Due Wednesday, Nov. 13

6.1. Let \((X, \mathcal{F}, \mu)\) be a measure space, and suppose \(g : X \to \mathbb{R}\) is integrable and \(\int_E gd\mu = 0\) for all \(E \in \mathcal{F}\). Show \(g = 0\) \(\mu\)-a.e.

6.2. Let \(\mu\) and \(\nu\) be finite measures on a measurable space \((X, \mathcal{F})\). Suppose \(\nu \ll \mu\), \(\lambda = \mu + \nu\) and the Radon-Nikodým derivative \(f = \frac{d\nu}{d\lambda}\). Show that \(0 \leq f < 1\) \(\mu\)-a.e., and

\[
\frac{d\nu}{d\mu} = \frac{f}{1 - f}.
\]

6.3. Let \((X, \mathcal{F})\) be a measurable space. Let \(\mathcal{M}\) be the vector space of all finite signed measures (charges) \(\mu : \mathcal{F} \to \mathbb{R}\), with the vector operations

\[
(c\mu)(E) = c\mu(E), \quad (\mu + \nu)(E) = \mu(E) + \nu(E).
\]

With norm \(\|\mu\| = |\mu|(X)\) given by total variation, show \(\mathcal{M}\) is a Banach space (i.e., it is complete).

6.4. Does there exist a \(\sigma\)-finite Borel measure on \((\mathbb{R}, \mathcal{B})\) such that \(\mu(I) = +\infty\) for all open \(I \neq \emptyset\)?

6.5. For \(j = 1, 2\) let \(\mu_j, \nu_j\) be \(\sigma\)-finite measures on \((X_j, \mathcal{F}_j)\) such that \(\nu_j \ll \mu_j\). Prove that \(\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2\) and

\[
\frac{d(\nu_1 \times \nu_2)}{d\mu_1 \times d\mu_2}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2) \quad \text{a.e.}
\]

Here \(d\nu_j/d\mu_j\) denotes the Radon-Nikodým derivative satisfying

\[
\nu_j(E) = \int_E \frac{d\nu_j}{d\mu_j} d\mu_j \quad \text{for } E \in \mathcal{F}_j.
\]