Lebesgue Decomposition and Radon-Nikodým Theorems: a clean way

Throughout, \((X, \mathcal{F})\) is a measurable space, and \(\mu, \nu : \mathcal{F} \to [0, \infty]\) are two fixed measures. We focus on the case of finite measures, as extending the results to \(\sigma\)-finite measures is rather straightforward. The goal is to avoid intricate constructions found in many sources.

**Theorem** (Lebesgue Decomposition for a finite measure) Assume \(\nu(X) < \infty\). Then there are unique measures \(\nu_1, \nu_2\) on \(\mathcal{F}\) such that

\[
\nu = \nu_1 + \nu_2, \quad \nu_1 \ll \mu, \quad \nu_2 \perp \mu.
\]  

Furthermore, there is a set \(B \in \mathcal{F}\) with \(\mu(B) = 0\) such that for all \(E \in \mathcal{F}\),

\[
\nu_1(E) = \nu(E \setminus B), \quad \nu_2(E) = \nu(E \cap B).
\]

**Proof.**

1. Define

\[
\beta = \sup \{ \nu(F) : F \in \mathcal{F} \text{ with } \mu(F) = 0 \}.
\]

Then \(\beta \leq \nu(X) < \infty\), and there is a sequence \((F_n)\) in \(\mathcal{F}\) with \(\mu(F_n) = 0\) and \(\beta = \lim \nu(F_n)\). It follows that the union \(B = \bigcup_{n=1}^{\infty} F_n\) satisfies

\[
B \in \mathcal{F}, \quad \mu(B) = 0 \quad \text{and} \quad \nu(B) = \beta.
\]  

2. Define \(\nu_1\) and \(\nu_2\) as in (2). It is straightforward to show \(\nu_1\) and \(\nu_2\) are measures on \(\mathcal{F}\). Then

\[
\nu_2 \perp \mu, \quad \text{because} \quad \nu_2(X \setminus B) = \nu(\emptyset) = 0 \quad \text{and} \quad \mu(B) = 0.
\]

3. We claim \(\nu_1 \ll \mu\). Suppose \(\mu(E) = 0\). Then \(\mu(B \cup E) = 0\), whence

\[
\beta \geq \nu(B \cup E) = \nu(B) + \nu(E \setminus B) = \beta + \nu_1(E) \geq \beta.
\]

Hence \(\nu_1(E) = 0\). Therefore \(\nu_1 \ll \mu\).

4. (uniqueness) Suppose that any two measures \(\hat{\nu}_1\) and \(\hat{\nu}_2\) on \(\mathcal{F}\) satisfy

\[
\nu = \hat{\nu}_1 + \hat{\nu}_2, \quad \hat{\nu}_1 \ll \mu, \quad \hat{\nu}_2 \perp \mu.
\]  

Because \(\hat{\nu}_2 \perp \mu\), there exists \(\hat{B} \in \mathcal{F}\) such that \(\hat{\nu}_2(X \setminus \hat{B}) = 0\) and \(0 = \mu(\hat{B}) = \hat{\nu}_1(\hat{B})\). For every \(E \in \mathcal{F}\), then we have

\[
\hat{\nu}_1(E) = \hat{\nu}_1(E \setminus \hat{B}) + \hat{\nu}_1(E \cap \hat{B}) = \hat{\nu}_1(E \setminus \hat{B}) + \hat{\nu}_2(E \setminus \hat{B}) = \nu(E \setminus \hat{B}), \\
\hat{\nu}_2(E) = \hat{\nu}_2(E \setminus \hat{B}) + \hat{\nu}_2(E \cap \hat{B}) = \hat{\nu}_2(E \setminus \hat{B}) + \hat{\nu}_1(E \cap \hat{B}) = \nu(E \cap \hat{B}).
\]
Using this and also (2) we can compute that
\[ \nu(B \setminus \hat{B}) = \hat{\nu}_1(B) = 0 \quad \text{since } \hat{\nu}_1 \ll \mu \text{ and } \mu(B) = 0, \]
\[ \nu(\hat{B} \setminus B) = \nu_1(\hat{B}) = 0 \quad \text{since } \nu_1 \ll \mu \text{ and } \mu(\hat{B}) = 0. \]

Therefore \( B \setminus \hat{B} \) and \( \hat{B} \setminus B \) are null sets for all the measures \( \mu, \nu, \nu_1, \nu_2, \hat{\nu}_1 \) and \( \hat{\nu}_2 \).

For every \( E \in \mathcal{F} \) we have \( E \cap B = (E \cap B \cap \hat{B}) \cup (E \cap B \setminus \hat{B}) \), hence
\[ \nu_2(E) = \nu(E \cap B) = \nu(E \cap B \cap \hat{B}), \]
and similarly
\[ \hat{\nu}_2(E) = \nu(E \cap \hat{B}) = \nu(E \cap B \cap \hat{B}). \]

Therefore \( \nu_2 = \hat{\nu}_2 \), and also \( \nu_1 = \nu - \nu_2 = \nu - \hat{\nu}_2 = \hat{\nu}_1 \). QED

The Radon-Nikodym theorem will provide a representation of the absolutely continuous part as an integral. Its proof will make use of the Hahn decomposition theorem.

**Theorem** (Radon-Nikodým for finite measures) Assume \( \mu(X) < \infty \) and \( \nu(X) < \infty \), and suppose \( \nu \ll \mu \). Then there exists an \( \mathcal{F} \)-measurable function \( f : X \to [0, \infty) \) such that
\[ \nu(E) = \int_E f \, d\mu \quad \text{for all } E \in \mathcal{F}. \]

**Proof.** 1. Let \( H \) be the set of all \( \mathcal{F} \)-measurable functions \( h : X \to [0, \infty] \) such that
\[ \int_E h \, d\mu \leq \nu(E) \quad \text{for all } E \in \mathcal{F}. \]

Then \( 0 \in H \), and if \( h_1, h_2 \in H \) then \( \max(h_1, h_2) \in H \), since with \( F = \{ x : h_1(x) \geq h_2(x) \} \), for any \( E \in \mathcal{F} \) we have
\[ \int_E \max(h_1, h_2) \, d\mu = \int_{E \cap F} h_1 \, d\mu + \int_{E \setminus F} h_2 \, d\mu \leq \nu(E \cap F) + \nu(E \setminus F) = \nu(E). \]

2. Define
\[ \alpha = \sup_{h \in H} \int h \, d\mu. \]

Then \( 0 \leq \alpha \leq \nu(X) < \infty \), and there is a sequence \( h_n \in H \) such that \( \alpha = \lim \int h_n \, d\mu \). Replacing \( h_n \) by \( \max(h_1, \ldots, h_n) \) we may suppose \( (h_n) \) is increasing. Letting \( f = \lim h_n \), the Monotone Convergence Theorem ensures
\[ \int_E f \, d\mu \leq \nu(E) \quad \text{for all } E \in \mathcal{F}, \quad \text{and} \quad \alpha = \int f \, d\mu \leq \nu(X) < \infty. \quad (6) \]

Therefore \( \{ x : f(x) = \infty \} \) is \( \mu \)-null; redefining \( f \) as 0 on this set, still (6) holds, so \( f \in H \).
3. We claim that equality always holds in (6). Supposing not, there exists $A \in \mathcal{F}$ and $\varepsilon > 0$ such that

$$\nu(A) > \int_A f \, d\mu + \varepsilon \mu(A). \quad (7)$$

This means $\lambda(A) > 0$, where $\lambda$ is a signed measure on $\mathcal{F}$ defined by

$$\lambda(E) = \nu(E) - \int_E f \, d\mu - \varepsilon \mu(E).$$

There is a Hahn decomposition $X = P \cup N$ with disjoint sets $P$ and $N$ such that (8) holds. In particular it follows that for all $E \in \mathcal{F}$, $\lambda(E \cap P) \geq 0$, meaning

$$\nu(E \cap P) \geq \int_{E \cap P} f \, d\mu + \varepsilon \mu(E \cap P).$$

Since anyway $\nu(E \cap N) \geq \int_{E \cap N} f \, d\mu$ because $f \in H$, we add and deduce

$$\nu(E) \geq \int_E (f + \varepsilon \mathbb{1}_P) \, d\mu.$$

This shows that $f + \varepsilon \mathbb{1}_P \in H$, which implies

$$\alpha = \int f \, d\mu = \int f \, d\mu + \varepsilon \mu(P).$$

We infer $\mu(P) = 0$, whence $\nu(P) = 0$ since $\nu \ll \mu$, hence $\lambda(P) = 0$. From (7) we now obtain

$$0 < \lambda(A) = \lambda(A \cap P) + \lambda(A \cap N) \leq \lambda(P \cap A) + \lambda(P \setminus A) = \lambda(P) = 0,$$

a contradiction. This proves the claim. QED

**Remark:** This account has merged ideas in these two papers:


together with the Hahn decomposition idea mentioned to me by Gautam Iyer.
To complete this treatment, we give a proof of the Hahn decomposition theorem that I find more straightforward than Folland’s, based upon the 1-page paper


**Theorem** (Hahn decomposition) Let \( \lambda \) be a signed measure on \( \mathcal{F} \). Then \( X = P \cup N \), where \( P \cap N = \emptyset \), \( P \) is positive and \( N \) is negative, meaning that

\[
\lambda(E \cap P) \geq 0 \geq \lambda(E \cap N) \quad \text{for all } E \in \mathcal{F}.
\]

(8)

We may suppose that \( +\infty \) is not in the range of \( \lambda \) (by replacing \( \lambda \) by \( -\lambda \) if necessary). For any \( \varepsilon > 0 \), we will say that \( A \in \mathcal{F} \) is \( \varepsilon \)-positive if

\[
\inf_{B \subseteq A} \lambda(B) \geq -\varepsilon.
\]

(9)

**Sublemma** Let \( E \in \mathcal{F} \) with \( \lambda(E) > 0 \) and let \( \varepsilon > 0 \). Then \( E \) contains some \( A \in \mathcal{F} \) with \( \lambda(A) \geq \lambda(E) \) such that \( A \) is \( \varepsilon \)-positive.

**Proof.** Fix \( E \) and \( \varepsilon \), and suppose the stated conclusion is false, meaning every \( \mathcal{F} \)-measurable \( A \subset E \) with \( \lambda(A) \geq \lambda(E) \) is not \( \varepsilon \)-positive. Then recursively, taking \( B_0 = \emptyset \), we can find \( B_k \subset A_k := E \setminus (B_0 \cup \ldots \cup B_{k-1}) \) with \( \lambda(B_k) < -\varepsilon \) for all \( k \in \mathbb{N} \). The \( B_k \) are disjoint, hence \( B = \bigcup_{k=1}^{\infty} B_k \) satisfies \( \lambda(B) = -\infty \). But then \( \infty > \lambda(E \setminus B) = +\infty \), a contradiction. QED

**Lemma** Let \( E \in \mathcal{F} \) with \( \lambda(E) > 0 \). Then \( E \) contains some \( Q \in \mathcal{F} \) with \( \lambda(Q) \geq \lambda(E) \) such that \( Q \) is positive.

**Proof.** Using the sublemma, recursively we can construct \((E_n)\) in \( E \) such that for all \( n \in \mathbb{N} \),

\[
E_{n+1} \subset E_n \subset E, \quad \lambda(E_{n+1}) \geq \lambda(E_n) \geq \lambda(E) > 0, \quad \text{and} \quad \inf_{B \subseteq E_n} \lambda(B) \geq -\frac{1}{n}.
\]

Let \( Q = \cap_{n=1}^{\infty} E_n \). Then \( \infty > \lambda(Q) = \lim \lambda(E_n) \geq \lambda(E) \) and \( \inf_{B \subseteq Q} \lambda(B) \geq -\frac{1}{n} \) for all \( n \). Hence \( Q \) is positive. QED

**Proof of the Hahn decomposition theorem.** Let \( M = \sup \{ \lambda(P) : P \in \mathcal{F} \text{ is positive} \} \). Then \( M \geq \lambda(\emptyset) = 0 \), and there is a sequence \((P_n)\) of positive sets with \( \lambda(P_n) \to M \). The set \( P = \bigcup_{n=1}^{\infty} P_n \) is then positive, so \( \lambda(P) \geq \lambda(P_n) \) for all \( n \), hence \( \lambda(P) = M < \infty \).

Now let \( N = X \setminus P \). Then \( N \) is negative. For if not, then \( N \) contains some \( E \in \mathcal{F} \) with \( \lambda(E) > 0 \). By the lemma, \( E \) contains some positive \( Q \) with \( \lambda(Q) \geq \lambda(E) \). But then \( P \cup Q \) is positive, and because \( P \cap Q = \emptyset \) we find \( \lambda(P \cup Q) > M \). Contradiction. QED

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