21-832 PDE II Fall 2016 R.L. Pego

Homework 2 Due Friday, Oct. 7

1. (Interpolation inequalities) Suppose $0 < \theta < 1$.

   (a) Assume $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$ with α_1, $\alpha_2 \in [0,1]$. Show that the Hölder seminorm satisfies
   $$|u|_\alpha \le |u|_{\alpha_1}^\theta |u|_{\alpha_2}^{1-\theta}$$
   for any $u \in C^{\alpha_1} \cap C^{\alpha_2}$.

   (b) Assume $s = \theta s_1 + (1-\theta)s_2$ with s_1, $s_2 \in \mathbb{R}$. Show that the H^s Sobolev norm satisfies
   $$\|u\|_{H^s} \le \|u\|_{H^{s_1}}^\theta \|u\|_{H^{s_2}}^{1-\theta}$$
   for all $u \in \mathcal{S}(\mathbb{R}^n)$, hence for all $u \in H^{s_1}(\mathbb{R}^n) \cap H^{s_2}(\mathbb{R}^n)$.

2. (Trace) For $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ write $z = (x,y) \in \mathbb{R}^{m+n}$. For $u \in \mathcal{S}(\mathbb{R}^{m+n})$ let $Tu(x) = u(x,0)$ be the restriction of u to the codimension-n hypersurface $y = 0$, $\mathbb{R}^m \times \{0\}$. Prove that if $r = s - \frac{n}{2} > 0$ then there is a constant C such that
   $$\|Tu\|_{H^r(\mathbb{R}^m)} \le C\|u\|_{H^s(\mathbb{R}^{m+n})}$$
   for all $u \in \mathcal{S}(\mathbb{R}^{m+n})$. Hence T extends to a bounded linear operator from $H^s(\mathbb{R}^{m+n})$ to $H^r(\mathbb{R}^m)$. Use the notation $\mathcal{F}_x$, $\mathcal{F}_y$ for the Fourier transforms in x and in y respectively. Hint: Show
   $$\sup_{a\ge 1} \int_{\mathbb{R}^n} a^r (a + |\xi|^2)^{-s}\, d\xi < \infty.$$

3. (a) (Evans 5.10.7) Suppose Ω is bounded and has C^1 boundary, and there exists a C^1 vector field $g : \bar{\Omega} \to \mathbb{R}^n$ such that $g \cdot \nu \ge 1$ on $\partial\Omega$, where ν is the outward unit normal to Ω. Assume $1 \le p < \infty$. Apply the divergence theorem and deduce there is a constant C such that
   $$\int_{\partial\Omega} |u|^p dS \le \int_{\partial\Omega} |u|^p g \cdot \nu\, dS \le C \int_\Omega \left(|Du|^p + |u|^p\right)\, dx$$
   for all $u \in C^1(\bar{\Omega})$.

   (b) (Evans 5.10.14) Verify that if $n > 1$ and $\Omega = B(0,1)$, the unbounded function $u(x) = \log\log(1 + 1/|x|)$ belongs to $W^{1,n}(\Omega)$.

4. For the first two parts, let H be a Hilbert space, and suppose $\{u_j\}$ is a sequence in H that converges *weakly* in H to some u in H. Recall that necessarily $\|u\|_H \leq \liminf \|u_j\|_H$.

   (a) Let K be a second Hilbert space, and let $A : H \to K$ be a bounded linear operator. Show that Au_j converges weakly to Au in K.

   (b) Assume in addition that $\|u_j\|_H^2 \to \|u\|_H^2$. Show that $\|u_j - u\|_H^2 \to 0$.

   (c) (A general Poincaré inequality) Let $\Omega \subset \mathbb{R}^n$ be open, connected, and bounded. Suppose X is a closed subspace of the Hilbert space $H^1(\Omega)$ such that $1 \notin X$. Prove that then there is a constant C such that

$$\int_\Omega |u|^2 \, dx \leq C \int_\Omega |Du|^2 \, dx$$

   for all $u \in X$. (Argue by contradiction: select u_j that violates the estimate with $C = j$, normalize so the H^1 norm of u_j is 1, and select a subsequence that converges *weakly* to some u in the Hilbert space $H^1(\Omega)$. Argue $Du = 0$ in the sense of distributions, and $u \in X$, and more...)

5. Let $\Omega \subset \mathbb{R}^n$ be open, connected, and bounded, and let X be the closed subspace of elements q of $H^1(\Omega)$ that satisfy $\int_\Omega q \, dx = 0$. Let $u \in L^2(\Omega, \mathbb{R}^n)$, and for $q \in H^1(\Omega)$ define

$$E(q) = \int_\Omega \left(\frac{1}{2} |\nabla q|^2 - u \cdot \nabla q \right) dx$$

   Show that $e_0 = \inf_{q \in X} E(q) > -\infty$. Suppose $\{q_j\}$ is a *minimizing sequence* for E, satisfying $E(q_j) \to e_0$. Then there is a subsequence converging weakly to some $q \in X$. Show that $E(q) = e_0$, and that q satisfies

$$\Delta q = \nabla \cdot u$$

   in the sense of distributions on Ω.

6. (An anisotropic Sobolev embedding) Let

$$W_2^{2,1}(\mathbb{R}^2) := \{u \in \mathcal{S}'(\mathbb{R}^2) \colon D^\alpha u \in L^2(\mathbb{R}^2) \text{ whenever } 0 \leq 2\alpha_1 + \alpha_2 \leq 2\}.$$

   The norm on the space $X = W_2^{1,2}(\mathbb{R}^2)$ is defined as $\|u\|_X = \left(\sum_{0 \leq 2\alpha_1 + \alpha_2 \leq 2} \|D^\alpha u\|_{L^2}^2 \right)^{1/2}$.
   (i) Let $g(\xi) = (1 + \xi_1^2 + \xi_2^4)^{1/2}$, $\xi \in \mathbb{R}^2$. Show that $X = \{u \in \mathcal{S}'(\mathbb{R}^2) \colon g\hat{u} \in L^2(\mathbb{R}^2)\}$, and for some $C > 0$, for all $u \in X$ we have

$$C^{-1} \|u\|_X \leq \|g\hat{u}\|_{L^2} \leq C \|u\|_X$$

   (ii) Show that there is some $C > 0$ such that for all $u \in X$,

$$\|u\|_\infty \leq C \|u\|_X$$