Due Friday April 27:

• **P1:** (cf. RS VI.6) Let $\mathcal{H}$ be a Hilbert space.

(a) Let $\{A_\alpha\}_{\alpha\in I}$ and $\{B_\alpha\}_{\alpha\in I}$ be nets in $\mathcal{L}(\mathcal{H})$. If $A_\alpha^* \xrightarrow{s} A^*$ and $B_\alpha \xrightarrow{s} B$, prove $A_\alpha B_\alpha \xrightarrow{w} AB$.

(b) Let $\{A_n\}$ and $\{B_n\}$ be sequences in $\mathcal{L}(\mathcal{H})$, and $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{s} B$. Prove $A_n B_n \xrightarrow{s} AB$.

(c) Find an example of sequences $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ such that $A_n \xrightarrow{w} A$ and $B_n \xrightarrow{w} B$, but it is false that $A_n B_n \xrightarrow{w} A_n B_n$.

• **P2:** (RS VI.42) Let X be a Banach space and let $A \in \mathcal{L}(X)$. Prove that the set of $\lambda \in \sigma(A)$ such that $\lambda I - A$ is injective and $\lambda I - A$ has closed range is an *open* subset of $\mathbb{C}$.

• **P3:** Suppose P and Q are bounded projections in a Banach space X, each with finite rank (rank is the dimension of the range). Prove that if $\|P - Q\| < 1$, then P and Q have the *same* rank.

• **P4:** Consider the following differential equation for 2π-periodic functions u, f on $\mathbb{R}$.

$$\lambda u(x) - u'(x) = f(x), \tag{1}$$

(a) Using Fourier series, show that whenever $\lambda \notin i\mathbb{Z} = \{ik : k \in \mathbb{Z}\}$, there is an operator $f \mapsto T_\lambda f$ bounded on L^2_{per}, such that whenever f is smooth, then $u = T_\lambda f$ solves (1). Find $\|T_\lambda\|$.

(b) Suppose $\lambda = \nu + i\mu$ with $\nu > 0$, μ real. Find a function $k_\lambda(x, y)$ such that whenever f is smooth and 2π-periodic and

$$u(x) = (K_\lambda f)(x) := \int_{-\infty}^{\infty} k_\lambda(x, y) f(y)\, dy, \tag{2}$$

then u is 2π-periodic and solves (1). Prove K_λ is bounded on L^2_{per} and $T_\lambda = K_\lambda$.

(c) Prove that for any bounded set $S \subset L^2_{\text{per}}$, the set $K_\lambda S = \{K_\lambda f : f \in S\}$ is an equicontinuous family of continuous functions. Conclude that K_λ is a compact operator on L^2_{per}.

(d) Find the point spectrum of K_λ.