Due Wednesday, Oct. 9

4.1. (a) If \( \mu(X) < \infty \) and \( 1 \leq p < q \leq \infty \), show \( L_q(X) \subset L_p(X) \) and the inclusion map from \( L_q(X) \to L_p(X) \) is continuous. (Hint: Show \( \|f\|_p \leq \mu(X)^\theta \|f\|_q \) for some \( \theta > 0 \).)

(b) Let \( \ell_p = L_p(\mathbb{N}) \) with respect to counting measure. If \( 1 \leq p < r \) show that \( \ell_p \subset \ell_r \) but \( \ell_p \neq \ell_r \). Is the inclusion map \( \ell_p \to \ell_r \) continuous? Prove your answer.

4.2. (a) Let \( p \in (1, \infty) \). Prove the Sobolev-type inequality

\[
\|f\|_\infty \leq p^{1/p} \|f\|^{(p-1)/p} \|f\|^{1/p}
\]

for every \( f : \mathbb{R} \to \mathbb{R} \) that is \( C^1 \) with compact support. (Hint: \( g(x) = |f(x)|^p = \int_{-\infty}^{x} g'(z) \, dz \).

(b) Find all possible values of \( p, q \in (1, \infty) \) and \( r \in (0, 1) \) such that there is a constant \( C \) independent of \( f \) such that

\[
\|f\|_\infty \leq C \|f\|_p^{1-r} \|f\|_q^r
\]

for every \( f : \mathbb{R} \to \mathbb{R} \) that is \( C^1 \) with compact support. (Hint: Identify necessary conditions by considering \( f \) of the form \( f(x) = h(ax) \) for arbitrary \( a > 0 \). Then consider \( g(x) = |f(x)|^r \).

4.3. Show that Fatou’s lemma remains valid if almost-everywhere convergence is replaced by convergence in measure.

4.4. Suppose \( \mu(X) < \infty \). For \( \hat{f} : X \to \mathbb{R} \) measurable define

\[
r(\hat{f}) = \int \frac{|\hat{f}|}{1 + |f|} \, d\mu,
\]

and define \( r(f) = r(\hat{f}) \) whenever \( f = [\hat{f}] \) is the equivalence class of measurable \( g \) with \( g = \hat{f} \mu \text{-a.e.} \)

(a) Show that \( d(f, g) = r(f - g) \) defines a metric on the set \( M \) of these equivalence classes.

(b) Given any sequence \( (\hat{f}_n)_{n \in \mathbb{N}} \) of such measurable functions, show that \( [\hat{f}_n] \to [\hat{f}] \) in the metric \( d \) if and only if \( \hat{f}_n \) converges to \( \hat{f} \) in measure.

4.5. Let \( (X, \mathcal{F}, \mu) \) be an arbitrary measure space, and let \( p \in [1, \infty) \). Suppose \( \varphi : \mathbb{R} \to \mathbb{R} \) is continuous and has the property that for some constant \( M \geq 0 \), \( |\varphi(t)| \leq M|t| \) for all real \( t \).

(a) Prove that if \( f \in L_p \), then \( \varphi \circ f \in L_p \).

(b) Let \( (f_n) \) be a sequence that converges to \( f \) in \( L_p \). Prove that \( \varphi \circ f_n \) converges to \( \varphi \circ f \) in \( L_p \).

4.6. Let \( B(X) \) be the vector space of (true) functions \( f : X \to \mathbb{R} \) that are bounded, and define

\[
|f|_\infty = \sup_x |f(x)| \quad \text{for } f \in B(X).
\]

Suppose that a given map \( F : [0, 1] \to L_\infty = L_\infty(X, \mathcal{F}, \mu) \) is continuous. (Recall: for each \( t \), \( F(t) \) is an equivalence class.) Prove that there is a function \( g : [0, 1] \to B(X) \) such that \( g(t) \in F(t) \) for all \( t \in [0, 1] \), and

\[
|g(t) - g(s)|_\infty = \|F(t) - F(s)\|_\infty \quad \text{for all } t, s \in [0, 1].
\]

(I suggest you start with a dense sequence \( (t_k) \) in \( [0, 1] \), elements \( f_k \in F(t_k) \), and find \( \mu \)-null sets \( D_k \supset D_{k-1} \) such that \( \sup_{x \in D_k} |f_k(x) - f_j(x)| = \|F(t_k) - F(t_j)\|_\infty \) for \( 1 \leq j < k \).)