1. (Interpolation inequalities) Suppose $0 < \theta < 1$.

(a) Assume $\alpha = \theta \alpha_1 + (1 - \theta) \alpha_2$ with $\alpha_1, \alpha_2 \in [0,1]$. Show that the Hölder seminorm satisfies

$$|u|_\alpha \leq |u|_{\alpha_1}^{\theta} |u|_{\alpha_2}^{1-\theta}$$

for any $u \in C^{\alpha_1} \cap C^{\alpha_2}$.

(b) Assume $s = \theta s_1 + (1 - \theta) s_2$ with $s_1, s_2 \in \mathbb{R}$. Show that the $H^s$ Sobolev norm satisfies

$$\|u\|_{H^s} \leq \|u\|_{H^{s_1}}^{\theta} \|u\|_{H^{s_2}}^{1-\theta}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$, hence for all $u \in H^{s_1}(\mathbb{R}^n) \cap H^{s_2}(\mathbb{R}^n)$.

2. (Trace) For $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ write $z = (x, y) \in \mathbb{R}^{m+n}$. For $u \in \mathcal{S}(\mathbb{R}^{m+n})$ let $Tu(x) = u(x,0)$ be the restriction of $u$ to the codimension-$n$ hypersurface $y = 0$, $\mathbb{R}^m \times \{0\}$. Prove that if $r = s - 2n > 0$ then there is a constant $C$ such that

$$\|Tu\|_{H^r(\mathbb{R}^m)} \leq C\|u\|_{H^s(\mathbb{R}^{m+n})}$$

for all $u \in \mathcal{S}(\mathbb{R}^{m+n})$. Hence $T$ extends to a bounded linear operator from $H^s(\mathbb{R}^{m+n})$ to $H^r(\mathbb{R}^m)$. Use the notation $\mathcal{F}_x, \mathcal{F}_y$ for the Fourier transforms in $x$ and in $y$ respectively. Hint: Show

$$\sup_{a \geq 1} \int_{\mathbb{R}^n} a^r (a + |\xi|^2)^{-s} d\xi < \infty.$$ 

3. (a) (Evans 5.10.7) Suppose $\Omega$ is bounded and there exists a $C^1$ vector field $g : \Omega \to \mathbb{R}^n$ such that $g \cdot \nu \geq 1$ on $\partial \Omega$, where $\nu$ is the outward unit normal to $\Omega$. Assume $1 \leq p < \infty$. Apply the divergence theorem and deduce there is a constant $C$ such that

$$\int_{\partial \Omega} |u|^p dS \leq \int_{\partial \Omega} |u|^p g \cdot \nu dS \leq C \int_{\Omega} (|Du|^p + |u|^p) dx$$

for all $u \in C^1(\bar{\Omega})$.

(b) (Evans 5.10.14) Verify that if $n > 1$ and $\Omega = B(0,1)$, the unbounded function $u(x) = \log \log(1 + 1/|x|)$ belongs to $W^{1,n}(\Omega)$. 


4. For the first two parts, let $H$ be a Hilbert space, and suppose $\{u_j\}$ is a sequence in $H$ that converges \textit{weakly} in $H$ to some $u$ in $H$. Recall that necessarily $\|u\|_H \leq \liminf \|u_j\|_H$.

(a) Let $K$ be a second Hilbert space, and let $A : H \to K$ be a bounded linear operator. Show that $Au_j$ converges weakly to $Au$ in $K$.

(b) Assume in addition that $\|u_j\|^2_H \to \|u\|^2_H$. Show that $\|u_j - u\|^2_H \to 0$.

(c) (A general Poincaré inequality) Let $\Omega \subset \mathbb{R}^n$ be open, connected, and bounded. Suppose $X$ is a closed subspace of the Hilbert space $H^1(\Omega)$ such that $1 \notin X$. Prove that then there is a constant $C$ such that

$$\int_{\Omega} |u|^2 \, dx \leq C \int_{\Omega} |Du|^2 \, dx$$

for all $u \in X$. (Argue by contradiction: select $u_j$ that violates the estimate with $C = j$, normalize so the $H^1$ norm of $u_j$ is 1, and select a subsequence that converges \textit{weakly} to some $u$ in the Hilbert space $H^1(\Omega)$. Argue $Du = 0$ in the sense of distributions, and $u \in X$, and more...)

5. Let $\Omega \subset \mathbb{R}^n$ be open, connected, and bounded, and let $X$ be the closed subspace of elements $q$ of $H^1(\Omega)$ that satisfy $\int_{\Omega} q \, dx = 0$. Let $u \in L^2(\Omega, \mathbb{R}^n)$, and for $q \in H^1(\Omega)$ define

$$E(q) = \int_{\Omega} \left( \frac{1}{2} |\nabla q|^2 - u \cdot \nabla q \right) \, dx$$

Show that $e_0 = \inf_{q \in X} E(q) > -\infty$. Suppose $\{q_j\}$ is a \textit{minimizing sequence} for $E$, satisfying $E(q_j) \to e_0$. Then there is a subsequence converging weakly to some $q \in X$. Show that $E(q) = e_0$, and that $q$ satisfies

$$\Delta q = \nabla \cdot u$$

in the sense of distributions on $\Omega$. 

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