1. Suppose that \(\{u_n\}\) is a sequence in a Hilbert space \(H\) that converges weakly to \(u \in H\): \(u_n \rightharpoonup u\) as \(n \to \infty\). If it is also true that \(\|u_n\| \to \|u\|\), then prove \(u_n \to u\) in the norm metric on \(H\).

2. (RS III.16) A subset \(S\) of a Banach space \(X\) is called weakly bounded if each \(\lambda \in X^*\) is bounded on \(S\); that is, \(\forall \lambda \in X^*, \sup_{x \in S} |\lambda(x)| < \infty\). \(S\) is called strongly bounded if \(\sup_{x \in S} \|x\| < \infty\). Prove that a set is strongly bounded if and only if it is weakly bounded.

3. Suppose that \(S\) is a bounded subset of \(\ell_2\) and let \(S^w\) denote the weak closure of \(S\) (the smallest subset of \(\ell_2\) that contains \(S\) and is closed in the weak topology on \(\ell_2\)). Show that if \(x \in S^w\) then there is a sequence \(\langle x_n \rangle_{n \in \mathbb{N}}\) in \(S\) that converges weakly to \(x\): \(x_n \rightharpoonup x\) as \(n \to \infty\).

4. Suppose \(\{x_n\}\) is sequence in a Banach space \(X\) converging weakly to \(x \in X\) and \(\{\ell_n\}\) is a sequence in the dual \(X^*\) converging weak-* to \(\ell \in X^*\).
   (a) Give an example to show that it is not necessarily true that \(\ell_n(x_n) \to \ell(x)\) as \(n \to \infty\).
   (b) Prove that if either \(\{x_n\}\) or \(\{\ell_n\}\) converges strongly, then necessarily \(\ell_n(x_n) \to \ell(x)\) as \(n \to \infty\).

5. (RS IV.40) Let \(X\) be an infinite-dimensional Banach space with the weak topology. Prove that the (weak) closure of the unit sphere \(S = \{x \in X : \|x\| = 1\}\) is the unit ball \(B = \{x \in X : \|x\| \leq 1\}\).

6. In \(H^1_{\text{per}}\), the Hilbert-space completion of the space \(S\) of 2\(\pi\)-periodic trig polynomials on \(\mathbb{R}\) with inner product
   \[
   (f, g)_{H^1} = \int_{-\pi}^{\pi} f(x)g(x) + f'(x)g'(x) \, dx,
   \]
   suppose \(\{f_k\}_{k \in \mathbb{N}}\) is a sequence which converges weakly to some \(f \in H^1_{\text{per}}\). I.e.,
   \[
   (f_k, g)_{H^1} \to (f, g)_{H^1} \quad \text{for all } g \in H^1_{\text{per}}.
   \]
   Prove that \(f_k \to f\) uniformly.

7. [Optional: do not turn in.] Suppose that \(\{x_\alpha\}_{\alpha \in I}\) is a Hamel basis for a Banach space \(X\); i.e., \(X\) is the algebraic span of the set \(\{x_\alpha : \alpha \in I\}\). Thus every \(x \in X\) has a unique representation as
   \[
   x = \sum_{\alpha \in I} c_\alpha x_\alpha, \quad c_\alpha \in \mathbb{C},
   \]
   where only finitely many terms are nonzero. It is straightforward to show that for each \(\alpha \in I\) the “coordinate map” \(x \mapsto c_\alpha\) is linear. Prove that only finitely many of these maps can be continuous.
   [Hint: if \(\{\alpha_j : j \in \mathbb{N}\}\) is a sequence in \(I\), consider the sequence of maps \(x \mapsto T_n x = \sum_{j \leq n} j c_{\alpha_j} x_{\alpha_j}\), and carefully interpret the statement of the uniform boundedness principle.]