4.1. Let \((X, \mathcal{F})\) be a measurable space. Let \(\mathcal{M}\) be the vector space of all finite signed measures \(\mu : \mathcal{F} \to \mathbb{R}\), with the vector operations
\[
(c\mu)(E) = c\mu(E), \quad (\mu + \nu)(E) = \mu(E) + \nu(E).
\]
With norm \(\|\mu\| = |\mu|(X)\) given by total variation, show \(\mathcal{M}\) is a Banach space (i.e., it is complete).

4.2. Let \(\mathcal{B}\) be the \(\sigma\)-algebra of Borel sets in \(\mathbb{R}\).

(a) Show every open set in \(\mathbb{R} \times \mathbb{R}\) is in \(\mathcal{B} \otimes \mathcal{B}\). Deduce \(\mathcal{B} \otimes \mathcal{B}\) is the \(\sigma\)-algebra of Borel sets in \(\mathbb{R} \times \mathbb{R}\).

(b) Let \(E \subset \mathbb{R}\) be a Borel set, and \(A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in E\}\). Show that \(A \in \mathcal{B} \otimes \mathcal{B}\).

4.3. Let \((X, \mathcal{F}, \mu)\) be a measure space and let \(f : X \to [0, \infty)\) be measurable. Consider the subgraph \(U = \{(x, t) \in X \times [0, \infty) : t < f(x)\}\).

Show \(U \in \mathcal{F} \otimes \mathcal{B}\) (\(\mathcal{B}\) the Borel sets as above), and that (with \(m = \text{Lebesgue measure}\))
\[
(\mu \times m)(U) = \int_X f \, d\mu.
\]

4.4. (a) Let \((X, \mathcal{F})\), \((Y, \mathcal{G})\) be measurable spaces and suppose \(f : X \to \mathbb{R}\) is \(\mathcal{F}\)-measurable and \(g : Y \to \mathbb{R}\) is \(\mathcal{G}\)-measurable. Define \(h(x, y) = f(x)g(y)\). Prove \(h\) is \(\mathcal{F} \otimes \mathcal{G}\) measurable. (Hint: The product of two measurable functions defined on the same space is measurable.)

(b) (convolution) Let \(\mathcal{B}\) be the \(\sigma\)-algebra of Borel sets in \(\mathbb{R}\), and let \(f\) and \(g\) be Lebesgue integrable functions on \((\mathbb{R}, \mathcal{B})\). From results above it follows that \(F(x, y) = f(x - y)g(y)\) is \(\mathcal{B} \otimes \mathcal{B}\)-measurable. If \(m\) denotes Lebesgue measure on \(\mathcal{B}\), use Tonelli’s theorem and the fact that
\[
\int_{\mathbb{R}} |f(x - y)| \, dm(x) = \int_{\mathbb{R}} |f(x)| \, dm(x)
\]
to show that \(h(x) = \int_{\mathbb{R}} f(x - y)g(y) \, dm(y)\) is almost everywhere well-defined and satisfies
\[
\int |h| \, dm \leq \left(\int |f| \, dm\right) \left(\int |g| \, dm\right).
\]

4.5. For \(j = 1, 2\) let \(\mu_j\), \(\nu_j\) be \(\sigma\)-finite measures on \((X_j, \mathcal{F}_j)\) such that \(\nu_j \ll \mu_j\). Prove that \(\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2\) and
\[
\frac{d(\nu_1 \times \nu_2)}{\mu_1 \times \mu_2}(x_1, x_2) = \frac{dn_1}{d\mu_1}(x_1) \frac{dn_2}{d\mu_2}(x_2) \quad \text{a.e.}
\]
Here \(dn_j/d\mu_j\) denotes the Radon-Nikodým derivative satisfying
\[
\nu_j(E) = \int_E \frac{dn_j}{d\mu_j} d\mu_j \quad \text{for } E \in \mathcal{F}_j.
\]