Due Friday, Oct. 16:

3.1. Show that Fatou’s lemma remains valid if almost-everywhere convergence is replaced by convergence in measure.

3.2. (a) Let \( p \in (1, \infty) \). Prove the Sobolev-type inequality

\[
\|f\|_\infty \leq p^{1/p}\|f\|_{L^p}^{(p-1)/p}\|f'\|_{L^p}^{1/p}
\]

for every \( f : \mathbb{R} \to \mathbb{R} \) that is \( C^1 \) with compact support. (Hint: \( g(x) = |f(x)|^p = \int_{-\infty}^x g'(z) dz \).)

(b) Find all possible values of \( p, q \in (1, \infty) \) and \( r \in (0,1) \) such that there is a constant \( C \) independent of \( f \) such that

\[
\|f\|_\infty \leq C\|f\|_p^{1-r}\|f'\|_q^r
\]

for every \( f : \mathbb{R} \to \mathbb{R} \) that is \( C^1 \) with compact support. (Hint: Identify necessary conditions by considering \( f \) of the form \( f(x) = h(ax) \) for arbitrary \( a > 0 \). Then consider \( g(x) = |f(x)|^s \).)

3.3. Let \( (X, \mathcal{F}, \mu) \) be a finite measure space, suppose \( \mathcal{G} \subset \mathcal{F} \) is a (sub)\( \sigma \)-algebra, and let \( \nu = \mu|_\mathcal{G} \).

(a) If \( f \in L^1(X, \mathcal{F}, \mu) \), prove there exists \( g \in L^1(X, \mathcal{G}, \nu) \) (in particular, a \( \mathcal{G} \)-measurable function) such that

\[
\int_E f \, d\mu = \int_E g \, d\nu \quad \text{for all } E \in \mathcal{G}.
\]

Also show that if \( \hat{g} \) is another such function then \( \hat{g} = g \) \( \nu \)-a.e. (In probability theory, \( g \) is called the conditional expectation of \( f \) on \( \mathcal{G} \)).

(b) If \( A \in \mathcal{F} \) and \( \mathcal{G} \) is the smallest \( \sigma \)-algebra such that \( \mathbb{1}_A \) is \( \mathcal{G} \)-measurable, explicitly list the members of \( \mathcal{G} \) and the values of \( \nu \) and \( g \).

3.4. Let \( (X, \mathcal{F}, \mu) \) be an arbitrary measure space, and let \( p \in [1, \infty) \). Suppose \( \varphi : \mathbb{R} \to \mathbb{R} \) is continuous and has the property that for some constant \( M \geq 0 \), \( |\varphi(t)| \leq M|t| \) for all real \( t \).

(a) Prove that if \( f \in L^p \), then \( \varphi \circ f \in L^p \).

(b) Conversely, if \( |\varphi(t)|/t \) is not bounded and \( X \) is the real line with Lebesgue measure, show there exists \( f \in L^p \) such that \( \varphi \circ f \) is not in \( L^p \).

3.5. Let \( (X, \mathcal{F}, \mu) \), \( p \), and \( \varphi \) be as in the previous problem. Suppose \( (f_n) \) is a sequence that converges to \( f \) in \( L^p \). Prove that \( \varphi \circ f_n \) converges to \( \varphi \circ f \) in \( L^p \).

3.6. Let \( B(X) \) be the vector space of (true) functions \( f : X \to \mathbb{R} \) that are bounded, and define

\[
|f|_\infty = \sup_x |f(x)| \quad \text{for } f \in B(X).
\]

Suppose that a given map \( F : [0,1] \to L^\infty = L^\infty(X, \mathcal{F}, \mu) \) is continuous. (Recall: for each \( t \), \( F(t) \) is an equivalence class.) Prove that there is a function \( g : [0,1] \to B(X) \) such that \( g(t) \in F(t) \) for all \( t \in [0,1] \), and

\[
|g(t) - g(s)|_\infty = \|F(t) - F(s)\|_\infty \quad \text{for all } t, s \in [0,1].
\]

(I suggest you start with a dense sequence \( (t_k) \) in \( [0,1] \), elements \( f_k \in F(t_k) \), and find \( \mu \)-null sets \( D_k \supset D_{k-1} \) such that \( \sup_{x \in D_k} |f_k(x) - f_j(x)| = \|F(t_k) - F(t_j)\|_\infty \) for \( 1 \leq j < k \).)