Due Wednesday, Sept. 11:

2.1. (i) Let $X$ be a set and let $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ be a partition of $X$, meaning $X = \bigcup_{i=1}^{\infty} A_i$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. Let

$$\mathcal{F} = \{ \bigcup_{i \in J} A_i : J \subset \mathbb{N} \},$$

where we define $\bigcup_{i \in J} A_i = \emptyset$ for $J = \emptyset$. Show that $\mathcal{F}$ is a $\sigma$-algebra.

(ii) If $X = \mathbb{N}$, is it the case that every $\sigma$-algebra contained in $\mathcal{P}(\mathbb{N})$ is of the form in part (i)?

2.2. Let $X$ be a set, let $k \in \mathbb{N}$ and let $\mathcal{B} = \{B_i : 1 \leq i \leq k\}$ be a finite collection of subsets $B_i \subset X$. Show that the $\sigma$-algebra $\mathcal{F}_\mathcal{B}$ generated by $\mathcal{B}$ is finite, and in particular describe a surjection from the iterated power set $\mathcal{P}(\mathcal{P}(\{1, \ldots, k\}))$ onto $\mathcal{F}_\mathcal{B}$.

2.3. For any set $E \subset \mathbb{R}^p$ recall $\text{diam } E = \sup \{|x - y| : x, y \in E\}$. (Here $|x|$ is the usual Euclidean norm of $x$.) Given $s \geq 0$, $\delta > 0$, for each $E \subset \mathbb{R}^p$ define $h_{s,\delta}(E) \in [0, \infty]$ by

$$h_{s,\delta}(E) = \inf \left\{ \sum_{n=1}^{\infty} (\text{diam } E_n)^s : E \subset \bigcup_{n=1}^{\infty} E_n, \text{ diam } E_n < \delta \right\}.$$

(i) Prove $h_{s,\delta}$ is countably subadditive on arbitrary subsets of $\mathbb{R}^p$.

(ii) Prove that if $0 < \delta < \hat{\delta}$ and $E \subset \mathbb{R}^p$, then $h_{s,\delta}(E) \geq h_{s,\hat{\delta}}(E)$

(iii) Define $h_s(E) := \sup_{\delta > 0} h_{s,\delta}(E)$. Prove that $h_s$ is countably subadditive on arbitrary subsets of $\mathbb{R}^p$.

Remark: $h_s$ is a constant multiple of $s$-dimensional Hausdorff outer measure on $\mathbb{R}^p$. For a discussion of the role of $\delta$ above (and much more), see Tao’s blog post at https://terrytao.wordpress.com/2009/05/19/245c-notes-5-hausdorff-dimension-optional/

2.4. Let $f : \mathbb{R} \to \mathbb{R}$ be increasing, and $\mu_f^s$ be the Lebesgue-Stieltjes outer measure generated by $f$:

$$\mu_f^s(E) := \inf \left\{ \sum_{k=1}^{\infty} f(b_k) - f(a_k) : E \subset \bigcup_{k=1}^{\infty} (a_k, b_k) \right\}.$$
If $E \subset \mathbb{R}$, we say $E$ is $\mu_f^*\text{-measurable}$ if for every $A \subset \mathbb{R}$ we have

$$
\mu_f^*(A) = \mu_f^*(A \cap E) + \mu_f^*(A \cap E^c).
$$

If $c \in \mathbb{R}$, show that $(-\infty, c)$ is $\mu_f^*$-measurable. (Hint: First consider the case that $f$ is continuous at $c$. You may use the fact that the collection of $\mu_f^*$-measurable sets is a $\sigma$-algebra; the proof goes just the same as that in class for Lebesgue-measurable sets.)

2.5. Let $X$, $Y$ be sets, $f : X \to Y$, and $\mathcal{X}$ be a $\sigma$-algebra of subsets of $X$.

(i) Let $\mathcal{Y} = \{ E \subset Y : f^{-1}(E) \in \mathcal{X} \}$ be the collection of all subsets of $Y$ whose pre-images are in $\mathcal{X}$. Show that $\mathcal{Y}$ is a $\sigma$-algebra.

(ii) Suppose $\mathcal{A}$ is a collection of subsets of $Y$ such that whenever $E \in \mathcal{A}$ then $f^{-1}(E) \in \mathcal{X}$. Show that $f^{-1}(F) \in \mathcal{X}$ for every set $F$ in the $\sigma$-algebra generated by $\mathcal{A}$. 
