Due Friday, November 13

4.1. Let \((X, \mathcal{F}, \mu)\) be a measure space, and suppose \(g : X \to \mathbb{R}\) is integrable and \(\int_E g \, d\mu = 0\) for all \(E \in \mathcal{F}\). Show \(g = 0\) \(\mu\)-a.e.

4.2. (a) Let \(A \subset \mathbb{R}^n\) be Lebesgue measurable with \(m(A) < \infty\) and let \(\varepsilon > 0\). Show that there exists an open set \(G \supset A\) with \(m(A) \leq m(G) < m(A) + \varepsilon\).

(b) Let \(B \subset \mathbb{R}^n\) be Lebesgue measurable with \(m(B) < \infty\) and let \(\varepsilon > 0\). Show there exists a compact set \(K \subset B\) such that \(m(K) \leq m(B) < m(K) + \varepsilon\).

(c) Deduce that for each Lebesgue measurable set \(A \subset \mathbb{R}^n\),
\[
m(A) = \inf \{m(G) : G \text{ is open and } A \subset G\} = \sup \{m(K) : K \text{ is compact and } K \subset A\}.
\]

4.3. (a) Let \(B\) be the \(\sigma\)-algebra of Borel sets in \(\mathbb{R}\). Show that every open set in \(\mathbb{R} \times \mathbb{R}\) is in \(B \otimes B\). Deduce that \(B \otimes B\) is the \(\sigma\)-algebra of Borel sets in \(\mathbb{R} \times \mathbb{R}\).

(b) Let \(E \subset \mathbb{R}\) be a Borel set, and \(A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in E\}\). Show that \(A \in B \otimes B\), where \(B\) \(\sigma\)-algebra of Borel sets in \(\mathbb{R}\).

4.4. (a) Let \((X, \mathcal{F})\), \((Y, \mathcal{G})\) be measurable spaces and suppose \(f : X \to \mathbb{R}\) is \(\mathcal{F}\)-measurable and \(g : Y \to \mathbb{R}\) is \(\mathcal{G}\)-measurable. Define \(h(x, y) = f(x)g(y)\). Prove \(h\) is \(\mathcal{F} \otimes \mathcal{G}\) measurable. (Hint: The product of two measurable functions defined on the same space is measurable.)

(b) (convolution) Let \(B\) be the \(\sigma\)-algebra of Borel sets in \(\mathbb{R}\), and let \(f\) and \(g\) be Lebesgue integrable functions on \((\mathbb{R}, B)\). From results above it follows that \(F(x, y) = f(x - y)g(y)\) is \(B \otimes B\)-measurable. If \(m\) denotes Lebesgue measure on \(B\), use Tonelli’s theorem and the fact that
\[
\int_{\mathbb{R}} |f(x-y)| \, dm(x) = m(\mathbb{R}) \int_{\mathbb{R}} |f(y)| \, dm(y)
\]
to show that \(h(x) = \int_{\mathbb{R}} f(x-y) g(y) \, dm(y)\) is almost everywhere well-defined and satisfies
\[
\int |h| \, dm \leq \left(\int |f| \, dm\right) \left(\int |g| \, dm\right)\]

4.5. For \(j = 1, 2\) let \(\mu_j\), \(\nu_j\) be \(\sigma\)-finite measures on \((X_j, \mathcal{F}_j)\) such that \(\nu_j \ll \mu_j\). Prove that \(\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2\) and
\[
\frac{d(\nu_1 \times \nu_2)}{\mu_1 \times \mu_2}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2) \quad \text{a.e.}
\]
Here \(d\nu_j/d\mu_j\) denotes the Radon-Nikodym derivative satisfying
\[
\nu_j(E) = \int_E \frac{d\nu_j}{d\mu_j} \, d\mu_j \quad \text{for } E \in \mathcal{F}_j.
\]