Due Wednesday, Sept. 9:

1.1. (i) Let \( a_{n,k} \geq 0 \) for \( n, k \in \mathbb{N} \). Prove that

\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k}.
\]

(ii) Suppose only that \( a_{n,k} \in \mathbb{R} \) for \( n, k \in \mathbb{N} \). Show that (1) does not necessarily follow. (Give an explicit counterexample.)

1.2. Suppose that \( \mathcal{F} \) is a \( \sigma \)-algebra of subsets of \( \mathbb{R}^p \) which contains every half-space of the form \( \{ x \in \mathbb{R}^p : x_j < c \} \), where \( j = 1, \ldots, p \) and \( c \in \mathbb{R} \) are fixed. Prove that \( \mathcal{F} \) contains every open cell in \( \mathbb{R}^p \) (i.e., every product of bounded open intervals).

1.3. For any set \( E \subset \mathbb{R}^p \) recall \( \text{diam} E = \sup\{|x - y| : x, y \in E\} \). (Here \( |x| \) is the usual Euclidean norm of \( x \).) Given \( s \geq 0, \delta > 0 \), for each \( E \subset \mathbb{R}^p \) define \( \eta_{s,\delta}(E) \in [0, \infty] \) by

\[
\eta_{s,\delta}(E) = \inf \left\{ \sum_{n=1}^{\infty} (\text{diam } E_n)^s : E \subset \bigcup_{n=1}^{\infty} E_n, \text{diam } E_n < \delta \right\}.
\]

(i) Prove \( \eta_{s,\delta} \) is countably subadditive on arbitrary subsets of \( \mathbb{R}^p \).

(ii) Prove that if \( 0 < \delta < \delta \) and \( E \subset \mathbb{R}^p \), then \( \eta_{s,\delta}(E) \geq \eta_{s,\delta}(E) \).

(iii) Define \( \eta_s(E) := \sup_{\delta > 0} \eta_{s,\delta}(E) \). Prove that \( \eta_s \) is countably subadditive on arbitrary subsets of \( \mathbb{R}^p \).

(\( \eta_s \) is a constant multiple of \textit{s-dimensional Hausdorff outer measure} on \( \mathbb{R}^p \).)

1.4. Suppose \( \mathcal{F} \) is a \( \sigma \)-algebra that has infinitely many elements.

(i) Show there is a countable family of pairwise disjoint sets in \( \mathcal{F} \).

(ii) Show \( \mathcal{F} \) is uncountable.

1.5. Given \( f : \mathbb{R} \to \mathbb{R} \) increasing, let \( \mu_f^* \) be the Lebesgue-Stieltjes outer measure generated by \( f \):

\[
\mu_f^*(E) := \inf \left\{ \sum_{k=1}^{\infty} f(b_k) - f(a_k) : E \subset \bigcup_{k=1}^{\infty} (a_k, b_k) \right\}.
\]

If \( E \subset \mathbb{R} \), we say \( E \) is \( \mu_f^* \)-measurable if for every \( A \subset \mathbb{R} \) we have

\[
\mu_f^*(A) = \mu_f^*(A \cap E) + \mu_f^*(A \cap E^c).
\]

If \( c \in \mathbb{R} \), show that \( (-\infty, c) \) is \( \mu_f^* \)-measurable. (Hint: First consider the case that \( f \) is continuous at \( c \). You may use the fact that the collection of \( \mu_f^* \)-measurable sets is a \( \sigma \)-algebra; the proof goes just the same as that in class for Lebesgue-measurable sets.)

1.6. Let \( X, Y \) be sets, \( f : X \to Y \), and \( \mathcal{X} \) be a \( \sigma \)-algebra of subsets of \( X \).

(i) Let \( \mathcal{Y} = \{ E \subset Y : f^{-1}(E) \in \mathcal{X} \} \) be the collection of all subsets of \( Y \) whose pre-images are in \( \mathcal{X} \). Show that \( \mathcal{Y} \) is a \( \sigma \)-algebra.

(ii) Suppose \( \mathcal{A} \) is a collection of subsets of \( Y \) such that whenever \( E \in \mathcal{A} \) then \( f^{-1}(E) \in \mathcal{X} \). Show that \( f^{-1}(F) \in \mathcal{X} \) for every set \( F \) in the \( \sigma \)-algebra generated by \( \mathcal{A} \).