Due Monday, September 28

2.1. Let \((X, \mathcal{F})\) be a measurable space and suppose \(f_n : X \rightarrow \mathbb{R}\) is \(\mathcal{F}\)-measurable for all \(n \in \mathbb{N}\). Prove that the set of points \(x\) where \((f_n(x))\) converges is measurable.

2.2. Let \((X, \mathcal{F}, \mu)\) be a measure space and suppose \(f : X \rightarrow [0, \infty]\) is \(\mathcal{F}\)-measurable. Prove that
\[
\int f \, d\mu = \int_{[0,\infty)} \mu(\{x : f(x) > t\}) \, dt.
\]

2.3. Let \((X, \mathcal{F}, \mu)\) be a measure space and let \(L\) denote the set of integrable functions \(f : X \rightarrow \mathbb{R}\). Suppose that \(f\) and \(f_n \in L\) for all \(n \in \mathbb{N}\) and \(f_n \rightarrow f\) \(\mu\)-a.e. Prove that \(\int \lvert f_n \rvert \, d\mu \rightarrow \int \lvert f \rvert \, d\mu\) if and only if \(\int \lvert f_n - f \rvert \, d\mu \rightarrow 0\).

2.4. (A generalized Dominated Convergence Theorem) Let \((X, \mathcal{F}, \mu)\) be a measure space and let \(L\) denote the set of integrable functions \(f : X \rightarrow \mathbb{R}\). Suppose that \(f, g, f_n, g_n \in L\) for all \(n \in \mathbb{N}\), that (i) \(f_n \rightarrow f\) \(\mu\)-a.e., (ii) \(f_n \leq g_n\) \(\mu\)-a.e. for all \(n\), and that (iii) \(\int g_n \, d\mu \rightarrow \int g \, d\mu\).

Prove \(\int f_n \, d\mu \rightarrow \int f \, d\mu\). (Rework the proof for the usual case when \(g_n = g\) for all \(n\).)

2.5. Define \(f : \mathbb{R} \rightarrow [0, \infty)\) by \(f(x) = x^{-1/2} \mathbb{1}_{(0,1)}\). Let \((r_n)_{n=1}^\infty\) be a list of all rationals, and set
\[
g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).
\]

Prove:
(i) \(g\) is Lebesgue integrable on \(\mathbb{R}\) (i.e., integrable on \((\mathbb{R}, \mathcal{L}, m)\) where \(m\) is Lebesgue measure).
(ii) \(g\) is discontinuous at every point and unbounded on every interval, and whenever \(g = h\) a.e., the same is true for \(h\).
(iii) \(g^2 < \infty\) a.e., but \(g^2\) is not integrable on any interval.

2.6. Let \((X, \mathcal{F})\), \((Y, \mathcal{G})\) be measurable spaces, and \(\mu : \mathcal{F} \rightarrow [0, \infty]\) a measure. Suppose \(f : X \rightarrow Y\) is measurable and define \(\nu : \mathcal{G} \rightarrow [0, \infty]\) by \(\nu(E) = \mu(f^{-1}(E))\). (i) Show that \(\nu\) is a measure on \(Y\).
(ii) Assuming that \(\nu\) is \(\sigma\)-finite, show that for each \(\mathcal{G}\)-measurable \(g : Y \rightarrow [0, \infty]\) we have
\[
\int_Y g(y) \, d\nu(y) = \int_X g(f(x)) \, d\mu(x).
\]