodic hyperfinite Borel subequivalence relation E_0 of E with respect to which μ is ergodic (see, for example, [Zim84, Lemma 9.3.2]). Then there is a Borel automorphism $T_0 \colon X \to X$ generating E_0 (see, for example, [DJK94, Theorem 5.1]).

Equality on X is Borel, so by subtracting this relation from E, we obtain a Borel graphing G of E. A *path* through a graph H is a sequence of the form $(x_i)_{i \leq n}$, where $n \in \mathbb{N}$ and $x_i \; H \; x_{i+1}$, for all $i < n$. Such a path is a *cycle* if $n \geq 3$, $(x_i)_{i<n}$ is injective, and $x_0 = x_n$. A graph is *acyclic* if it has no cycles. The cycle-cutting lemma of Kechris-Miller and Pichot (see, for example, [KM04, Lemma 28.11]) yields an acyclic Borel subgraph H of G with the property that $\operatorname{graph}(T_0) \subseteq H$ and $C_\mu(H) \geq C_\mu(E)$, thus $C_\mu(H \setminus \operatorname{graph}(T_0^{\pm 1})) \geq 1$.

An *oriented graph* on X is an antisymmetric irreflexive subset K of $X \times X$. The graph *induced* by such an oriented graph is the smallest graph containing K. The fact that X admits a Borel linear order easily yields a Borel oriented graph K on X generating $H \setminus \operatorname{graph}(T_0^{\pm 1})$.

The *in-degree* of K at a point y is given by $|K^y|$, whereas the *out-degree* of K at a point x is given by $|K_x|$. As the E-invariance of μ ensures that the average in-degree of K with respect to μ equals the average out-degree of K with respect to μ, they must be at least one.

Lemma 1.2. *After throwing out an E-invariant μ-null Borel set, there is a Borel function $f \colon X \to X$, whose graph is contained in E_0, such that $\forall x \in X \; |f^{-1}(x)| \leq |K_x|$.*

Proof. If the set $B = \{x \in X \mid |K_x| = \aleph_0\}$ is μ-positive, then since the uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that $[B]_{E_0}$ is Borel, the E_0-ergodicity of μ implies that $[B]_{E_0}$ is μ-conull, so its complement is μ-null. As another appeal to the uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that the E-saturation of the latter set is Borel, the E-quasi-invariance of μ implies that it is also μ-null. By throwing out this set, we can assume that B is E_0-*complete*, in the sense that it intersects every E_0-class. But then one more appeal to the uniformization theorem for Borel subsets of the plane with countable vertical sections yields a Borel function $f \colon X \to B$ whose graph is contained in E_0, and clearly any such function is as desired.

If B is not μ-positive, then the uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that both B and $[B]_E$ are Borel, so the former is μ-null, thus the E-quasi-invariance of μ ensures that so too is the latter. By throwing out this set, we can assume that the out-degree of K at every point is finite.

Define $D = \{x \in X \mid |K_x| = 0\}$ and $R = \{x \in X \mid |K_x| > 0\}$. The uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that these sets are Borel. By [KM04, Lemma 7.3], there is a maximal Borel function g from a Borel subset of D to R, whose graph is contained in E_0, such that $\forall x \in R' \ |g^{-1}(x)| = |K_x| - 1$, where $D' = \mathrm{dom}(g)$ and $R' = g(D')$.

As the average out-degree of K on $D' \cup R'$ is one, the average out-degree of K off of $D' \cup R'$ is at least one. In particular, it follows from the E_0-quasi-invariance of μ that $[D \setminus D']_{E_0} \setminus [R \setminus R']_{E_0}$ is not μ-conull. As μ is E_0-ergodic and E-quasi-invariant, it follows from the uniformization theorem for Borel subsets of the plane with countable vertical sections that both this set and its E-saturation are Borel and μ-null. By throwing out the latter, we can assume that every E_0-class intersecting $D \setminus D'$ also intersects $R \setminus R'$.

As the maximality of g ensures that $|(D \setminus D') \cap [x]_{E_0}| < |K_x| - 1$ for all $x \in R \setminus R'$, it follows that $D \setminus D'$ intersects every E_0-class in a finite set, and is therefore μ-null. As the uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that the E-saturation of this set is Borel, the E-quasi-invariance of μ ensures that it is also μ-null, so by throwing it out, we can assume that $D = D'$. Then the function f agreeing with g on D and with the identity function on R is as desired. ⊠

By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel functions $g, h \colon X \to X^{\leq \mathbb{N}}$ with the property that $(g_n(x))_{n < |f^{-1}(x)|}$ is an enumeration of $f^{-1}(x)$, and $(h_n(x))_{n < |K_x|}$ is an enumeration of K_x, for all $x \in X$. Let K' denote the set of all pairs of the form $(g_n(x), h_n(x))$, where $x \in X$ and $n < |f^{-1}(x)|$.

The analog of Lemma 1.2 and the comments thereafter with $(K')^{-1}$ in place of K yield a new Borel oriented graph K'' with the property that the in-degree and out-degree of K'' at every point is one. Let T_1 be the Borel automorphism whose graph is K''. The acyclicity of H ensures that the action of $\mathbb{F}_2$ generated by T_0 and T_1 is free, and is therefore as desired. ⊠

Remark 1.3. After throwing out an E-invariant μ-null Borel set, the conclusion of Proposition 1.1 can be established from the weaker hypothesis that there is a Borel subequivalence relation F of E, for which $C_\mu(F) > 1$, with respect to which μ is ergodic. To see this, note that we can assume that $C_\mu(E) > 1$. As μ is necessarily continuous, there is a Borel set $B \subseteq X$ for which there is a natural number $k \geq 1/(C_\mu(E) - 1)$ such that $\mu(B) = 1/k$, thus $\mu(B) \leq C_\mu(E) - 1$. Gaboriau's formula for the cost of the restriction of a countable Borel equivalence relation (see, for example, [KM04, Theorem 21.1]) then ensures that $C_{\mu \restriction B}(E \restriction B) = (C_\mu(E) - 1) + \mu(B) \geq 2\mu(B)$. The proof of Proposition 1.1 therefore yields the desired action of $\mathbb{F}_2$, albeit on B, not on X. To rectify this problem, note that by the proof of [KM04, Lemma 7.10], after throwing out an E-invariant μ-null Borel set, we can assume that there is a partition of X into Borel sets $B_1, \ldots, B_k$, as well as Borel isomorphisms $\pi_i \colon B \to B_i$ whose graphs are contained in E, for all $1 \leq i \leq k$. Let T_0' denote the Borel automorphism of X which agrees with $\pi_{i+1} \circ \pi_i^{-1}$ on B_i, for all $1 \leq i < k$, and which agrees with $\pi_1 \circ T_0 \circ \pi_k^{-1}$ on B_k. Let T_1' denote the Borel automorphism of X which agrees with $\pi_i \circ T_1^i \circ T_0 \circ T_1^{-i} \circ \pi_i^{-1}$ on B_i, for all $1 \leq i \leq k$. Then T_0' and T_1' yield the desired action of $\mathbb{F}_2$.

We equip the space of Borel probability measures μ on a standard Borel space X with the (standard) Borel structure generated by the functions $\mathrm{eval}_B(\mu) = \mu(B)$, where B varies over all Borel subsets of X.

A *uniform ergodic decomposition* of a countable Borel equivalence relation E on a standard Borel space X is a sequence $(\mu_x)_{x \in X}$ of E-ergodic E-invariant Borel probability measures on X such that (1) $\mu_x = \mu_y$ whenever $x \mathrel{E} y$, (2) $\mu(\{x \in X \mid \mu = \mu_x\}) = 1$ for every E-ergodic E-invariant Borel probability measure μ on X, and (3) $\mu = \int \mu_x \, d\mu(x)$ for every E-invariant Borel probability measure μ on X. The Farrell-Varadarajan uniform ergodic decomposition theorem (see, for example, [KM04, Theorem 3.3]) ensures the existence of Borel such decompositions. We next establish an ergodicity-free analog of Proposition 1.1, by uniformly pasting together actions obtained from the latter along such a decomposition.

Proposition 1.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, μ is an E-invariant Borel probability measure on X, $(\mu_x)_{x \in X}$ is a Borel uniform ergodic decomposition of E, and for μ-almost all $x \in X$ there is a free Borel action of a countable non-abelian free group generating a subequivalence relation of E with respect to which μ_x is ergodic. Then for every non-abelian group Γ freely generated by a countable set S and for every $\gamma \in S$, there is a* *free Borel action of* Γ *on* X *generating a subequivalence relation of* E *such that for* μ*-almost all* $x \in X$*, the measure* μ_x *is ergodic with respect to the equivalence relation generated by* γ*.*

Proof. As before, it is sufficient to construct the desired action for $\mathbb{F}_2$, freely discarding E-invariant μ-null Borel sets as we proceed. Fix (γ_0, γ_1) freely generating $\mathbb{F}_2$.

Fix a countable basis $\{U_n \mid n \in \mathbb{N}\}$ for X which is closed under finite unions, and define $G \subseteq \mathbb{N}^{\mathbb{N}\times\mathbb{N}} \times X$ by $G_\phi = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} U_{\phi(m,n)}$. By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel functions $f_n \colon X \to X$ such that $E = \bigcup_{n\in\mathbb{N}} \operatorname{graph}(f_n)$.

Define $R \subseteq (\mathbb{N}^{\mathbb{N}\times\mathbb{N}})^{\mathbb{N}} \times (X \times X)$ by $R_\phi = \bigcup_{n\in\mathbb{N}} \operatorname{graph}(f_n \restriction G_{\phi(n)})$ for all $\phi \in (\mathbb{N}^{\mathbb{N}\times\mathbb{N}})^{\mathbb{N}}$, and let B denote the Borel set consisting of all $((\phi_0, \phi_1), x) \in ((\mathbb{N}^{\mathbb{N}\times\mathbb{N}})^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}\times\mathbb{N}})^{\mathbb{N}}) \times X$ with the property that the sets $R_{\phi_0} \cap ([x]_E \times [x]_E)$ and $R_{\phi_1} \cap ([x]_E \times [x]_E)$ are graphs of functions inducing a free action of $\mathbb{F}_2$ on $[x]_E$. For each pair $\phi = (\phi_0, \phi_1)$ in $(\mathbb{N}^{\mathbb{N}\times\mathbb{N}})^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}\times\mathbb{N}})^{\mathbb{N}}$, let $\cdot_\phi$ denote the action of $\mathbb{F}_2$ on B_ϕ for which the graph of the function $x \mapsto \gamma_i \cdot_\phi x$ is $R_{\phi_i} \cap (B_\phi \times B_\phi)$, for all $i < 2$. Let F denote the equivalence relation on B generated by the function $(\phi, x) \mapsto (\phi, \gamma_0 \cdot_\phi x)$, and fix a Borel uniform ergodic decomposition $(\nu_{\phi,x})_{(\phi,x)\in B}$ of F.

A subset of a standard Borel space is *analytic* if it is the image of a Borel subset of a standard Borel space under a Borel function. We use $\sigma(\mathbf{\Sigma}^1_1)$ to denote the smallest σ-algebra containing all such sets, and we say that a function is $\sigma(\mathbf{\Sigma}^1_1)$*-measurable* if pre-images of open sets are $\mathbf{\Sigma}^1_1$. Define $\operatorname{proj}_X \colon X \times Y \to X$ by $\operatorname{proj}_X(x, y) = x$. A *uniformization* of a set $R \subseteq X \times Y$ is a function $\phi \colon \operatorname{proj}_X(R) \to Y$ whose graph is contained in R.

Let Λ denote the push-forward of μ through the function $x \mapsto \mu_x$. The regularity of Borel probability measures on Polish spaces ensures that if ν is an E-invariant Borel probability measure on X concentrating on B_ϕ, then every free Borel action of $\mathbb{F}_2$ on an E-invariant ν-conull Borel subset of X agrees with some $\cdot_\phi$ on an E-invariant ν-conull Borel subset of B_ϕ. And $\nu \restriction B_\phi$ is ergodic with respect to the equivalence relation generated by the action of γ_0 on B_ϕ if and only if the measures induced by ν and $\nu_{\phi,x} \restriction (\{\phi\} \times B_\phi)$ on B_ϕ agree, for ν-almost all $x \in X$. As the set of pairs (ν, ϕ) satisfying this latter property is Borel (see, for example, [Kec95, Proposition 12.4 and Theorem 17.25]), the Jankov-von Neumann uniformization theorem for analytic subsets of the plane (see, for example, [Kec95, Theorem 18.1]) yields a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformization ϕ. As $\mathrm{dom}(\phi)$ is analytic, and a result of Lusin's ensures that every analytic set is Λ-measurable (see, for example, [Kec95, Theorem 21.10]), Proposition 1.1 implies that $\mathrm{dom}(\phi)$ is Λ-conull. As ϕ is Λ-measurable, there is a Λ-conull Borel set $M \subseteq \mathrm{dom}(\phi)$ on which it is Borel.

Define $C = \bigcup_{\nu \in M} \{x \in B_{\phi(\nu)} \mid \mu_x = \nu\}$, and observe that the action of $\mathbb{F}_2$ on C given by $\gamma_i \cdot x = \gamma_i \cdot_{\phi(\mu_x)} x$, for $i < 2$, is as desired. ⊠

Remark 1.5. By Remark 1.3, after throwing out an E-invariant μ-null Borel set, the conclusion of Proposition 1.4 follows from the weaker assumption that for μ-almost all $x \in X$ there is a Borel subequivalence relation F of E, for which $C_{\mu_x}(F) > 1$, with respect to which μ_x is ergodic.

2. Stratification

We begin this section by noting that strong proper inclusion of equivalence relations often gives rise to a measure-theoretic analog.

Proposition 2.1. *Suppose that X is a Polish space, E and F are countable Borel equivalence relations on X such that every E-class is properly contained in an F-class, and μ is an E-ergodic F-quasi-invariant Borel probability measure on X. Then for every Borel set $B \subseteq X$, the $(E \restriction B)$-class of $(\mu \restriction B)$-almost every point of B is properly contained in an $(F \restriction B)$-class.*

Proof. The uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that $[B]_F$ is Borel, thus so too is the set $A = \{x \in [B]_F \mid B \cap [x]_E = B \cap [x]_F\}$. As the fact that every E-class is properly contained in an F-class ensures that A is contained in the F-saturation of its complement, the F-quasi-invariance of μ implies that A is not μ-conull. As A is E-invariant, the E-ergodicity of μ therefore ensures that A is μ-null. It only remains to note that if $x \in B \setminus A$, then x is necessarily $(F \setminus E)$-related to some other point in B, thus the $(E \restriction B)$-class of x is properly contained in an $(F \restriction B)$-class. ⊠

We say that a sequence $(E_r)_{r \in \mathbb{R}}$ of subequivalence relations of an equivalence relation E on X is a *stratification* of E if for all real numbers $r < s$, every E_r-class is properly contained in an E_s-class. We say that a sequence $(\cdot_r)_{r \in \mathbb{R}}$ of actions of a group Γ on X is a *stratification* of E if the corresponding sequence $(E_r)_{r \in \mathbb{R}}$ of equivalence relations generated by the actions is a stratification.

We say that a stratification $(E_r)_{r \in \mathbb{R}}$ of E is a *μ-stratification* if for all μ-positive Borel sets $B \subseteq X$ and all real numbers $r < s$, the

$(E_r \restriction B)$-class of $(\mu \restriction B)$-almost every point of B is properly contained in an $(E_s \restriction B)$-class. We say that a stratification $(\cdot_r)_{r \in \mathbb{R}}$ of E is a *μ-stratification* if the corresponding sequence $(E_r)_{r \in \mathbb{R}}$ of equivalence relations generated by the actions is a μ-stratification.

We say that a (μ-)stratification $(E_r)_{r \in \mathbb{R}}$ by equivalence relations is *Borel* if it is Borel as a sequence of subsets of $X \times X$. Analogously, a (μ-)stratification $(\cdot_r)_{r \in \mathbb{R}}$ by actions of Γ is *Borel* if the corresponding sequence $(\mathrm{graph}(\cdot_r))_{r \in \mathbb{R}}$ is Borel as a sequence of subsets of $(\Gamma \times X) \times X$.

Proposition 2.2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, μ is an E-invariant Borel probability measure on X, and there is a free Borel action of a countable non-abelian free group on X generating a subequivalence relation of E with respect to which μ is ergodic. Then for every countable non-abelian free group Γ, there is a Borel μ-stratification of E by free actions of Γ on X, generating equivalence relations with respect to which μ is ergodic.*

Proof. Fix a countable set S freely generating Γ, as well as some $\gamma \in S$. By Proposition 1.1, there is a free Borel action $\cdot$ of Γ on X, generating a subequivalence relation of E, such that μ is ergodic with respect to the equivalence relation generated by γ. Fix $\delta \in S \setminus \{\gamma\}$, appeal to [Mil12, Proposition 5.2] to obtain a Borel stratification $(*_r)_{r \in \mathbb{R}}$ by actions of $\mathbb{Z}$ of the equivalence relation generated by δ, and for each $r \in \mathbb{R}$, let $\cdot_r$ denote the action of Γ on X given by $\delta \cdot_r x = \delta *_r x$ and $\lambda \cdot_r x = \lambda \cdot x$, for $\lambda \in S \setminus \{\delta\}$. Proposition 2.1 then ensures that $(\cdot_r)_{r \in \mathbb{R}}$ is the desired μ-stratification of E. ⊠

Remark 2.3. By Remark 1.3, after throwing out an E-invariant μ-null Borel set, the conclusion of Proposition 2.2 follows from the weaker assumption that there is a Borel subequivalence relation F of E, for which $C_\mu(F) > 1$, with respect to which μ is ergodic.

We next establish an analogous result in the absence of ergodicity.

Proposition 2.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, μ is an E-invariant Borel probability measure on X, and there is a free Borel action of a countable non-abelian free group on X generating a subequivalence relation of E. Then for every countable non-abelian free group Γ, there is a Borel μ-stratification of E by free actions of Γ on X.*

Proof. Fix a countable set S freely generating a non-abelian group Γ, as well as some $\gamma \in S$. Clearly we can assume that E is itself generated by a free Borel action of a countable non-abelian free group on X. By the uniform ergodic decomposition theorem, there is a Borel uniform ergodic decomposition $(\mu_x)_{x \in X}$ of E. By Proposition 1.4, there is a free Borel action $\cdot$ of Γ on X, generating a subequivalence relation of E, such that for μ-almost all $x \in X$, the measure μ_x is ergodic with respect to the equivalence relation generated by γ. Fix $\delta \in S \setminus \{\gamma\}$, appeal to [Mil12, Proposition 5.2] to obtain a Borel stratification $(*_r)_{r \in \mathbb{R}}$ by actions of $\mathbb{Z}$ of the equivalence relation generated by δ, and for each $r \in \mathbb{R}$, let $\cdot_r$ denote the action of Γ on X given by $\delta \cdot_r x = \delta *_r x$ and $\lambda \cdot_r x = \lambda \cdot x$, for $\lambda \in S \setminus \{\delta\}$. Proposition 2.1 ensures that $(\cdot_r)_{r \in \mathbb{R}}$ is a μ_x-stratification of E for μ-almost all $x \in X$, and is therefore a μ-stratification of E. ⊠

Remark 2.5. After throwing out an E-invariant μ-null Borel set, the conclusion of Proposition 2.4 follows from the weaker assumption that for all μ-positive Borel sets $B \subseteq X$, there is a Borel subequivalence relation F of $E \restriction B$ for which $C_{\mu \restriction B}(F) > \mu(B)$. In light of Remark 1.4 and the proof of Proposition 2.4, to see this, it is sufficient to show that there is a Borel subequivalence relation F of E with the property that if $(\mu_x)_{x \in X}$ is an ergodic decomposition of F, then $C_{\mu_x}(F) > 1$ for μ-almost all $x \in X$. In fact, it is enough to produce such an F on an E-invariant μ-positive Borel subset of X. Towards this end, fix a Borel subequivalence relation F' of E such that $C_\mu(F') > 1$, and appeal to the uniform ergodic decomposition theorem to produce a Borel uniform ergodic decomposition $(\mu_x)_{x \in X}$ of F'. As [KM04, Proposition 18.1] ensures that the set $C = \{x \in X \mid C_{\mu_x}(F') > 1\}$ is co-analytic, and every co-analytic subset of X is μ-measurable, the cost integration formula (see [KM04, Corollary 18.6]) ensures that it is μ-positive. Fix a μ-positive Borel set $B \subseteq X$ contained in C. As μ is F'-invariant, we can assume that B is F'-invariant. By the uniformization theorem for Borel subsets of the plane with countable vertical sections, the set $[B]_E$ is Borel and there is a Borel function $\phi \colon [B]_E \to B$ whose graph is contained in E. Then the formula for the cost of a restriction of a countable Borel equivalence relation ensures that the equivalence relation F on $[B]_E$ given by $x \mathrel{F} y \iff \phi(x) \mathrel{F'} \phi(y)$ is as desired.

A *treeing* of an equivalence relation is an acyclic graphing. We say that a Borel equivalence relation E on X is *treeable* if it has a Borel treeing. We say that a countable Borel equivalence relation E on X is *compressible* if there is a Borel injection $\phi \colon X \to X$, whose graph is contained in E, such that $X \setminus \phi(X)$ is E-complete.

Proposition 2.6. *Suppose that X is a Polish space, E is a compressible treeable countable Borel equivalence relation on X, and Γ is a countable non-abelian free group. Then there is a free Borel action of Γ on X generating E.*

Proof. A straightforward modification of the proof of [JKL02, Corollary 3.11] reveals that there is a Borel treeing G of E which is generated by a Borel automorphism $T_0 \colon X \to X$ and a Borel isomorphism $T_1 \colon A \to B$, where $A, B \subseteq X$ are disjoint Borel sets. Set $C = X \setminus (A \cup B)$, and for each $x \in X$, let $D(x)$ denote the unique set in $\{A, B, C\}$ containing x.

Suppose now that $k \in \{2, 3, \ldots, \aleph_0\}$ and Γ is freely generated by $(\gamma_i)_{i<k}$. The idea is to again use compressibility to replace E with $E \times I(\mathbb{N})$, and to use the right-hand coordinate to accomodate the generators other than the first two, as well as the points at which T_1 and T_1^{-1} are not defined.

Towards this end, fix bijections $\gamma_0^D \colon \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ for all $D \in \{A, B, C\}$, $\gamma_1^A \colon \mathbb{N} \setminus \{0\} \to \mathbb{N}$, $\gamma_1^B \colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$, $\gamma_1^C \colon \mathbb{N} \to \mathbb{N}$, as well as $\gamma_i^D \colon \mathbb{N} \to \mathbb{N}$ for all $D \in \{A, B, C\}$ and $1 < i < k$, with the property that for all $D \in \{A, B, C\}$, the corresponding approximation to an action of $\mathbb{F}_k$ on $\mathbb{N}$, given by

$$(\gamma_{s(0)}^{t(0)} \cdots \gamma_{s(n)}^{t(n)})^D \cdot x = ((\gamma_{s(0)}^D)^{t(0)} \circ \cdots \circ (\gamma_{s(n)}^D)^{t(n)})(x),$$

is both *free* and *transitive*, in the sense that:

(1) $\forall n \in \mathbb{N} \forall \gamma \in \mathbb{F}_k \ (\gamma^D \cdot n = n \iff \gamma = \mathrm{id})$.
(2) $\forall m, n \in \mathbb{N} \exists \gamma \in \mathbb{F}_k \ (\gamma^D \cdot m = n)$.

Let γ_0 act on $X \times \mathbb{N}$ via

$$\gamma_0 \cdot (x, n) = \begin{cases} (T_0(x), n) & \text{if } n = 0, \text{ and} \\ (x, \gamma_0^{D(x)} \cdot n) & \text{otherwise.} \end{cases}$$

Similarly, let γ_1 act on $X \times \mathbb{N}$ via

$$\gamma_1 \cdot (x, n) = \begin{cases} (T_1(x), n) & \text{if } n = 0 \text{ and } x \in A, \text{ and} \\ (x, \gamma_1^{D(x)} \cdot n) & \text{otherwise.} \end{cases}$$

And finally, let γ_i act on $X \times \mathbb{N}$ via $\gamma_i \cdot (x, n) = (x, \gamma_i^{D(x)} \cdot n)$, for all $1 < i < k$. This defines a free Borel action of Γ generating $E \times I(\mathbb{N})$, so the proposition follows from the fact that the compressibility of E is equivalent to the existence of a Borel isomorphism between E and $E \times I(\mathbb{N})$ (see, for example, [DJK94, Proposition 2.5]). ⊠

In particular, this yields the following stratification result.

Proposition 2.7. *Suppose that X is a Polish space, E is a compressible treeable countable Borel equivalence relation on X, and μ is a Borel probability measure on X such that E is μ-nowhere hyperfinite. Then there is a Borel μ-stratification of E by μ-nowhere hyperfinite equivalence relations, each of which is generated by free Borel actions of every countable non-abelian free group on X.*

Proof. Fix a Borel isomorphism $\pi\colon X \to X \times \mathbb{N}$ of E with $E \times I(\mathbb{N})$, and let ν denote the push-forward of μ through π. Fix an $(E \times I(\mathbb{N}))$-invariant Borel probability measure $\nu' \gg \nu$ on $X \times \mathbb{N}$ (see, for example, the proof of [KM04, Corollary 10.2]). Define a Borel measure ν'_0 on X by $\nu'_0(B) = \nu'(B \times \{0\})$, for all Borel sets $B \subseteq X$. By [CM14, Theorem 5.7], there is a Borel μ-stratification $(E_r)_{r \in \mathbb{R}}$ of E by ν'_0-nowhere hyperfinite countable Borel equivalence relations. Then the pullback of $(E_r \times I(\mathbb{N}))_{r \in \mathbb{R}}$ through π yields a Borel stratification of E by μ-nowhere hyperfinite countable Borel equivalence relations. As these relations are necessarily compressible, Proposition 2.6 implies that they are induced by free Borel actions of every countable non-abelian free group on X. ⊠

The following ensures that Borel μ-stratifications by equivalence relations generated by free Borel actions of countable groups give rise to Borel μ-stratifications by actions.

Proposition 2.8. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, Γ is a countable group, μ is a Borel probability measure on X, and $(E_r)_{r \in \mathbb{R}}$ is a Borel μ-stratification of E by equivalence relations generated by free Borel actions of Γ on X. Then there is a Borel μ-stratification $(\cdot_r)_{r \in \mathbb{R}}$ of E by free Borel actions of Γ on X for which there is a Borel function $\pi\colon \mathbb{R} \to \mathbb{R}$ such that $E_{\pi(r)}$ is the equivalence relation generated by $\cdot_r$ on an E-invariant μ-conull Borel set, for all $r \in \mathbb{R}$.*

Proof. Fix a countable basis $\{U_n \mid n \in \mathbb{N}\}$ for X which is closed under finite unions, and define $G \subseteq \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \times X$ by $G_\phi = \bigcap_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} U_{\phi(m,n)}$. By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel functions $f_n\colon X \to X$ such that $E = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$.

Define $R \subseteq (\mathbb{N}^{\mathbb{N} \times \mathbb{N}})^{\mathbb{N}} \times (X \times X)$ by $R_\phi = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n \restriction G_{\phi(n)})$, and let B denote the set of all pairs $(\phi, x) \in ((\mathbb{N}^{\mathbb{N} \times \mathbb{N}})^{\mathbb{N}})^\Gamma \times X$ for which the sets of the form $R_{\phi(\gamma)} \cap ([x]_E \times [x]_E)$ are graphs of functions inducing a free action of Γ on $[x]_E$. For each function $\phi\colon \Gamma \to (\mathbb{N}^{\mathbb{N} \times \mathbb{N}})^{\mathbb{N}}$, let $\cdot_\phi$ denote the action of Γ on B_ϕ with the property that the graph of the function $x \mapsto \gamma \cdot_\phi x$ is $R_{\phi(\gamma)} \cap (B_\phi \times B_\phi)$, for all $\gamma \in \Gamma$.

The regularity of Borel probability measures on Polish spaces ensures that for all $r \in \mathbb{R}$, the equivalence relation generated by an action of the form $\cdot_\phi$ is E_r on an E-invariant μ-conull Borel subset of B_ϕ. As the set of pairs (r, ϕ) satisfying this latter property is Borel, the uniformization theorem for analytic subsets of the plane yields a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformization ϕ. Fix a continuous Borel probability measure m on

$\mathbb{R}$. As ϕ is necessarily m-measurable, there is an m-conull Borel set $R \subseteq \mathbb{R}$ on which ϕ is Borel. As R is necessarily uncountable, the proof of the perfect set theorem (see, for example, [Kec95, Theorem 13.6]) yields an order-preserving continuous embedding of $2^{\mathbb{N}}$ (equipped with the lexicographic order) into R. And by composing such an embedding with any order-preserving Borel embedding of $\mathbb{R}$ into $2^{\mathbb{N}}$, we obtain an order-preserving Borel embedding π of $\mathbb{R}$ into R, in which case the sequence $(\cdot_{\pi(r)})_{r \in \mathbb{R}}$ is as desired. ⊠

Finally, we establish an analog of Proposition 2.4 without invariance.

Proposition 2.9. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, μ is a Borel probability measure on X, and there is a free Borel action of a countable non-abelian free group on X generating a μ-nowhere hyperfinite subequivalence relation of E. Then for every countable non-abelian free group Γ, there is a Borel μ-stratification of E by free actions of Γ on X generating μ-nowhere hyperfinite equivalence relations.*

Proof. As a result of Hopf's ensures that X is compressible off of an E-invariant Borel set on which μ is equivalent to an E-invariant Borel probability measure (see, for example, [Nad98, §10]), the desired result is a consequence of Propositions 2.4, 2.7, and 2.8. ⊠

Remark 2.10. By Remark 2.5, after throwing out an E-invariant μ-null Borel set, the conclusion of Proposition 2.9 follows from the weaker assumption that for all μ-positive Borel subsets $B \subseteq X$ such that $\mu \restriction B$ is equivalent to an E-invariant Borel probability measure ν, there is a Borel subequivalence relation F of $E \restriction B$ for which $C_{\nu \restriction B}(F) > \nu(B)$.

3. Antichains

We are now prepared to establish our primary results. Although these can be obtained from the arguments of [CM14, §8] by substituting the stratification results of the previous section for those of [CM14], we will nevertheless provide the full proofs, both for the convenience of the reader and because somewhat simpler versions are sufficient to obtain the results we consider here.

Theorem 3.1. *Suppose that X is a standard Borel space, E is a projectively separable countable Borel equivalence relation on X, μ is an E-invariant Borel probability measure on X, and there is a free Borel action of a countable non-abelian free group on X generating a subequivalence relation of E with respect to which μ is ergodic. Then for every countable non-abelian free group Γ, there is a Borel sequence*

$(\cdot_r)_{r \in \mathbb{R}}$ of free actions of Γ on X, generating subequivalence relations E_r of E with respect to which μ is ergodic, with the further property that $(E_r)_{r \in \mathbb{R}}$ is an increasing sequence of relations which are pairwise incomparable under μ-reducibility.

Proof. By Proposition 2.2, there is a Borel μ-stratification $(\cdot_r)_{r \in \mathbb{R}}$ of E by free actions of Γ on X generating equivalence relations E_r with respect to which μ is ergodic. Clearly we can assume that $\bigcap_{r \in \mathbb{R}} E_r$ is μ-nowhere hyperfinite. Let R denote the relation on $\mathbb{R}$ in which two real numbers r and s are related if E_r is μ-reducible to E_s.

Lemma 3.2. *Every horizontal section of R is countable.*

Proof. Suppose, towards a contradiction, that there exists $t \in \mathbb{R}$ for which R^t is uncountable. For each $s \in R^t$, fix a μ-conull Borel set $B_s \subseteq X$ on which there is a Borel reduction $\phi_s \colon B_s \to X$ of E_s to E_t. As each ϕ_s is a homomorphism from $(\bigcap_{r \in \mathbb{R}} E_r) \restriction B_s$ to E, the μ-nowhere hyperfiniteness of $\bigcap_{r \in \mathbb{R}} E_r$ and the projective separability of E ensure the existence of distinct $r, s \in R^t$ for which $d_\mu(\phi_r, \phi_s) < 1$. Then $\{x \in B_r \cap B_s \mid \phi_r(x) = \phi_s(x)\}$ is a μ-positive Borel set on which E_r and E_s coincide, a contradiction. ⊠

As R is analytic (see, for example, [CM14, Proposition I.15]), a result of Lusin-Sierpiński ensures that it has the Baire property (see, for example, [Kec95, Theorem 21.6]). As the horizontal sections of R are countable and therefore meager, a result of Kuratowski-Ulam implies that R is meager (see, for example, [Kec95, Theorem 8.41]), in which case a result of Mycielski's yields a continuous order-preserving embedding $\phi \colon 2^{\mathbb{N}} \to \mathbb{R}$ such that pairs of distinct sequences in $2^{\mathbb{N}}$ are mapped to R-unrelated pairs of real numbers (see, for example, [CM14, Theorem B.5]). Fix a Borel embedding $\psi \colon \mathbb{R} \to 2^{\mathbb{N}}$ of the usual ordering of $\mathbb{R}$ into the lexicographical ordering of $2^{\mathbb{N}}$, and observe that the sequence $(\cdot_{\pi(r)})_{r \in \mathbb{R}}$ is as desired, where $\pi = \phi \circ \psi$. ⊠

Remark 3.3. By Remark 2.3, after throwing out an E-invariant μ-null Borel set, the conclusion of Theorem 3.1 follows from the weaker assumption that there is a Borel subequivalence relation F of E, for which $C_\mu(F) > 1$, with respect to which μ is ergodic.

Remark 3.4. In particular, this gives a simple new proof of the existence of such actions for the equivalence relation generated by the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$, a result which originally appeared in [Hjo12].

We next establish an analogous result in the absence of ergodicity.

Theorem 3.5. *Suppose that X is a standard Borel space, E is a projectively separable countable Borel equivalence relation on X, μ is an E-invariant Borel probability measure on X, and there is a free Borel action of a countable non-abelian free group on X generating a subequivalence relation of E. Then for every countable non-abelian free group Γ, there is a Borel sequence $(\cdot_r)_{r \in \mathbb{R}}$ of free actions of Γ on X, generating subequivalence relations E_r of E, with the further property that $(E_r)_{r \in \mathbb{R}}$ is an increasing sequence of relations which are pairwise incomparable under μ-somewhere reducibility.*

Proof. By Proposition 2.4, there is a Borel μ-stratification $(\cdot_r)_{r \in \mathbb{R}}$ of E by free actions of Γ on X. Letting E_r denote the equivalence relation generated by $\cdot_r$, we can again assume that $\bigcap_{r \in \mathbb{R}} E_r$ is μ-nowhere hyperfinite. Let R denote the relation on $\mathbb{R}$ in which two real numbers r and s are related if E_r is μ-somewhere reducible to E_s.

Lemma 3.6. *Every horizontal section of R is countable.*

Proof. Suppose, towards a contradiction, that there exists $t \in \mathbb{R}$ for which R^t is uncountable. For each $s \in R^t$, fix a μ-positive Borel set $B_s \subseteq X$ on which there is a Borel reduction $\phi_s \colon B_s \to X$ of E_s to E_t. Then there exists $\epsilon > 0$ with $\mu(B_s) \geq \epsilon$ for uncountably many $s \in R^t$. As each ϕ_s is a homomorphism from $(\bigcap_{r \in \mathbb{R}} E_r) \restriction B_s$ to E, the μ-nowhere hyperfiniteness of $\bigcap_{r \in \mathbb{R}} E_r$ and the projective separability of E ensure the existence of distinct $r, s \in R^t$ for which $\mu(B_r), \mu(B_s) \geq \epsilon$ and $d_\mu(\phi_r, \phi_s) < \epsilon$. Then $\{x \in B_r \cap B_s \mid \phi_r(x) = \phi_s(x)\}$ is a μ-positive Borel set on which E_r and E_s coincide, a contradiction. ⊠

One can now proceed exactly as in the proof of Theorem 3.1 to obtain the desired sequence. ⊠

Remark 3.7. By Remark 2.5, after throwing out an E-invariant μ-null Borel set, the conclusion of Theorem 3.5 follows from the weaker assumption that for all μ-positive Borel sets $B \subseteq X$, there is a Borel subequivalence relation F of $E \restriction B$ for which $C_{\mu \restriction B}(F) > \mu(B)$.

Finally, we establish an analogous result in the absence of invariance.

Theorem 3.8. *Suppose that X is a standard Borel space, E is a projectively separable countable Borel equivalence relation on X, μ is an Borel probability measure on X, and there is a free Borel action of a countable non-abelian free group on X generating a μ-nowhere hyperfinite subequivalence relation of E. Then for every countable non-abelian free group Γ, there is a Borel sequence $(\cdot_r)_{r \in \mathbb{R}}$ of free actions of Γ on X, generating subequivalence relations E_r of E, with the further property* *that $(E_r)_{r \in \mathbb{R}}$ is an increasing sequence of relations which are pairwise incomparable under μ-somewhere reducibility.*

Proof. This follows from the proof of Theorem 3.5, using Proposition 2.9 in place of Proposition 2.2. ⊠

Remark 3.9. By Remark 2.10, after throwing out an E-invariant μ-null Borel set, the conclusion of Proposition 2.9 follows from the weaker assumption that for all μ-positive Borel subsets $B \subseteq X$ such that $\mu \upharpoonright B$ is equivalent to an E-invariant Borel probability measure ν, there is a Borel subequivalence relation F of $E \upharpoonright B$ for which $C_{\nu \upharpoonright B}(F) > \nu(B)$.

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