21-640 S07: Functional Analysis Assignment 2 R. Pego

Due Wednesday February 28:

Below we use the notation $\boldsymbol{a} = \{a_k\}_{k\in\mathbb{N}}$ to represent a sequence of complex numbers, and set

$$\begin{aligned} \ell_\infty &= \{\boldsymbol{a} \mid \|\boldsymbol{a}\|_\infty := \sup_k |a_k| < \infty\}, \\ \ell_p &= \{\boldsymbol{a} \mid \|\boldsymbol{a}\|_p := \left(\sum_{k=1}^\infty |a_k|^p\right)^{1/p} < \infty\} \quad (p > 0), \\ c_0 &= \{\boldsymbol{a} \mid \lim_{k\to\infty} a_k = 0\}. \end{aligned}$$

- **P1:** Prove that if $p > 1$ and $\boldsymbol{a} \in \ell_p \cap \ell_\infty$, then
  - (a) $\boldsymbol{a} \in \ell_q$ whenever $p < q \le \infty$, with $\|\boldsymbol{a}\|_q \le \|\boldsymbol{a}\|_p$.
  - (b) $\lim_{q\to\infty} \|\boldsymbol{a}\|_q = \|\boldsymbol{a}\|_\infty$.

- **P2:** (RS III.2a) If $p \ge 1$, prove ℓ_p and c_0 are separable, but ℓ_∞ is not.

- **P3:** (RS III.7) Prove that $\ell_\infty^* \ne \ell^1$ by using the Hahn-Banach theorem.

- **P4:** (RS III.15) Let H be a separable Hilbert space, with orthonormal basis $\{x_n\}_{n\in\mathbb{N}}$, and suppose $\{y_n\}$ a sequence of elements of H. Prove that the following statements are equivalent:
  - (a) For each $x \in H$, $(x, y_n) \to 0$ as $n \to \infty$.
  - (b) $\{\|y_n\|\}_{n\in\mathbb{N}}$ is bounded, and $\forall m \in \mathbb{N}$, $(x_m, y_n) \to 0$ as $n \to \infty$.

- **P5:** (RS III.16) A subset S of a Banach space X is called *weakly bounded* if each $\lambda \in X^*$ is bounded on S; that is, $\forall\lambda \in X^*$, $\sup_{x\in S} |\lambda(x)| < \infty$. S is called *strongly bounded* if $\sup_{x\in S} \|x\| < \infty$. Prove that a set is strongly bounded if and only if it is weakly bounded.

- **P6:** (a) For any $u \in L^2_{\text{per}}$, the Hilbert-space completion of the space of 2π-periodic trigonometric polynomials with inner product $(u, v) = \int_{-\pi}^{\pi} \overline{u(x)} v(x)\, dx$, define $\hat{u}(k) = (e_k, u)$, $e_k(x) = e^{ikx}/\sqrt{2\pi}$. Prove Plancherel's identity:

$$(u, v) = \sum_{k\in\mathbb{Z}} \overline{\hat{u}(k)} \hat{v}(k).$$

(b) (The isoperimetric inequality) Suppose we have a smooth closed curve in the (complex) plane which encloses an area A and has perimeter P. We wish to prove that

$$P^2 \ge 4\pi A\ . \qquad (**)$$

To do this, assume that the curve is parametrized by a smooth 2π-periodic complex valued function $f(x) = u(x) + iv(x)$ such that $(u')^2 + (v')^2 = c^2$ is constant. Using that $c((u')^2 + (v')^2)^{1/2} = |f'|^2$, relate P^2 to $\int_{-\pi}^{\pi} |f'(x)|^2\, dx$. Relate $A = \int u\, dv$ to the L^2-inner product

$$(f', f) = \int_{-\pi}^{\pi} \overline{f'(x)} f(x)\, dx.$$

Using Plancherel's identity you should be able to deduce $(**)$.