21-832 PDE II   Fall 2016   R.L. Pego

Homework 1   Due Friday, Sept. 16

1. Find the distributional derivative of the function $f(x) = \ln|x|$ in $\mathcal{D}'(\mathbb{R})$.

2. (Distribution limit of harmonic functions) Let Ω be open in $\mathbb{R}^n$, and let $\{f_j\}$ be a sequence of harmonic functions in Ω that converges *in the sense of distributions* to some $T \in \mathcal{D}'(\Omega)$, meaning
$$\forall \phi \in \mathcal{D}(\Omega), \qquad (T, \phi) = \lim_{j\to\infty} \int_\Omega f_j(x)\phi(x)\,dx.$$
It is a consequence of the Banach-Steinhaus theorem on Fréchet spaces that this convergence is uniform in ϕ for ϕ in any *bounded* subset of $\mathcal{D}(\Omega)$. (A set $E \subset \mathcal{D}(\Omega)$ is bounded if all $\phi \subset E$ are supported in the same compact set $K \subset \Omega$ and $\sup_{\phi\in E}\sup_{x\in K}|D^\alpha\phi(x)| < \infty$ for every multiindex α.)

   Prove that f_j converges uniformly on every compact subset of Ω to a harmonic function f where $f = T$ in $\mathcal{D}'(\Omega)$. (Suggestion: recall the use of mean values and mollifiers last term.)

3. In $\mathbb{R}^3$, find a fundamental solution of the biharmonic equation $\Delta\Delta u = f$, i.e., find a distribution G on $\mathbb{R}^3$ satisfying $\Delta\Delta G = \delta$. (Suggestion: Seek $G = g(|x|)$ where $g(r)$ is a radially symmetric bi-harmonic function away from $r = 0$.)

4. Let Ω be open in $\mathbb{R}^n$, and suppose $f, g : \Omega \to \mathbb{R}$ are continuous and have compact support in Ω. Show that if some partial derivative $\partial_{x_i} f = g$ in the sense of distributions, then the partial derivative $\partial_{x_i} f(x)$ exists for each $x \in \Omega$ in the classical sense and $\partial_{x_i} f(x) = g(x)$. (Hint: Mollify.)

5. (a) Let ϕ be a test function on $(-\pi, \pi)$. Regarding e^{ikx} as a (complex-valued) distribution, show that
$$|(e^{ikx}, \phi)| \le 2\pi|k|^{-n}\|D^n\phi\|_\infty\ .$$
   (b) *Schrödinger's equation* $iu_t = u_{xx}$ is linear and has solutions of the form $u_k(x,t) = e^{itk^2+ikx}$. Show that for fixed $t \ge 0$, the infinite series
$$u(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{itk^2+ikx}$$
   converges in the sense of distributions on $(-\pi, \pi)$. (Note $u(0) = \delta$. $u(t)$ is the (very strange) fundamental solution of Schrödinger's equation for 2π-periodic solutions.)

6. Let $f(x) = e^x$ and $g(x) = e^x\cos(e^x)$. Show g is a tempered distribution but that f is not.