Rank-one Generated Spectral Cones Defined by Two Homogeneous Linear Matrix Inequalities

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Joint work with Fatma Kılınç-Karzan

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Nonconvex quadratic program

\[
\max_{x} \, x^\top Qx \\
\text{s.t. } x^\top M_i x \geq 0, \ i = 1, \ldots, k
\]
Introduction – Motivation

Nonconvex quadratic program

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\end{align*}
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We have \( x^\top A x = \text{Tr}(x^\top A x) = \text{Tr}(Axx^\top) = \langle A, xx^\top \rangle \).

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\begin{align*}
\max_{x} & \quad \langle Q, xx^\top \rangle \\
\text{s.t.} & \quad \langle M_i, xx^\top \rangle \geq 0, \ i = 1, \ldots, k
\end{align*}
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Nonconvex quadratic program

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s.t. $$x^\top M_i x \geq 0, \ i = 1, \ldots, k$$

We have $$x^\top A x = \text{Tr}(x^\top A x) = \text{Tr}(A xx^\top) = \langle A, xx^\top \rangle.$$  

$$\max_x \langle Q, xx^\top \rangle$$

s.t. $$\langle M_i, xx^\top \rangle \geq 0, \ i = 1, \ldots, k$$

Can write $$X = xx^\top$$ if and only if $$X \succeq 0$$ and $$\text{rank}(X) = 1.$$  

$$\max_X \langle Q, X \rangle$$

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$$X \succeq 0$$

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The condition \(\text{rank}(X) = 1\) is nonconvex.
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- When is this relaxation tight?
- Feasible set perspective.

Tight for every objective function if and only if every extreme ray is rank one (Rank-One Generated/ROG).

Analogous to integral polyhedra/total unimodularity.

Burer '15, Hildebrand '16, Blekherman et al. '16.
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Introduction – Our Question

Question
Let $M_1, M_2$ be $n \times n$ symmetric matrices.

When is
\[ S := \{ Y \succeq 0 : \langle Y, M_1 \rangle \geq 0, \langle Y, M_2 \rangle \geq 0 \} \]

an ROG cone?
Two geometric perspectives.
Each perspective gives a sufficient condition for $\mathcal{S}$ to be ROG. Together these conditions are also necessary.
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- $\langle M, Y \rangle = 0$ as a hyperplane in $\mathbb{S}^n := \{n \times n$ symmetric matrices$\}$. 
Two geometric perspectives.
Each perspective gives a sufficient condition for $S$ to be ROG. Together these conditions are also necessary.

- $\langle M, Y \rangle = 0$ as a hyperplane in $\mathbb{S}^n := \{n \times n \text{ symmetric matrices}\}$.
- $\langle M, xx^\top \rangle = x^\top M_1 x$ as a quadratic form in $\mathbb{R}^n$. 

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A set/cone is ROG if all its extreme points/rays have rank 1.
SDP relaxations of quadratic programs are tight for every objective function if and only if the feasible set is ROG.
Consider $\mathcal{S} := \{Y \succeq 0 : \langle Y, M_1 \rangle \geq 0, \langle Y, M_2 \rangle \geq 0\}$ (two LMI).
Two geometric perspectives – $\mathbb{S}^n$ and $\mathbb{R}^n$. 
Geometry of $\mathbb{S}^n_+$ – Rank

Consider $\mathbb{S}^n_+ := \{\text{positive semidefinite } n \times n \text{ matrices}\} \subseteq \mathbb{S}^n$.

- Red ray: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Green ray: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Rank 1 $\leftrightarrow$ extreme.
- Rank $\geq 2 \leftrightarrow$ not extreme.
Fact [Ye, Zhang ’03]

\[ S := \{ Y \succeq 0 : \langle M, Y \rangle \geq 0 \} \text{ is ROG for any } M \in S^3. \]
Geometry of $S^n_+$ – Two LMIs

Interacting inside $S^n_+$.  

Non-interacting inside $S^n_+$. 

ROG spectral cones defined by two LMIs
If $M_1$ and $M_2$ are non-interacting, then every extreme ray of

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is an extreme ray of either

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$\Rightarrow \mathcal{S}$ is ROG.
Non-interacting inside $\mathbb{S}_n^+$ when:

- One LMI does not intersect $\mathbb{S}_n^+$, i.e. when $\pm M_i \succeq 0$.

Using $\mathbb{S}$-lemma, this is true when $\lambda (\pm M_1) - (\pm M_2) \succeq 0$ for some $\lambda \geq 0$.

In sum, non-interacting when $\alpha M_1 + \beta M_2 \succeq 0$ for some $(\alpha, \beta) \neq (0,0)$.
Non-interacting inside $S^n_+$ when:

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- $\langle \pm M_1, X \rangle \geq 0$ is a consequence of $\langle \pm M_2, X \rangle \geq 0$ for $X \succeq 0$. 

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Geometry of $\mathbb{S}_+^n$ – Non-interacting LMIs

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In sum, non-interacting when $\alpha M_1 + \beta M_2 \succeq 0$ for some $(\alpha, \beta) \neq (0, 0)$. 
Non-interacting LMIs yield ROG cones.

$M_1, M_2$ are non-interacting if $\langle \pm M_2, Y \rangle \geq 0$ along with $Y \succeq 0$ implies $\langle \pm M_1, Y \rangle \geq 0$.

Proposition 1

If $\alpha M_1 + \beta M_2 \succeq 0$ has a nontrivial solution, i.e. $(\alpha, \beta) \neq (0, 0)$ then $S$ is ROG.
Question

In general, how do we show that a cone is ROG?
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Show that \( Y \notin \text{Ext}(S) \) when:
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Show that $Y \notin \text{Ext}({\mathcal{S}})$ when:

- $\text{rank}(Y) \geq 2$.
- $\langle Y, M_1 \rangle = \langle Y, M_2 \rangle = 0$. 
Showing $Y \not\in \text{Ext}(\mathcal{S})$

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Find $x \in \mathbb{R}^n$ such that $Y \pm xx^\top \in \mathcal{S}$. 

For $Y - xx^\top \succeq 0$, need $x \in \text{Range}(Y)$. Since $\langle Y, M_i \rangle = 0$, need $0 = \langle xx^\top, M_i \rangle = x^\top M_i x$. 

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Showing $Y \notin \text{Ext}(S)$

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Fix a candidate extreme ray $Y$. Define

$$\mathcal{N}_1 := \{ x \in \mathbb{R}^n : x^\top M_1 x = 0 \}.$$  
$$\mathcal{N}_2 := \{ x \in \mathbb{R}^n : x^\top M_2 x = 0 \}.$$  

When is $\text{Range}(Y) \cap \mathcal{N}_1 \cap \mathcal{N}_2 \neq \{0\}$?
Quadratic Forms

Start with $n = 3$.
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- If $\mathcal{N}_1 \cap \mathcal{N}_2 \subseteq \mathbb{R}^3$ contains a plane, then it intersects every plane nontrivially.
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- Range$(Y)$ is a plane.
- If $\mathcal{N}_1 \cap \mathcal{N}_2 \subseteq \mathbb{R}^3$ contains a plane, then it intersects every plane nontrivially.

Observation

$\mathcal{S}$ is ROG when $\mathcal{N}_1 \cap \mathcal{N}_2$ contains a plane.
Quadratic Forms – $\mathcal{N}_1 \cap \mathcal{N}_2$ contains a plane

**Question**

When does $\mathcal{N}_1 \cap \mathcal{N}_2$ contain a plane?

\[
\begin{align*}
\text{For any } (\alpha, \beta), \quad \mathcal{N}_{\alpha,\beta} := \{ x \in \mathbb{R}^3 : x^\top (\alpha M_1 + \beta M_2) x = 0 \} \supseteq \mathcal{N}_1 \cap \mathcal{N}_2. 
\end{align*}
\]

In particular, $\mathcal{N}_{\alpha,\beta}$ contains a plane for all $(\alpha, \beta)$.

**Answer**

$\star$ When $\text{rank}(\alpha M_1 + \beta M_2) \leq 2$ for all $(\alpha, \beta)$.
Quadratic Forms – $\mathcal{N}_1 \cap \mathcal{N}_2$ contains a plane

Question

When does $\mathcal{N}_1 \cap \mathcal{N}_2$ contain a plane?

- $\{x \in \mathbb{R}^3 : x^\top M x = 0\}$ contains a plane when $\text{rank}(M) \leq 2$ and $M$ is indefinite.
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Answer*
When \( \text{rank}(\alpha M_1 + \beta M_2) \leq 2 \) for all \( (\alpha, \beta) \).
$Y$ is not an extreme ray when $\text{Range}(Y) \cap \mathcal{N}_1 \cap \mathcal{N}_2$ has a nonzero element.
Geometry of Quadratic Forms – Recap

- $Y$ is not an extreme ray when $\text{Range}(Y) \cap \mathcal{N}_1 \cap \mathcal{N}_2$ has a nonzero element.
- In $\mathbb{R}^3$, if $\mathcal{N}_1 \cap \mathcal{N}_2$ contains a plane, no rank 2 extreme rays.
Geometry of Quadratic Forms – Recap

- $Y$ is not an extreme ray when $\text{Range}(Y) \cap N_1 \cap N_2$ has a nonzero element.
- In $\mathbb{R}^3$, if $N_1 \cap N_2$ contains a plane, no rank 2 extreme rays.
- $N_1 \cap N_2$ contains a plane when $\text{rank}(\alpha M_1 + \beta M_2) \leq 2$ for all $\alpha, \beta$.
Y is not an extreme ray when \( \text{Range}(Y) \cap \mathcal{N}_1 \cap \mathcal{N}_2 \) has a nonzero element.

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**Proposition 2**

\( S \) is ROG when \( \text{rank}(\alpha M_1 + \beta M_2) \leq 2 \) for all \( (\alpha, \beta) \), \( \text{Span}\{\text{Range}(M_1) \cup \text{Range}(M_2)\} \) has dimension 3, and \( \alpha M_1 + \beta M_2 \succeq 0 \) has only the trivial solution \( (\alpha, \beta) = (0, 0) \).
Main Result

**Theorem 3 (A, Kılınç-Karzan, '17)**

\[ \{ Y \succeq 0 : \langle M_1, Y \rangle \succeq 0, \langle M_2, Y \rangle \succeq 0 \} \text{ is ROG iff one of the following holds} \]

(i) \( \alpha M_1 + \beta M_2 \succeq 0 \) for some \( (\alpha, \beta) \neq (0, 0) \).

(ii) \( \text{rank}(\alpha M_1 + \beta M_2) \leq 2 \) for all \( (\alpha, \beta) \) and \( \text{Span}\{\text{Range}(M_1) \cup \text{Range}(M_2)\} \) has dimension 3.
Proving Necessity (Sketch)

First consider the case of $\mathbb{S}^3$. Suppose that:

(i) $\alpha M_1 + \beta M_2 \not\succeq 0$ for any $(\alpha, \beta) \neq (0, 0)$.

(ii) $\text{rank}(aM_1 + bM_2) \geq 3$ for some $(a, b)$. 
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- $\text{rank}(a M_1 + b M_2) = 3$ implies that $\mathcal{N}_1 \cap \mathcal{N}_2$ is “sparse.”
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- Use infeasibility of $\alpha M_1 + \beta M_2 \succeq 0$ for $(\alpha, \beta) \neq (0, 0)$ to get $w$ such that $Y = zz^T + ww^T$ is tight for both LMI's ($w \neq \lambda z$ for $\lambda \in \mathbb{R}$).
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We reduce the general case of $\mathbb{S}^n$ to the case of $\mathbb{S}^3$. 
Extensions/Questions

- Necessary and sufficient conditions for more than 2 LMIs.
- Use results to analyze conic constraints.
  - Alternate analysis of Burer’s work on extensions of the Trust Region Subproblem.
  - Necessary and sufficient conditions for more general conic constraints.
Thank you!
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Further Reading

A Gentle Geometric Introduction to Copositive Optimization.
*Mathematical Programming, June 2015, Volume 151, Issue 1, pp 89-116.*

Roland Hildebrand (2016).
Spectrahedral Cones Generated by Rank-1 Matrices

Grigoriy Blekherman et al. (2016).
Do Sums of Squares Dream of Free Resolutions?