21-720 F09: Measure and Integration, Assignment 2 R. Pego

Due Monday, September 28

2.1. Let $(X,\mathcal{F})$ be a measurable space and suppose $f_n : X \to \mathbb{R}$ is $\mathcal{F}$-measurable for all $n \in \mathbb{N}$. Prove that the set of points x where $(f_n(x))$ converges is measurable.

2.2. Let $(X,\mathcal{F},\mu)$ be a measure space and suppose $f : X \to [0,\infty]$ is $\mathcal{F}$-measurable. Prove that

$$\int f\,d\mu = \int_{[0,\infty)} \mu(\{x : f(x) > t\})\,dm(t).$$

2.3. Let $(X,\mathcal{F},\mu)$ be a measure space and let L denote the set of integrable functions $f : X \to \mathbb{R}$. Suppose that f and $f_n \in L$ for all $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e. Prove that

$$\int |f_n|\,d\mu \to \int |f|\,d\mu \quad \text{if and only if} \quad \int |f_n - f|\,d\mu \to 0.$$

2.4. (A generalized Dominated Convergence Theorem) Let $(X,\mathcal{F},\mu)$ be a measure space and let L denote the set of integrable functions $f : X \to \mathbb{R}$. Suppose that f, g, f_n, $g_n \in L$ for all $n \in \mathbb{N}$, that (i) $f_n \to f$ μ-a.e., (ii) $|f_n| \le g_n$ μ-a.e. for all n, and that (iii) $\int g_n\,d\mu \to \int g\,d\mu$.
Prove $\int f_n\,d\mu \to \int f\,d\mu$. (Rework the proof for the usual case when $g_n = g$ for all n.)

2.5. Define $f : \mathbb{R} \to [0,\infty)$ by $f(x) = x^{-1/2}\mathbb{1}_{(0,1)}$. Let $(r_n)_{n=1}^{\infty}$ be a list of all rationals, and set

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove:
(i) g is Lebesgue integrable on $\mathbb{R}$ (i.e., integrable on $(\mathbb{R},\mathcal{L},m)$ where m is Lebesgue measure).
(ii) g is discontinuous at every point and unbounded on every interval, and whenever $g = h$ a.e., the same is true for h.
(iii) $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.

2.6. Let $(X,\mathcal{F})$, $(Y,\mathcal{G})$ be measurable spaces, and $\mu : \mathcal{F} \to [0,\infty]$ a measure. Suppose $f : X \to Y$ is measurable and define $\nu : \mathcal{G} \to [0,\infty]$ by $\nu(E) = \mu(f^{-1}(E))$. (i) Show that ν is a measure on Y. (ii) Assuming that ν is σ-finite, show that for each $\mathcal{G}$-measurable $g : Y \to [0,\infty]$ we have

$$\int_Y g(y)\,d\nu(y) = \int_X g(f(x))\,d\mu(x).$$