21-720 F09: Measure and Integration, Assignment 3 R. Pego

Due Friday, October 23

3.1. (A generalized Hölder inequality) Suppose $p, q, r \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Show that if $f \in L_p$ and $g \in L_q$ then $fg \in L_r$ and $\|fg\|_r \le \|f\|_p \|g\|_q$.

3.2. Let $(X, \mathcal{F}, \mu)$ be a finite measure space (i.e.,$\mu(X) < \infty$). Show that $L_\infty \subset L_p$ for all $p \in [1, \infty)$, and

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p \quad \text{for all } f \in L_\infty.$$

3.3. Show that Fatou's lemma remains valid if almost-everywhere convergence is replaced by convergence in measure.

3.4. Let $(X, \mathcal{F}, \mu)$ be a finite measure space. If $f : X \to \mathbb{R}$ is $\mathcal{F}$-measurable, define

$$\rho(f) = \int \frac{|f|}{1 + |f|}\, d\mu.$$

Show that a sequence (f_n) of $\mathcal{F}$-measurable functions converges to f in measure if and only if $\rho(f_n - f) \to 0$.

3.5. Let $(X, \mathcal{F}, \mu)$ be a finite measure space, suppose $\mathcal{G} \subset \mathcal{F}$ is a (sub)σ-algebra, and let $\nu = \mu|_{\mathcal{G}}$.
(a) If $f \in L_1(X, \mathcal{F}, \mu)$, prove there exists $g \in L_1(X, \mathcal{G}, \nu)$ (in particular, a $\mathcal{G}$-measurable function) such that

$$\int_E f\, d\mu = \int_E g\, d\nu \qquad \text{for all } E \in \mathcal{G}.$$

Also show that if $\hat{g}$ is another such function then $\hat{g} = g$ ν-a.e. (In probability theory, g is called the *conditional expectation* of f on $\mathcal{G}$).
(b) If $A \in \mathcal{F}$ and $\mathcal{G}$ is the smallest σ-algebra such that $\mathbb{1}_A$ is $\mathcal{G}$-measurable, explicitly list the members of $\mathcal{G}$ and the values of ν and g.

3.6. Let $B(X)$ be the vector space of (true) functions $f : X \to \mathbb{R}$ that are bounded, and define

$$|f|_\infty = \sup_x |f(x)| \qquad \text{for } f \in B(X).$$

Suppose that a given map $F : [0, 1] \to L_\infty = L_\infty(X, \mathcal{F}, \mu)$ is *continuous.* (Recall: for each t, $F(t)$ is an equivalence class.) Prove that there is a function $g : [0, 1] \to B(X)$ such that $g(t) \in F(t)$ for all $t \in [0, 1]$, and

$$|g(t) - g(s)|_\infty = \|F(t) - F(s)\|_\infty \quad \text{for all } t, s \in [0, 1].$$

(I suggest you start with a dense sequence (t_k) in $[0, 1]$, elements $f_k \in F(t_k)$, and find μ-null sets $D_k \supset D_{k-1}$ such that $\sup_{x\in D_k^c} |f_k(x) - f_j(x)| = \|F(t_k) - F(t_j)\|_\infty$ for $1 \le j < k$.)