21-720 F09: Measure and Integration, Assignment 5 R. Pego

Due Wednesday, December 2

5.1. Let ν be a regular signed Borel measure on $\mathbb{R}^n$, with $\nu = \nu_1 + \nu_2$ its Lebesgue decomposition ($|\nu_1| \ll m$, $|\nu_2| \perp m$). Prove (a) $|\nu| = |\nu_1| + |\nu_2|$, and (b) $|\nu_1|$ and $|\nu_2|$ are regular.

5.2. Suppose $\{\mu_k\}$ is a sequence of regular Borel measures on $\mathbb{R}^n$, and

$$\mu(E) = \sum_{k=1}^{\infty} \mu_k(E).$$

Assume $\mu(\mathbb{R}^n) < \infty$. (a) Show that μ is a regular Borel measure. (b) What is the relation between the Lebesgue decompositions of the μ_k and that of μ? If we use the notation $D\mu(x)$ to denote the function defined for a.e. x by

$$D\mu(x) = \lim_{r \to 0} \frac{\mu(B(r,x))}{m(B(r,x))},$$

show that $D\mu(x) = \sum_{k=1}^{\infty} D\mu_k(x)$ a.e.

5.3. Let F equal the Cantor function on $[0,1]$ (a continuous increasing function with $F(0) = 0$, $F(1) = 1$, $F'(x) = 0$ for all $x \in [0,1] \setminus \mathcal{C}$ where $\mathcal{C}$ is the Cantor set), and let $F(x) = 0$ for $x < 0$, $F(x) = 1$ for $x > 1$. Let $\{[a_n, b_n]\}$ be an enumeration of the closed nonempty subintervals of $[0,1]$ with rational endpoints, and let $F_n(x) = F((x - a_n)/(b_n - a_n))$. Show $G = \sum_{n=1}^{\infty} 2^{-n} F_n$ is continuous and *strictly* increasing on $[0,1]$, and $G'(x) = 0$ a.e.

5.4. If $E \subset \mathbb{R}^n$ is a Borel set in $\mathbb{R}^n$, the (Lebesgue) *density* of E at x is defined as

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))},$$

whenever the limit exists. Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

5.5. If X is a Banach space, a set $A \subset X$ is called *precompact* if each sequence $\{f_k\}$ in S has some subsequence that converges to some $f \in X$. Let $A \subset L_p(\mathbb{R}^n, m)$, where $1 < p < \infty$, and define

$$\tau(R) = \sup_{f \in A} \int_{|x| > R} |f|^p \, dm.$$

Prove that if A is precompact, then $\tau(R) \to 0$ as $R \to \infty$.

5.6. (a) Let $f \in L_p(\mathbb{R}^n, m)$, where $1 < p < \infty$. For $h \in \mathbb{R}^n$ define $S_h f(x) = f(x + h)$. Prove $\|S_h f - f\|_p \to 0$ as $|h| \to 0$.
(b) Suppose $A \subset L_p(\mathbb{R}^n, m)$, and for $h \in \mathbb{R}^n$ define

$$\sigma(h) = \sup_{f \in A} \|S_h f - f\|_p.$$

Prove that if A is precompact, then $\sigma(h) \to 0$ as $h \to 0$.
(Remarks: Together, the two necessary criteria for precompactness in 4.5 and 4.6 are also sufficient. This is a classical result due to Riesz and Tamarkin. A 1985 result of mine says that the two criteria are mapped each to the other by the Fourier transform.)