

Due Friday, November 13

4.1. Let (X, \mathcal{F}, μ) be a measure space, and suppose $g : X \rightarrow \mathbb{R}$ is integrable and $\int_E g d\mu = 0$ for all $E \in \mathcal{F}$. Show $g = 0$ μ -a.e.

4.2. (a) Let $A \subset \mathbb{R}^n$ be Lebesgue measurable with $m(A) < \infty$ and let $\varepsilon > 0$. Show that there exists an open set $G \supset A$ with $m(A) \leq m(G) < m(A) + \varepsilon$.

(b) Let $B \subset \mathbb{R}^n$ be Lebesgue measurable with $m(B) < \infty$ and let $\varepsilon > 0$. Show there exists a compact set $K \subset B$ such that $m(K) \leq m(B) < m(K) + \varepsilon$.

(c) Deduce that for each Lebesgue measurable set $A \subset \mathbb{R}^n$,

$$\begin{aligned} m(A) &= \inf\{m(G) : G \text{ is open and } A \subset G\} \\ &= \sup\{m(K) : K \text{ is compact and } K \subset A\}. \end{aligned}$$

4.3. (a) Let \mathcal{B} be the σ -algebra of Borel sets in \mathbb{R} . Show that every open set in $\mathbb{R} \times \mathbb{R}$ is in $\mathcal{B} \otimes \mathcal{B}$. Deduce that $\mathcal{B} \otimes \mathcal{B}$ is the σ -algebra of Borel sets in $\mathbb{R} \times \mathbb{R}$.

(b) Let $E \subset \mathbb{R}$ be a Borel set, and $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in E\}$. Show that $A \in \mathcal{B} \otimes \mathcal{B}$, where \mathcal{B} σ -algebra of Borel sets in \mathbb{R} .

4.4. (a) Let (X, \mathcal{F}) , (Y, \mathcal{G}) be measurable spaces and suppose $f : X \rightarrow \mathbb{R}$ is \mathcal{F} -measurable and $g : Y \rightarrow \mathbb{R}$ is \mathcal{G} -measurable. Define $h(x, y) = f(x)g(y)$. Prove h is $\mathcal{F} \otimes \mathcal{G}$ measurable. (Hint: The product of two measurable functions defined on the same space is measurable.)

(b) (convolution) Let \mathcal{B} be the σ -algebra of Borel sets in \mathbb{R} , and let f and g be Lebesgue integrable functions on $(\mathbb{R}, \mathcal{B})$. From results above it follows that $F(x, y) = f(x - y)g(y)$ is $\mathcal{B} \otimes \mathcal{B}$ -measurable. If m denotes Lebesgue measure on \mathcal{B} , use Tonelli's theorem and the fact that

$$\int_{\mathbb{R}} |f(x - y)| dm(x) = \int_{\mathbb{R}} |f(x)| dm(x)$$

to show that $h(x) = \int_{\mathbb{R}} f(x - y)g(y) dm(y)$ is almost everywhere well-defined and satisfies

$$\int |h| dm \leq \left(\int |f| dm \right) \left(\int |g| dm \right).$$

4.5. For $j = 1, 2$ let μ_j, ν_j be σ -finite measures on (X_j, \mathcal{F}_j) such that $\nu_j \ll \mu_j$. Prove that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2) \quad \text{a.e.}$$

Here $d\nu_j/d\mu_j$ denotes the Radon-Nikodým derivative satisfying

$$\nu_j(E) = \int_E \frac{d\nu_j}{d\mu_j} d\mu_j \quad \text{for } E \in \mathcal{F}_j.$$