Due Monday March 26:

- **P1:** (RS IV.39) Find a Banach space and a weakly convergent net which is not norm bounded.

- **P2:** (RS IV.40) Let X be an infinite-dimensional Banach space with the weak topology. Prove that the closure of the unit sphere $S = \{x \in X : \|x\| = 1\}$ is the unit ball $B = \{x \in X : \|x\| \le 1\}$.

- **P3:** (cf. RS IV.12) In the Banach space $X = \ell_\infty$ consider the sequence $(\delta_n)_{n\in\mathbb{N}}$ in X^* given by

$$\delta_n(\boldsymbol{a}) = a_n \quad \text{whenever} \quad \boldsymbol{a} = (a_k)_{k\in\mathbb{N}} \in \ell_\infty.$$

Prove that (δ_n) has no weak-$*$ convergent subsequence. (This does not contradict the separable Banach-Alaoglu theorem since ℓ_∞ is not separable.) If possible (and I don't know that it is!), describe a cluster point of the net $\{\delta_n\}_{n\in\mathbb{N}}$ in X^*.

- **P4:** (RS IV.36) Let X be a Banach space. Prove that under the standard embedding of X into X^{**}, X is dense under the weak-$*$ topology of X^{**}.

- **P5:** In H^1_{per}, the Hilbert-space completion of the space $\mathcal{S}$ of 2π-periodic trig polynomials on $\mathbb{R}$ with inner product

$$(f,g)_{H^1} = \int_{-\pi}^{\pi} \overline{f(x)}g(x) + \overline{f'(x)}g'(x)\,dx,$$

suppose $\{f_k\}_{k\in\mathbb{N}}$ is a sequence which converges *weakly* to some $f \in H^1_{\text{per}}$. I.e.,

$$(f_k, g)_{H^1} \to (f,g)_{H^1}$$

for all $g \in H^1_{\text{per}}$. Prove that $f_k \to f$ *uniformly.*

- **P6:** For nonzero $h \in \mathbb{R}$ define the difference quotient operator D_h by

$$D_h f(x) = \frac{f(x+h) - f(x)}{h}$$

for $f \in \mathcal{S}$, then extended to L^2_{per} by continuity.

(a) On L^2_{per}, show that $\|hD_h\| = \sqrt{2}$ for all $h \ne 0$, but that $hD_h f \to 0$ as $h \to 0$ for all $f \in L^2_{\text{per}}$.

(b) Let $f \in L^2_{\text{per}}$ and suppose $\{D_h f : h \ne 0\}$ is bounded. Show that f has a weak derivative $g \in L^2_{\text{per}}$ and $D_h f \rightharpoonup g$ (weakly) as $h \to 0$.

(c) With f as in part (b), show that $D_h f \to g$ (in norm) as $h \to 0$.

(d) Show further that $f \in H^1_{\text{per}}$, i.e., f is the H^1-limit of a sequence in $\mathcal{S}$.