Homework Assignment 3

Below we use the notation $a = \{a_k\}_{k \in \mathbb{N}}$ to represent a sequence of complex numbers, and set

$$\ell_0 = \{a \mid \lim_{k \to \infty} a_k = 0\}, \quad \ell_\infty = \{a \mid \|a\|_\infty := \sup_k |a_k| < \infty\},$$

$$\ell_p = \{a \mid \|a\|_p := \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} < \infty\} \quad (p > 0).$$

1. (RS III.2a) If $p \geq 1$, prove $\ell_p$ and $\ell_0$ are separable, but $\ell_\infty$ is not.

2. (a) Prove that $\ell_1^* = \ell_\infty$.
   (b) Using the Hahn-Banach theorem, prove there is a linear functional $T$ on $\ell_\infty$ with the property that for all $a \in \ell_\infty$ whose components $a_j$ are all real,

$$\liminf_{n \to \infty} a_n \leq T(a) \leq \limsup_{n \to \infty} a_n.$$

[Hint: Taking limits is a linear operation. Or is it?]

(c) Deduce that the dual $\ell_\infty^* \neq \ell_1$. I.e., deduce that not every bounded linear functional $S : \ell_\infty \to \mathbb{C}$ has the form $S(a) = \sum b_j a_j$ for some sequence $b = \{b_j\} \in \ell_1$.

3. Let $X$ be a Banach space and let $(\varepsilon_n)$ be a sequence of positive numbers converging to zero. Suppose that $(f_n)$ is a sequence in the dual $X^*$ having the property that for all $x \in X$ with $\|x\| < 1$, there exists $C(x) > 0$ such that

$$|f_n(x)| \leq \varepsilon_n\|f_n\| + C(x).$$

Prove that $(f_n)$ is bounded. [Hint: let $g_n = f_n/(1 + \varepsilon_n\|f_n\|)]$

4. Let $X$ be a Banach space and suppose $T : X \to X^*$ is linear and has the property that whenever $f = T(x)$ then $f(x) \geq 0$. Prove $T$ is bounded.

5. (a) For any $u \in L^2_{\text{per}}$, the Hilbert-space completion of the space of $2\pi$-periodic trigonometric polynomials with inner product $(u, v) = \int_{-\pi}^{\pi} \overline{u(x)} v(x)\, dx$, define $\hat{u}(k) = (e_k, u)$, $e_k(x) = e^{ikx}/\sqrt{2\pi}$. Prove Plancherel's identity:

$$(u, v) = \sum_{k \in \mathbb{Z}} \overline{\hat{u}(k)} \hat{v}(k).$$

(b) (The isoperimetric inequality) Suppose we have a smooth closed curve in the (complex) plane which encloses an area $A$ and has perimeter $P$. We wish to prove that

$$P^2 \geq 4\pi A.$$  (***)

To do this, assume that the curve is parametrized by a smooth $2\pi$-periodic complex valued function $f(x) = u(x) + iv(x)$ such that $(u')^2 + (v')^2 = c^2$ is constant. Using that $c((u')^2 + (v')^2)^{1/2} = |f'|^2$, relate $P^2$ to $\int_{-\pi}^{\pi} |f'(x)|^2\, dx$. Relate $A = \int u\, dv$ to the $L^2$-inner product

$$(f', f) = \int_{-\pi}^{\pi} \overline{f'(x)} f(x)\, dx.$$

Using Plancherel's identity you should be able to deduce (***).
6. (a) Show that the $2\pi$-periodic function $u(x) = |\sin \frac{x}{2}|$ belongs to the periodic Sobolev space $H = H^1_{\text{per}}$, and is a weak solution of a problem of the form

$$-u''(x) + u(x) = f,$$

where $f \in H^*$ is a linear combination of the Dirac delta distribution and a map of the form $v \mapsto \int_0^{2\pi} g(x)v(x) \, dx$.

(b) Show that the function $w(x) = \sqrt{u(x)}$ does not belong to $H^1_{\text{per}}$. In particular, you must show that $w$ has no weak derivative in $L^2_{\text{per}}$. That is, there is no linear map $T : C^\infty_{\text{per}} \to \mathbb{C}$ that is bounded with respect to the $L^2$ norm such that

$$T(f) = -\int_0^{2\pi} \frac{w(x)}{w(x)} f'(x) \, dx \quad \text{for all } f \in C^\infty_{\text{per}}.$$

[Q: Why must such a map necessarily take the form you naturally guess?]