Hedging of Credit Derivatives in Models with Totally Unexpected Default

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Based on

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- T.R. Bielecki, M. Jeanblanc and M. Rutkowski: Hedging of Defaultable Claims. In: *Paris-Princeton Lectures on Mathematical Finance 2003.* R. Carmona et al., eds. Springer-Verlag, 2004, 1-132.
- T.R. Bielecki, M. Jeanblanc and M. Rutkowski: Hedging of Credit Derivatives in Models with Totally Unexpected Default. Forthcoming in *Proceedings of Ritsumeikan Conference*, 2005.
- Bielecki, T., Jeanblanc, M. and Rutkowski, M.: Completeness of a General Semimartingale Market under Constrained Trading. In: *Stochastic Finance,* M. do Rosário Grossinho et al., eds. Springer-Verlag, 2006, 83-106.

Other related papers

- T.R. Bielecki, M. Jeanblanc and M. Rutkowski: PDE Approach to Valuation and Hedging of Credit Derivatives. *Quantitative Finance* 5 (2005), 1–14.
- T. Bielecki, M. Jeanblanc and M. Rutkowski (2004) Completeness of a Reduced-form Credit Risk Model with Discontinuous Asset Prices. Working paper.
- T. Bielecki, M. Jeanblanc and M. Rutkowski (2005) Pricing and Trading Credit Default Swaps. Working paper.
- T. Bielecki, S. Crépey, M. Jeanblanc and M. Rutkowski (2005) Hedging of Convertible Bonds in the Default Intensity Set-up. Working paper.

Outline

1. Risk-Neutral Valuation of Defaultable Claims
2. Trading Strategies: Default-Free Assets
3. Trading Strategies: Defaultable Assets
4. Replication of a Generic Defaultable Claim
5. Examples of Replication
6. Two Defaultable Assets with Total Default

Introduction

Terminology

- A defaultable claim is any financial contract with features related to the credit risk of some entity, e.g., a corporate bond, a vulnerable option, a defaultable swap, etc.
- A credit derivative is a special device that is tailored to transfer (buy or sell) the credit risk of a reference name.
- Plain vanilla credit derivatives: credit default swaps on a single name and related options (credit default swaptions).
- Multi-name (structured) credit derivatives: CDOs (collateralized debt obligations), basket swaps, index swaps.
- The term default time refers to the time of occurrence of some credit event.

Main issues

- How to hedge a defaultable claim within the framework of a given intensity-based model of credit risk?
- How to construct a model for given a set of liquid credit risk sensitive assets and given “practically acceptable” hedging strategies for credit derivatives?

Objectives

- To analyze unconstrained and constrained trading strategies with default-free and defaultable assets.
- To examine the arbitrage-free property and completeness of a model via the existence and uniqueness of a martingale measure.
- To study replicating strategies for a generic defaultable claim in terms of traded default-free and defaultable assets.
- To derive of explicit formulae for prices and replicating strategies of credit derivatives.
- To derive PDE approach to valuation and hedging of credit derivatives in a Markovian setup.

Risk-Neutral Valuation of Defaultable Claims

1 Risk-Neutral Valuation of Defaultable Claims

We adopt throughout the framework of the intensity-based approach.

1.1 Defaultable claims

A generic defaultable claim (X, Z, τ) consists of:

- The promised contingent claim X representing the payoff received by the owner of the claim at time T, if there was no default prior to or at maturity date T.

- The recovery process Z representing the recovery payoff at time of default, if default occurs prior to or at maturity date T.

- The default time τ. The name default time is merely a convention.

1.2 Default time within the intensity-based approach

The default time τ is a non-negative random variable on $(\Omega, \mathcal{G}, \mathbb{Q})$.

The default process equals $H_t = \mathbb{1}_{\{\tau \le t\}}$ and the filtration generated by this process is denoted by $\mathbb{H}$.

We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathbb{F} = (\mathcal{F}_t)$ is a reference filtration. The choice of $\mathbb{F}$ depends on a problem at hand.

Define the risk-neutral survival process G_t as

$$G_t = 1 - F_t = \mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}, \quad F_t = \mathbb{Q}\{\tau \le t \,|\, \mathcal{F}_t\}.$$

Then the risk-neutral hazard process Γ equals

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t.$$

1.3 Dividend process

The dividend process D represents all cash flows associated with a defaultable claim (X, Z, τ).

Formally, the dividend process D_t, $t \in [0, T]$, is defined through the formula

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{\{t = T\}} + \int_{]0,t]} Z_u \, dH_u$$

where the integral is the (stochastic) Stieltjes integral.

Recall that the filtration $\mathbb{G}$ models the full information, that is, the observations of the default-free market and the default event.

1.4 Ex-dividend price

The ex-dividend price process U of a defaultable claim (X, Z, τ) that settles at time T is given as

$$U_t = B_t\,\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1}\,dD_u \,\Big|\, \mathcal{G}_t\Big), \quad \forall\, t \in [0, T[,$$

where $\mathbb{Q}$ is the spot martingale measure (also known as the risk-neutral probability) and B is the savings account.

In addition, at maturity date T we set

$$U_T = U_T(X) + U_T(Z) = X\mathbb{1}_{\{\tau > T\}} + Z_T\mathbb{1}_{\{\tau = T\}}.$$

1.5 Pre-default values

1.5.1 Valuation of a survival claim $(X, 0, \tau)$

For an $\mathcal{F}_T$-measurable r.v. X and any $t \leq T$ the value $U_t(X)$ equals

$$U_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau > T\}} B_T^{-1} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \, B_t \, \mathbb{E}_{\mathbb{Q}}(e^{\Gamma_t - \Gamma_T} B_T^{-1} X \,|\, \mathcal{F}_t).$$

1.5.2 Valuation of a recovery process $(0, Z, \tau)$

For an $\mathbb{F}$-predictable process Z and any $t \leq T$ the value $U_t(Z)$ equals

$$U_t(Z) = B_t \, \mathbb{E}_{\mathbb{Q}}(B_\tau^{-1} Z_\tau \mathbb{1}_{\{t < \tau \leq T\}} | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} B_t \, e^{\Gamma_t} \, \mathbb{E}_{\mathbb{Q}} \left(\int_t^T B_u^{-1} Z_u \, dF_u \,\middle|\, \mathcal{F}_t \right).$$

Note that $U_t(X) = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t(X)$ and $U_t(Z) = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t(Z)$ for some $\mathbb{F}$-predictable processes $\widetilde{U}(X)$ and $\widetilde{U}(Z)$.

1.6 Comments

- The process $\widetilde{U}(X)$ is called the pre-default value of a survival claim $(X, 0, \tau)$.
- The process $\widetilde{U}(Z)$ is termed the pre-default value of a recovery process $(0, Z, \tau)$.
- Valuation results for defaultable claims presented in this section were not supported by replication arguments. It was assumed, somewhat arbitrarily, that $\mathbb{Q}$ is the pricing measure (risk-neutral probability).
- In what follows, we shall examine separately on replication of $(X, 0, \tau)$ and $(0, Z, \tau)$.
- Replication will hold on the closed random interval $[\![0, \tau \wedge T]\!]$, where $\tau \wedge T$ represents the effective maturity of a defaultable claim.

Trading Strategies: Default-Free Assets

2 Trading Strategies: Default-Free Assets

- First, we shall recall the properties of the wealth process of a standard self-financing trading strategy without and with constraints.

- In this section, we concentrate on trading in default-free assets.

- Let $Y_t^1, Y_t^2, \dots, Y_t^k$ represent the cash values at time t of k traded assets. We postulate that Y^1, $Y^2, \dots, Y^k$ are (possibly discontinuous) semimartingales with respect to a filtration $\mathbb{F}$. Usually $\mathbb{F} = \mathbb{F}^Y$.

- We recall the properties of the wealth process of self-financing strategies without and with constraints in a semimartingale set-up.

2.1 Self-financing strategies

Definition. The wealth process $V(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ equals

$$V_t(\phi) = \sum_{i=1}^{k} \phi_t^i Y_t^i.$$

A process ϕ is in Φ, i.e. is a self-financing strategy if for every $t \in [0, T]$

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{k} \int_0^t \phi_u^i \, dY_u^i.$$

Remark. Let Y^1 be strictly positive. The last two formulae yield

$$dV_t(\phi) = \Big(V_t(\phi) - \sum_{i=2}^{k} \phi_t^i Y_t^i\Big)(Y_t^1)^{-1} \, dY_t^1 + \sum_{i=2}^{k} \phi_t^i \, dY_t^i.$$

2.2 Discounted wealth

The last representation of $V(\phi)$ shows that the wealth process depends only on $k-1$ components of the process ϕ.

Let us choose Y^1 as a numeraire asset. Then, writing

$$V_t^1(\phi) = V_t(\phi)/Y_t^1, \quad Y_t^{i,1} = Y_t^i/Y_t^1,$$

we get the following well-known result.

Lemma. Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing strategy. Then

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{k} \int_0^t \phi_u^i \, dY_u^{i,1}, \quad \forall\, t \in [0,T].$$

2.3 Replication

Proposition. Let X be an $\mathcal{F}_T$-measurable random variable. Assume that there exists $x \in \mathbb{R}$ and $\mathbb{F}$-predictable processes ϕ^i, $i = 2, 3, \ldots, k$ such that

$$\frac{X}{Y_T^1} = x + \sum_{i=2}^{k} \int_0^T \phi_t^i \, dY_t^{i,1}.$$

Then there exists a $\mathbb{F}$-predictable process ϕ^1 such that the strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ is self-financing and replicates X. Moreover

$$\frac{\pi_t(X)}{Y_t^1} = x + \sum_{i=2}^{k} \int_0^t \phi_u^i \, dY_u^{i,1}$$

where $\pi_t(X)$ is the arbitrage price (cost of replication) of X.

2.4 Constrained trading strategies

- By definition, a constrained strategy ϕ satisfies

$$\sum_{i=\ell+1}^{k} \phi_t^i Y_{t-}^i = Z_t$$

where Z is a predetermined $\mathbb{F}$-predictable process.

- The constraint above is referred to as the balance condition and the class of all constrained self-financing strategies is denoted by $\Phi_l(Z)$.

- For any $\phi \in \Phi_l(Z)$ we have, for every $t \in [0,T]$,

$$V_{t-}(\phi) = \sum_{i=1}^{k} \phi_t^i Y_{t-}^i = \sum_{i=1}^{\ell} \phi_t^i Y_{t-}^i + Z_t.$$

2.5 Dynamics of a relative wealth

Lemma. The relative wealth $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ of any strategy $\phi \in \Phi_l(Z)$ satisfies

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \, dY_u^{i,k,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1}$$

where we denote $Z_t^1 = Z_t / Y_t^1$ and

$$Y_t^{i,k,1} = \int_0^t \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right).$$

2.6 Replication under balance condition

Let X be a $\mathcal{F}_T$-measurable random variable.

Proposition. Assume that there exist $\mathbb{F}$-predictable processes ϕ^i, $i = 2, 3, \dots, k-1$ such that

$$\frac{X}{Y_T^1} = x + \sum_{i=2}^{l} \int_0^T \phi_t^i \, dY_t^{i,1} + \sum_{i=l+1}^{k-1} \int_0^T \phi_t^i \, dY_t^{i,k,1} + \int_0^T \frac{Z_t^1}{Y_{t-}^{k,1}} \, dY_t^{k,1}.$$

Then there exist the $\mathbb{F}$-predictable processes ϕ^1 and ϕ^k such that the strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k) \in \Phi_l(Z)$ replicates X. Moreover

$$V_t^1(\phi) = x + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \, dY_u^{i,k,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1}.$$

2.7 Synthetic assets

- Processes $Y^{i,k,1}$ given by

$$Y_t^{i,k,1} = \int_0^t \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}}\, dY_u^{k,1} \right)$$

  represent relative prices of synthetic assets.

- More specifically, for any $i = l+1, l+2, \ldots, k-1$ the process $\bar{Y}^{i,k,1} = Y^1 Y^{i,k,1}$ is the cash price of the ith synthetic asset. We write briefly $\bar{Y}^i = \bar{Y}^{i,k,1}$.

- The ith synthetic asset $\bar{Y}^i$ can be obtained by continuous trading in primary assets Y^1, Y^i and Y^k.

2.8 Case of continuous semimartingales

Lemma. Assume that the prices Y^1, Y^i and Y^k follow strictly positive continuous semimartingales. Then we have

$$Y_t^{i,k,1} = \int_0^t (Y_u^{1,k})^{-1} e^{\alpha_u^{i,k,1}} \, d\widehat{Y}_u^{i,k,1}$$

where

$$\widehat{Y}_t^{i,k,1} = Y_t^{i,k} e^{-\alpha_t^{i,k,1}}$$

and

$$\alpha_t^{i,k,1} = \langle \ln Y^{i,k}, \ln Y^{1,k} \rangle_t = \int_0^t (Y_u^{i,k})^{-1} (Y_u^{1,k})^{-1} \, d\langle Y^{i,k}, Y^{1,k} \rangle_u.$$

2.9 Further properties

- Each primary asset Y^i, $i = l+1, \ldots, k-1$ can be obtained by trading in primary assets Y^1, Y^k and a synthetic asset $\bar{Y}^i$.

- Constrained market models

$$\mathcal{M}_l(Z) = (Y^1, Y^2, \ldots, Y^k; \Phi_l(Z))$$

$$\bar{\mathcal{M}}_{k-1}(Z) = (Y^1, Y^2, \ldots, Y^l, \bar{Y}^{l+1}, \ldots, \bar{Y}^{k-1}, Y^k; \Phi_{k-1}(Z))$$

  are equivalent. Note that $\phi \in \Phi_{k-1}(Z)$ if $\phi_t^k Y_t^k = Z_t$.

- Instead of using primary assets $Y^1, Y^2, \ldots, Y^k$, it is more convenient to replicate a contingent claim using the assets $Y^1, Y^2, \ldots, Y^l, \bar{Y}^{l+1}, \ldots, \bar{Y}^{k-1}, Y^k$.

2.10 Remarks

- In the case of a single constraint, the wealth $V(\phi)$ of a self-financing trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ is completely specified by the $k - 2$ components $\phi^2, \dots, \phi^{k-1}$ of ϕ.

- The coefficients $\alpha_t^{i,k,1}$ represent the correlations between relative prices $Y^{i,k}$ and $Y^{1,k}$ (in so-called volatility-based models they are given as integrals of products of the respective volatilities).

- The concept of a self-financing constrained strategy allows us to deal with the recovery process Z. For a survival claim $(X, 0, \tau)$ we set $Z = 0$.

- It remains to specify the behavior of defaultable tradeable assets at the time of default (recovery rule).

Trading Strategies: Defaultable Assets

Standing assumptions

- Zero recovery scheme for all defaultable assets.
- Pre-default values of all defaultable assets follow continuous processes.
- All defaultable assets have a common default time.
- Prices of default-free assets follow continuous processes.

These assumptions are not realistic and too restrictive, but each of them can be subsequently relaxed.

3 Trading Strategies: Defaultable Assets

- Let Y^i, $i = 1, \ldots, m$ be prices of defaultable assets traded in the market. A random time τ is the common default time for all defaultable assets.

- If Y^i is subject to zero recovery then

$$Y_t^i = \mathbb{1}_{\{\tau > t\}} \widetilde{Y}_t^i,$$

  where the process $\widetilde{Y}^i$, representing the pre-default value of Y^i, is adapted to the reference filtration $\mathbb{F}$.

- We assume that the pre-default price processes $\widetilde{Y}_t^i$, $t \in [0, T]$, are continuous semimartingales and $\widetilde{Y}^1$ is strictly positive.

3.1 Self-financing strategies

Let $Y^1, \dots, Y^m$ be prices of m defaultable assets, and let $Y^{m+1}, \dots, Y^k$ represent prices of $k-m$ default-free assets. Processes $Y^{m+1}, \dots, Y^k$ are continuous semimartingales and Y^k is strictly positive.

We postulate here that the processes $\phi^1, \dots, \phi^k$ are $\mathbb{G}$-predictable.

Definition. The wealth $V_t(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals $V_t(\phi) = \sum_{i=1}^k \phi_t^i Y_t^i$ for every $t \in [0,T]$. A strategy ϕ is said to be self-financing if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^m \int_0^t \phi_u^i \, dY_u^i + \sum_{i=m+1}^k \int_0^t \phi_u^i \, dY_u^i, \quad \forall \, t \in [0,T].$$

3.2 Pre-default wealth

Definition. The pre-default wealth $\widetilde{V}(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ equals

$$\widetilde{V}_t(\phi) = \sum_{i=1}^{m} \phi_t^i \widetilde{Y}_t^i + \sum_{i=m+1}^{k} \phi_t^i Y_t^i, \quad \forall\, t \in [0, T].$$

A strategy ϕ is said to be self-financing prior to default if

$$\widetilde{V}_t(\phi) = \widetilde{V}_0(\phi) + \sum_{i=1}^{m} \int_0^t \phi_u^i \, d\widetilde{Y}_u^i + \sum_{i=m+1}^{k} \int_0^t \phi_u^i \, dY_u^i, \quad \forall\, t \in [0, T].$$

Comments:

- If a trading strategy is self-financing on $[\![0, \tau \wedge T[\![$ then it is also self-financing on $[\![0, \tau \wedge T]\!]$.
- We may and do assume that the processes $\phi^1, \ldots, \phi^k$ are $\mathbb{F}$-predictable.

Replication of a Generic Defaultable Claim

4 Replication of a Generic Defaultable Claim

Recall that $\widetilde{U}_t(X)$ and $\widetilde{U}_t(Z)$ stand for pre-default values of defaultable claims $(X, 0, \tau)$ and $(0, Z, \tau)$, respectively.

Definition. A self-financing trading strategy ϕ is a replicating strategy for a defaultable claim (X, Z, τ) if and only if the following hold:

- $V_t(\phi) = \widetilde{U}_t(X) + \widetilde{U}_t(Z)$ on the random interval $[\![0, \tau \wedge T[\![$,
- $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \leq T\}$,
- $V_T(\phi) = X$ on the set $\{\tau > T\}$.

We say that a defaultable claim is attainable if it admits at least one replicating strategy.

4.1 Replication of a survival claim

It is enough to deal with the pre-default wealth process $\widetilde{V}(\phi)$.

Proposition. Let a constant $\widetilde{V}^1_0$ and $\mathbb{F}$-predictable processes ψ^i for $i = 2, \ldots, m$ and $\widetilde{\psi}^{i,k,1}$ for $i = m+1, \ldots, k-1$ be such that

$$\widetilde{Y}^1_T \left(\widetilde{V}^1_0 + \sum_{i=2}^{m} \int_0^T \psi^i_u \, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^T \widetilde{\psi}^{i,k,1}_u \, d\widehat{Y}^{i,k,1}_u \right) = X.$$

Let $\widetilde{V}_t = \widetilde{Y}^1_t \widetilde{V}^1_t$ where

$$\widetilde{V}^1_t = \widetilde{V}^1_0 + \sum_{i=2}^{m} \int_0^t \psi^i_u \, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\psi}^{i,k,1}_u \, d\widehat{Y}^{i,k,1}_u .$$

Proposition (continued).

Then the trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ defined by

$$
\begin{aligned}
\phi_t^1 &= \left(\widetilde{V}_t - \sum_{i=2}^{m} \phi_t^i \widetilde{Y}_t^i \right) (\widetilde{Y}_t^1)^{-1}, \\
\phi_t^i &= \psi_t^i, \quad i = 2, \ldots, m, \\
\phi_t^i &= \widetilde{\psi}_t^{i,k,1} \widetilde{Y}_t^{1,k} e^{-\widetilde{\alpha}_t^{i,k,1}}, \quad i = m+1, \ldots, k-1, \\
\phi_t^k &= -\sum_{i=m+1}^{k-1} \phi_t^i Y_t^i (Y_t^k)^{-1},
\end{aligned}
$$

is self-financing and it replicates $(X, 0, \tau)$. We have $\widetilde{V}_t(\phi) = \widetilde{V}_t = \widetilde{U}_t(X)$, that is, the process $\widetilde{V}$ represents the pre-default value of a survival claim.

4.2 Remarks

- To completely specify a strategy ϕ it suffices to specify (in fact, to find if we wish to replicate a given defaultable claim) $k-2$ components only, namely, the components $\phi^2, \ldots, \phi^{k-1}$.

- The coefficients $\widetilde{\alpha}_t^{i,k,1}$ are correlations between the relative asset prices $Y^{i,k} = Y^i/Y^k$ and $\widetilde{Y}^{1,k} = \widetilde{Y}^1/Y^k$ and, typically, they equal to integrals of products of the respective volatilities.

- The volatility of $\widetilde{Y}^{1,k} = \widetilde{Y}^1/Y^k$ will depend on the properly defined volatility $\beta(t,T)$ of the hazard process (if Γ is deterministic then $\beta(t,T)$ vanishes).

4.3 Replication of a recovery payoff

Proposition. Let a constant $\widetilde{V}^1_0$ and $\mathbb{F}$-predictable processes ψ^i for $i = 2, \dots, m$ and $\widetilde{\psi}^{i,k,1}$ for $i = m+1, \dots, k-1$ be such that

$$\widetilde{V}^1_0 + \sum_{i=2}^{m} \int_0^T \psi^i_u \, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^T \widetilde{\psi}^{i,k,1}_u \, d\widehat{Y}^{i,k,1}_u$$

$$+ \int_0^T Z_u (Y^k_u)^{-1} \, d(\widetilde{Y}^{1,k}_u)^{-1} = 0.$$

Let $\widetilde{V}_t = \widetilde{Y}^1_t \widetilde{V}^1_t$ where

$$\widetilde{V}^1_t = \widetilde{V}^1_0 + \sum_{i=2}^{m} \int_0^t \psi^i_u \, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\psi}^{i,k,1}_u \, d\widehat{Y}^{i,k,1}_u$$

$$+ \int_0^t Z_u (Y^k_u)^{-1} \, d(\widetilde{Y}^{1,k}_u)^{-1}.$$

Proposition (continued).

The replicating strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ for a recovery process Z is given by the following expressions

$$
\begin{aligned}
\phi^1_t &= \left(\widetilde{V}_t - Z_t - \sum_{i=2}^{m} \phi^i_t \widetilde{Y}^i_t \right) (\widetilde{Y}^1_t)^{-1}, \\
\phi^i_t &= \psi^i_t, \quad i = 2, \ldots, m, \\
\phi^i_t &= \widetilde{\psi}^{i,k,1}_t \widetilde{Y}^{1,k}_t e^{-\widetilde{\alpha}^{i,k,1}_t}, \quad i = m+1, \ldots, k-1, \\
\phi^k_t &= \left(Z_t - \sum_{i=m+1}^{k-1} \phi^i_t Y^i_t \right) (Y^k_t)^{-1}.
\end{aligned}
$$

Moreover, $\widetilde{V}_t(\phi) = \widetilde{V}_t = \widetilde{U}_t(Z)$, that is, the process $\widetilde{V}$ represents the pre-default value of a recovery payoff.

Examples of Replication

5 Examples of Replication

5.1 Standing assumptions

- We are given an arbitrage-free term structure model driven by a Brownian motion W. The reference filtration $\mathbb{F}$ is the Brownian filtration: $\mathbb{F} = \mathbb{F}^W$.
- We are given the $\mathbb{F}$-hazard process Γ of default time τ.
- Default-free discount bonds $B(t,T)$ and defaultable bonds with zero recovery $D^0(t,T)$ are traded assets. For a fixed $T > 0$, we define

$$D^0(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}}(B_T^{-1} \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \widetilde{D}^0(t,T),$$

where $\widetilde{D}^0(t,T)$ stands for the pre-default value of the defaultable bond.

5.2 Forward martingale measure

Let $\mathbb{Q}_T$ stand for the forward martingale measure on $(\Omega, \mathcal{G}_T)$

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{1}{B_T B(0,T)} \quad \mathbb{Q}\text{-a.s.}$$

so that the process $W_t^T = W_t - \int_0^t b(u,T)\,du$ is a Brownian motion under $\mathbb{Q}_T$. Denote by $F(t,U,T) = B(t,U)(B(t,T))^{-1}$ the forward price of U-maturity bond, so that

$$dF(t,U,T) = F(t,U,T)\big(b(t,U) - b(t,T)\big)dW_t^T$$

where W^T is a Brownian motion under $\mathbb{Q}_T$.

Since the savings account B_t and the bond price $B(t,T)$ are $\mathbb{F}$-adapted, it can be shown that Γ is also the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_T$

$$\mathbb{Q}_T\{t < \tau \leq T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}_T}(e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t).$$

5.3 Volatility process $\beta(t,T)$

Observe that

$$D^0(t,T) = \mathbb{1}_{\{\tau>t\}} \widetilde{D}^0(t,T) = \mathbb{1}_{\{\tau>t\}} B(t,T)\, \mathbb{E}_{\mathbb{Q}_T}(e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t).$$

We set

$$\Gamma(t,T) = \widetilde{D}^0(t,T)(B(t,T))^{-1} = \mathbb{E}_{\mathbb{Q}_T}(e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t).$$

Lemma. Assume that the $\mathbb{F}$-hazard process Γ is continuous. The process $\Gamma(t,T)$, $t \in [0,T]$, is a continuous $\mathbb{F}$-submartingale and

$$d\Gamma(t,T) = \Gamma(t,T)\big(d\Gamma_t + \beta(t,T)\, dW_t^T\big)$$

for some $\mathbb{F}$-predictable process $\beta(t,T)$. The process $\Gamma(t,T)$ is of finite variation if and only if the hazard process Γ is deterministic. In this case, we have that $\Gamma(t,T) = e^{\Gamma_t - \Gamma_T}$.

5.4 Example 1: Vulnerable option on a default-free bond

For a fixed $U > T$, we assume that the U-maturity default-free bond is also traded, and we consider a vulnerable European call option with the terminal payoff

$$\widehat{C}_T = \mathbb{1}_{\{\tau > T\}}(B(T,U) - K)^+.$$

We thus deal with a survival claim $(X, 0, \tau)$ with the promised payoff $X = (B(T,U) - K)^+$.

We take $Y_t^1 = D^0(t,T)$, $Y_t^2 = B(t,U)$ and $Y_t^3 = B(t,T)$ as traded assets.

Let us denote

$$f(t) = \beta(t,T)(b(t,U) - b(t,T)), \quad \forall\, t \in [0,T],$$

and let us assume that f is a deterministic function.

5.5 Replication of a vulnerable option

Proposition. Let us set $F(t,U,T) = B(t,U)/B(t,T)$. The pre-default price $\widetilde{C}_t$ of a vulnerable call option written on a default-free zero-coupon bond equals

$$\widetilde{C}_t = \widetilde{D}^0(t,T)\Big(F(t,U,T)e^{\int_t^T f(u)\,du}N\big(h_+(t,U,T)\big) - KN\big(h_-(t,U,T)\big)\Big)$$

where

$$h_\pm(t,U,T) = \frac{\ln F(t,U,T) - \log K + \int_t^T f(u)\,du \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and $v^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2\,du$. The replicating strategy satisfies

$$\phi_t^1 = \frac{\widetilde{C}_t}{\widetilde{D}^0(t,T)},\ \phi_t^2 = e^{\widetilde{\alpha}_T^{2,3,1} - \widetilde{\alpha}_t^{2,3,1}}\Gamma(t,T)N\big(h_+(t,U,T)\big),\ \phi_t^3 = -\phi_t^2 F(t,U,T).$$

5.6 Remarks

- The pricing formula is structurally similar to the pricing formula for a bond futures option in the Gaussian HJM setup.
- The promised payoff is attainable in a non-defaultable market with traded assets $B(t,T)$ and $B(t,U)$.
- To replicate the option we assume, in addition, that a defaultable bond $D^0(t,T)$ is traded.
- If Γ is deterministic then the price $\widetilde{C}_t = \Gamma(t,T)C_t$, where C_t is the price of an equivalent non-defaultable option. Moreover

$$\phi_t^1 = C_t B(t,T)^{-1},\ \phi_t^2 = \Gamma(t,T)N\big(h_+(t,U,T)\big),\ \phi_t^3 = -\phi_t^2 F(t,U,T).$$

- The method is quite general and thus it applies to other claims as well.

5.7 Example 2: Vulnerable option on a default-free asset

We shall now analyze a vulnerable call option with the payoff

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - K)^+.$$

Our goal is to find a replicating strategy for this claim, interpreted as a survival claim $(X, 0, \tau)$ with the promised payoff $X = C_T = (Y_T^2 - K)^+$, where C_T is the payoff of an equivalent non-vulnerable option.

Method presented below is quite general, however, so that it can be applied to any survival claim with the promised payoff $X = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$ satisfying the usual integrability assumptions.

We assume that $Y_t^1 = B(t,T)$, $Y_t^3 = D(t,T)$, and the price of a default-free asset Y^2 is governed by

$$dY_t^2 = Y_t^2\big(\mu_t\, dt + \sigma_t\, dW_t\big)$$

with $\mathbb{F}$-predictable coefficients μ and σ.

5.8 Credit-risk-adjusted forward price: definition

Definition. Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the credit-risk-adjusted forward price of Y if the pre-default value at time t of the vulnerable forward contract represented by the claim $\mathbb{1}_{\{T<\tau\}}(Y-K)$ equals 0.

Then we have the following result.

Lemma. The credit-risk-adjusted forward price $\widehat{F}_Y(t,T)$ of an attainable survival claim $(X,0,\tau)$, represented by a $\mathcal{G}_T$-measurable claim $Y = X\mathbb{1}_{\{T<\tau\}}$, equals $\widetilde{\pi}_t(X,0,\tau)(\widetilde{D}(t,T))^{-1}$, where $\widetilde{\pi}_t(X,0,\tau)$ is the pre-default price of $(X,0,\tau)$.

5.9 Credit-risk-adjusted forward price: computation

Let us now focus on default-free assets. Manifestly, the credit-risk-adjusted forward price of the bond $B(t,T)$ equals 1. To find the credit-risk-adjusted forward price of Y^2, let us denote

$$\widehat{F}_{Y^2}(t,T) := F_{Y^2}(t,T)\, e^{\alpha_T - \alpha_t} = Y_t^{2,1} e^{\alpha_T - \alpha_t},$$

where $\alpha_t := \langle \ln Y^{2,1}, \ln Y^{3,1} \rangle_t$ satisfies

$$\alpha_t = \int_0^t \big(\sigma_u - b(u,T)\big)\beta(u,T)\, du = \int_0^t \big(\sigma_u - b(u,T)\big)\big(\widetilde{d}(u,T) - b(u,T)\big)\, du.$$

Lemma. Assume that α_t, $t \in [0,T]$, is a deterministic function. Then the credit-risk-adjusted forward price of Y^2 equals $\widehat{F}_{Y^2}(t,T)$ for every $t \in [0,T]$.

5.10 Replication of a vulnerable option

Proposition. Suppose that the volatilities σ, b and β are deterministic. Then the credit-risk-adjusted forward price of a vulnerable call option written on a default-free asset Y^2 equals

$$\widehat{F}_{C^d}(t,T) = \widehat{F}_{Y^2}(t,T)N(d_+(\widehat{F}_{Y^2}(t,T),t,T)) - KN(d_-(\widehat{F}_{Y^2}(t,T),t,T))$$

where

$$d_{\pm}(\widehat{f},t,T) = \frac{\ln \widehat{f} - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_u - b(u,T))^2 \, du.$$

The replicating strategy ϕ in the spot market satisfies, on the set $\{t < \tau\}$,

$$\phi_t^1 B(t,T) = -\phi_t^2 Y_t^2, \;\; \phi_t^2 = \widetilde{D}(t,T)(B(t,T))^{-1} N(d_+(t,T))e^{\alpha_T - \alpha_t}$$

and $\phi_t^3 \widetilde{D}(t,T) = \widetilde{C}_t^d$, where $d_+(t,T) = d_+(\widehat{F}_{Y^2}(t,T),t,T)$.

5.11 Example 3: Option on a defaultable bond

Consider a (non-vulnerable) call option on a defaultable bond with maturity date U and zero recovery. Let T be the expiration date and let $K > 0$ stand for the strike. The terminal payoff equals

$$\bar{C}_T = (D^0(T,U) - K)^+.$$

Note that

$$\bar{C}_T = \big(\mathbb{1}_{\{\tau>T\}}\widetilde{D}^0(T,U) - K\big)^+ = \mathbb{1}_{\{\tau>T\}}\big(\widetilde{D}^0(T,U) - K\big)^+ = \mathbb{1}_{\{\tau>T\}}X$$

where $X = (\widetilde{D}^0(T,U) - K)^+$, so that we deal again with a survival claim $(X, 0, \tau)$. Since the underlying asset is defaultable here, the replicating strategy will have different features. We now postulate that defaultable bonds of maturities U and T are the only tradeable assets.

5.12 Replication of an option on a defaultable bond

Proposition. Let $\beta(t,U)+b(t,U)-b(t,T)$ be deterministic. Then the pre-default price $\widetilde{C}_t$ of a call option written on a U-maturity defaultable bond equals

$$\widetilde{C}_t = \widetilde{D}^0(t,U)N\big(k_+(t,U,T)\big) - K\widetilde{D}^0(t,T)N\big(k_-(t,U,T)\big)$$

where

$$k_\pm(t,U,T) = \frac{\ln \widetilde{D}^0(t,U) - \ln \widetilde{D}^0(t,T) - \log K \pm \frac{1}{2}\widetilde{v}^2(t,T)}{\widetilde{v}(t,T)}$$

and $\widetilde{v}^2(t,T) = \int_t^T |\beta(u,U)+b(u,U)-b(u,T)|^2\,du$. The replicating strategy $\phi = (\phi^1,\phi^2)$ is given by

$$\phi^1_t = (\widetilde{C}_t - \phi^2_t\widetilde{D}^0(t,U))(\widetilde{D}^0(t,T))^{-1}, \quad \phi^2_t = N\big(k_+(t,U,T)\big).$$

5.13 Remarks

- The payoff is attainable in a defaultable market with traded assets $D^0(t,T)$ and $D^0(t,U)$. Default-free assets are not used for replication.

- If Γ is deterministic then

$$\widetilde{C}_t = e^{\Gamma_t - \Gamma_U} B(t,U) N\big(k_+(t,U,T)\big) - K e^{\Gamma_t - \Gamma_T} B(t,T) N\big(k_-(t,U,T)\big)$$

where

$$k_\pm(t,U,T) = \frac{\ln B(t,U) - \ln B(t,T) - \log K - \Gamma_T + \Gamma_U \pm \frac{1}{2} v^2(t,T)}{v(t,T)}$$

and $v^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2 \, du$.

- This corresponds to credit-risk-adjusted interest rate $\widehat{r}_t = r_t + \gamma(t)$.

Two Defaultable Assets with Total Default

6 Two Defaultable Assets with Total Default

We shall now assume that we have only two assets, and both are defaultable assets with total default. This special case is also examined in a recent work by P. Carr (2005) *Dynamic replication of a digital default claim*.

We postulate that under the statistical probability $\mathbb{Q}$ we have, for $i = 1, 2$,

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t - dM_t \big)$$

where W is a d-dimensional Brownian motion, so that

$$Y_t^1 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^2.$$

6.1 Pre-default values

The pre-default values are governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_{i,t} + \gamma_t)\, dt + \sigma_{i,t}\, dW_t \big).$$

The wealth process $V(\phi)$ associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies, for every $t \in [0, T]$,

$$V_t(\phi) = Y_t^1 \left(V_0^1(\phi) + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,1} \right)$$

where $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2 / \widetilde{Y}_t^1$. Since both primary traded assets are subject to total default, it is clear that the present model is incomplete, in the sense, that not all defaultable claims can be replicated.

6.2 Completeness

- We shall check that, under the assumption that the driving Brownian motion W is one-dimensional, all survival claims satisfying natural technical conditions are hedgeable, however.

- In the more realistic case of a two-dimensional noise, we will still be able to hedge a large class of survival claims, including options on a defaultable asset and options to exchange defaultable assets.

- We shall argue that in a model with two defaultable assets governed, replication of a survival claim $(X, 0, \tau)$ is in fact equivalent to replication of the promised payoff X using the pre-default processes.

6.3 Replication with pre-default values

Lemma. If a strategy ϕ^i, $i = 1, 2$, based on pre-default values $\widetilde{Y}^i$, $i = 1, 2$, is a replicating strategy for an $\mathcal{F}_T$-measurable claim X, that is, if ϕ is such that the process $\widetilde{V}_t(\phi) = \phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2$ satisfies, for every $t \in [0, T]$,

$$d\widetilde{V}_t(\phi) = \phi_t^1 \, d\widetilde{Y}_t^1 + \phi_t^2 \, d\widetilde{Y}_t^2, \quad \widetilde{V}_T(\phi) = X,$$

then for the process $V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2$ we have, for $t \in [0, T]$,

$$dV_t(\phi) = \phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2, \quad V_T(\phi) = X \mathbb{1}_{\{T < \tau\}}.$$

This means that a strategy ϕ replicates a survival claim $(X, 0, \tau)$.

6.4 Replication of a survival claim

We conclude that a strategy (ϕ^1, ϕ^2) replicates a survival claim $(X, 0, \tau)$ whenever

$$\widetilde{Y}_T^1 \Big(x + \int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,1} \Big) = X$$

for some constant x and some $\mathbb{F}$-predictable process ϕ^2.

Note that

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1} \Big(\big(\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t}) \big) \, dt + (\sigma_{2,t} - \sigma_{1,t}) \, dW_t \Big)$$

and introduce a probability measure $\widetilde{\mathbb{Q}}$, equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$, and such that $\widetilde{Y}^{2,1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$.

6.5 Complete case: one-dimensional noise

We argue that a survival claim is attainable if the random variable $X(\widetilde{Y}^1_T)^{-1}$ is $\widetilde{\mathbb{Q}}$-integrable. The pre-default value $\widetilde{V}_t$ of a survival claim equals

$$\widetilde{V}_t = \widetilde{Y}^1_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(X(\widetilde{Y}^1_T)^{-1} \,|\, \mathcal{F}_t\big)$$

and from the predictable representation theorem, we deduce that there exists a process ϕ^2 such that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(X(\widetilde{Y}^1_T)^{-1} \,|\, \mathcal{F}_t\big) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(X(\widetilde{Y}^1_T)^{-1}\big) + \int_0^t \phi^2_u \, d\widetilde{Y}^{2,1}_u.$$

The component ϕ^1 of the self-financing strategy $\phi = (\phi^1, \phi^2)$ is chosen in such a way that $\phi^1_t \widetilde{Y}^1_t + \phi^2_t \widetilde{Y}^2_t = \widetilde{V}_t$ for $t \in [0, T]$.

6.6 Incomplete case: multi-dimensional noise

We work here with the two correlated one-dimensional Brownian motions, so that

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t} \, dt + \sigma_{i,t} \, dW_t^i - dM_t \big), \quad i = 1, 2,$$

where $d\langle W^1, W^2 \rangle_t = \rho_t \, dt$ for some correlation coefficient ρ.

The model is incomplete, but the exchange option $(Y_T^2 - KY_T^1)^+$ is attainable and the option pricing formula in terms of pre-default values is exactly the same as the standard formula for an option to exchange non-defaultable assets.

It is remarkable that in the next result we make no assumption about the behavior of stochastic default intensity.

6.7 Example 4: option to exchange defaultable assets

Proposition. Let the volatilities σ_1, σ_2 and the correlation coefficient ρ be deterministic. Then the pre-default price of the exchange option equals

$$\widetilde{C}_t = \widetilde{Y}_t^2 N\big(d_+(\widetilde{Y}_t^{2,1}, t, T)\big) - K\widetilde{Y}_t^1 N\big(d_-(\widetilde{Y}_t^{2,1}, t, T)\big),$$

where

$$d_\pm(\widetilde{y}, t, T) = \frac{\ln \widetilde{y} - \ln K \pm \frac{1}{2} v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T \big(\sigma_{1,u}^2 + \sigma_{2,u}^2 - 2\rho_u \sigma_{1,u}\sigma_{2,u}\big)\, du.$$

The replicating strategy ϕ satisfies for $t \in [0,T]$, on $\{t < \tau\}$,

$$\phi_t^1 = -KN\big(d_-(\widetilde{Y}_t^{2,1}, t, T)\big), \quad \phi_t^2 = N\big(d_+(\widetilde{Y}_t^{2,1}, t, T)\big).$$

6.8 Conclusions

- Pricing and hedging of any attainable survival claim with the promised payoff $X = g(\widetilde{Y}^1_T, \widetilde{Y}^2_T)$ depends on the choice of a default intensity only through the pre-default prices $\widetilde{Y}^1_t$ and $\widetilde{Y}^2_t$.
- The model considered here is incomplete, even if the notion of completeness is reduced to survival claims. Basically, a survival claim can be hedged if its promised payoff can be represents as $X = \widetilde{Y}^1_T h(\widetilde{Y}^{2,1}_T)$.
- The number of traded (default-free and defaultable) assets $Y^1, Y^2, \ldots, Y^k$ is arbitrary.

7 Open Problems

- Explicit necessary and sufficient conditions for the completeness of a model in terms of tradeable assets.
- Selection of tradeable assets for a given class of credit derivatives. The choice should be motivated by practical considerations (liquidity).
- The case of discontinuous prices of default-free and defaultable assets.
- The case of a general recovery scheme for defaultable assets.

8 Related Works

- In *"PDE approach to valuation and hedging of credit derivatives"* we develop the PDE approach in a Markovian set-up.
- In *"Pricing and trading credit default swaps"* we examine, in particular, hedging strategies for basket credit derivatives based on single name CDSs.
- In *"Hedging of convertible bonds in the default intensity set-up"* we study the valuation and hedging of convertible bonds with credit risk.