

Due Wednesday February 7:

• **P1:** Recall ℓ^2 is the vector space of complex sequences $\mathbf{x} = \{x_k\}_{k=1}^\infty$ such that $\|\mathbf{x}\| < \infty$, where $\|\mathbf{x}\|^2 = \sum_{k=1}^\infty |x_k|^2$. Prove that ℓ^2 is complete. (Hence it is a Hilbert space.)

• **P2:** Any compact set S in a metric space must be closed and bounded. Moreover, for any $\mathbf{x} = \{x_k\}_{k=1}^\infty \in \ell^2$ we have

$$\sum_{k=N}^{\infty} |x_k|^2 \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (1)$$

Suppose S is a closed and bounded subset of ℓ^2 . Prove: S is compact if and only if (1) holds *uniformly* for all $\mathbf{x} \in S$.

• **P3:** Let $a > 0$ and let E be the Banach space $C[0, a]$ of continuous functions $u : [0, a] \rightarrow \mathbb{C}$, with norm $\|u\|_\infty = \sup_{x \in [0, a]} |u(x)|$.

Define $T : E \rightarrow E$ by

$$(Tu)(x) = \int_0^x u(s) ds.$$

(i) Find $\|T\|$.

(ii) Show that T is injective but not surjective, and show T^{-1} is not bounded from the image $\text{Ran}(T) \rightarrow E$.

(iii) Find $\|T^n\|$ for $n = 2, 3, \dots$, and show that T is *quasinilpotent*, meaning

$$\|T^n\|^{1/n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

(iv) Show that T has *no eigenvalues*. I.e., show that for any $\lambda \in \mathbb{C}$ and $u \in E$, if $Tu = \lambda u$ then $u = 0$.

• Reed-Simon p33: #11 (In (c), the goal is to extend I_α to $PC[0, 1]$. You may use the result of problem #10. See page 10 for the definition of $S[a, b]$ and $PC[a, b]$.)

• Reed-Simon p34: #27

• Reed-Simon p63: #4(b)